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Weingarten Timelike Tube Surfaces around a Spacelike Curve

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Abstract

The subject of this paper is the study of a timelike tube surface around the spacelike curve with timelike and spacelike binormal vectors in a three-dimensional Minkowski space E_1^3. Moreover, we have discussed Weingarten and linear Weingarten conditions for this surface with respect to their curvatures; the mean curvature H , Gaussian curvature K and the second Gaussian curvature K_{II} .

Keywords: Tube surfaces, Weingarten property, Minkowski space-time, Mean and Gaussian curvatures, Second Gaussian curvature

Mathematics Subject Classification: 53A05, 53B25

1 Introduction

In the study of the differential geometry of surfaces, it is common to determine some surfaces satisfying curvature conditions. An interesting curvature property to study for surfaces in a Minkowski space E_1^3 is the one that requires the existence of a non-trivial functional relationship between the principal curvatures. The resulting surfaces are called *Weingarten surfaces*. In particular, a surface S (in either R^3 or E_1^3) is called a Weingarten surface if there is some (smooth) relation $U(\kappa_1, \kappa_2) = 0$ between its two principal curvatures κ_1 and κ_2, or equivalently, if there exists a non -trivial functional relation $\Omega(K, H) = 0$ with respect to its Gaussian curvature K and its mean curvature H. The existence of a non-trivial functional relation $\Omega(K, H) = 0$ on the surface S parametrized by $\Phi(u, v)$ is equivalent to the vanishing of the corresponding Jacobian determinant, namely $\left|\frac{\partial(K,H)}{\partial(u,v)}\right| = 0$. Also, the linear Weingarten surfaces, these are Weingarten surfaces satisfying a linear equation between Gaussian and mean curvatures, that is, $aK + bH = c$, $(a, b, c) \in R$, $(a, b, c) \neq (0, 0, 0)$.

There are many studies of these surfaces [3],[4] gave a classification of ruled Weingarten surfaces in a Minkowski 3- space E_1^3. Recently [8],[9] investigated polynomial translation (K, H)- Weingarten surfaces and translation (H, K_{II})- linear Weingarten surfaces in a Euclidean 3- space.

Besides, several geometers [1, 2, 11],[6],[7] have studied Weingarten surfaces and linear Weingarten surfaces.

On account of the study that considered in[1],[11], we develop a corresponding work by studying a timelike tube surface around a spacelike curve in E_1^3 , which satisfy the Weingarten and linear Weingarten conditions with respect to its curvatures.

2 Preliminaries

Let $R^3 = \{(x, y, z) : x, y, z \in R\}$ be a 3-dimensional space, and let $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ be two vectors in R^3. The Lorentz scalar product of X and Y is defined by

$$\langle X\ , Y\ \ \rangle = x_1y_1 + x_2y_2 - x_3y_3, \tag{2.1}$$

$E_1^3 = (R^3,\ \langle X, Y\ \rangle)$ is called Minkowski 3−space. Since the metric is indefinite, recall that a vector x of E_1^3 can have one of three causal characters: it can be spacelike vector, null(lightlike) vector or timelike vector if $\langle x, x\ \rangle > 0$ or $x = 0$, $\langle\ x, x\ \rangle = 0$ and $x \neq 0$, $\langle x, x\ \rangle \lessdot 0$, respectively. For $x \in E_1^3$, the norm of the vector x is given by $\|x\| = \sqrt{|\langle x, x\ \rangle|}$. Therefore, x is a unit vector if $\langle x, x\rangle = \pm 1$. Similarly, an arbitrary curve $\alpha = \alpha(s) \subset E_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\acute{\alpha}(s)$ are respectively spacelike, timelike or null (lightlike) (i.e., $\langle \acute{\alpha}(s), \acute{\alpha}(s)\rangle > 0$, $\langle \acute{\alpha}(s), \acute{\alpha}(s)\rangle \lessdot 0$, $\langle \acute{\alpha}(s), \acute{\alpha}(s)\rangle = 0$). So, $\alpha(s)$ is a unit speed curve if $\langle \acute{\alpha}(s), \acute{\alpha}(s)\rangle = \pm 1$, where s is the arc length parameter of α. Any two vectors $X, Y \in E_1^3$ are called orthogonal [10] if $\langle X\ , Y\ \ \rangle = 0$.

The vector product of two vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$ belong to E_1^3, is defined as

$$X \wedge Y = (\ x_3y_2 - x_2y_3,\ x_1y_3 - x_3y_1,\ x_1y_2 - x_2y_1).$$

We also recall that the pseudosphere of radius 1 and center at the origin is the hyperquadric in E_1^3 defined by $S_1^2(1) = \{v \in E_1^3 : \langle v, v\ \rangle = 1\}$.

3 Timelike tube surface around a spacelike curve with a timelike binormal

In this section we define the notion of timelike tube surfaces.

Let $\delta = \delta(u) : (\zeta, \eta) \longrightarrow E_1^3$ be **a spacelike unit speed curve with a timelike binormal $\mathbf{e}_3$,** where u is the arc length parameter of δ.

Considering that $\|\mathbf{e}_1\| = 1$, $\mathbf{e}_2 = \frac{\mathbf{e}_1'}{\|\mathbf{e}_1'\|}$ and $\mathbf{e}_3 = \frac{\mathbf{e}_1 \wedge \mathbf{e}_1'}{\|\mathbf{e}_1 \wedge \mathbf{e}_1'\|}$, we obtain the orthonormal frame field $\{\mathbf{e}_1(u), \mathbf{e}_2(u), \mathbf{e}_3(u)\}$.This frame satisfies the following conditions:

$$\begin{aligned} \langle \mathbf{e}_1, \mathbf{e}_1 \rangle &= \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = 1, \\ \langle \mathbf{e}_3, \mathbf{e}_3 \rangle &= -1, \\ \langle \mathbf{e}_1, \mathbf{e}_2 \rangle &= \langle \mathbf{e}_2, \mathbf{e}_3 \rangle = \langle \mathbf{e}_3, \mathbf{e}_1 \rangle = 0, \end{aligned} \tag{3.1}$$

and vector product is defined to be

$$\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \wedge \mathbf{e}_3 = -\mathbf{e}_1, \quad \mathbf{e}_3 \wedge \mathbf{e}_1 = -\mathbf{e}_2. \tag{3.2}$$

Differential formula for this orthonormal system is expressed by

$$\mathbf{e}_1' = \kappa(u)\mathbf{e}_2(u), \qquad \mathbf{e}_2' = \kappa(u)\mathbf{e}_1(u) - \tau(u)\mathbf{e}_3(u), \qquad \mathbf{e}_3' = \tau(u)\mathbf{e}_2(u), \tag{3.3}$$

where the prime " ′ " denotes the derivative with respect to the u −parameter and $\kappa(u)$ and $\tau(u)$ are the curvature and the torsion of the curve $\delta(u)$ respectively.

Consider M is a timelike tube surface parametrized by $\Psi : j \times R \longrightarrow E_1^3$. Then, the position vector of Ψ can be written in the following form

$$\Psi(u, v) = \Psi_{(\delta,\, \mathbf{e}_2,\, \mathbf{e}_3,\, r)}(u, v) = \delta(u) + r\ (\mathbf{e}_2(u)\cosh v\ + \mathbf{e}_3(u)\sinh v\ ), \tag{3.4}$$

where $\delta, \mathbf{e}_2$, $\mathbf{e}_3 : j \longrightarrow E_1^3$ and $r : j \longrightarrow R_{.>0}$. We call δ a base curve and a pair of two vectors $\mathbf{e}_2$, $\mathbf{e}_3$ a director frame of the timelike tube surface Ψ. We must have

$$\| \Psi(u, v) - \delta(u) \|^2 = r^2.$$

The last equation expresses analytically the geometric fact that $\Psi(u, v)$ lies on a Lorentzian sphere $S_1^2(u)$ of radius r centered at $\delta(u)$.

The unit normal vector field σ of the timelike tube surface Ψ is given by

$$\sigma = \frac{\Psi_u \ \wedge \ \Psi_v}{\| \ \Psi_u \ \wedge \Psi_v \ \|}. \tag{3.5}$$

Accordingly to parametrization (3.4) of a timelike tube surface Ψ, one easily defines its first fundamental form

$$I = Edu^2 + 2Fdudv + Gdv^2,$$

where E, F, G - the coefficients of I - are given by

$$E = \langle \ \Psi_u, \Psi_u \ \rangle, \qquad F = \langle \ \Psi_u, \Psi_v \ \rangle, \qquad G = \langle \ \Psi_v, \Psi_v \ \rangle, \qquad \Psi_u = \frac{\partial \Psi(u,v)}{\partial u}. \tag{3.6}$$

For the spacelike surface in E_1^3, $EG-F^2 > 0$; for the timelike surface in E_1^3, $EG-F^2 < 0$.

We define the second fundamental form II of M by

$$II = Pdu^2 + 2Qdudv + Wdv^2,$$

with the coefficients given by

$$P = \langle \Psi_{uu} \ , \sigma \rangle, \ Q = \langle \Psi_{uv} \ , \sigma \rangle, \ W = \langle \Psi_{vv} \ , \sigma \rangle. \tag{3.7}$$

Moreover, the Gaussian and mean curvatures of the timelike tube surface $\Psi(u,v)$ are given by, respectively

$$K = \frac{PW - Q^2}{EG - F^2}. \tag{3.8}$$

$$H = \frac{EW \ - 2FQ \ + GP}{2(EG \ - F^2)}. \tag{3.9}$$

If the second fundamental form is non-degenerate; $PW - Q^2 \neq 0$. In this case, one define formally the second Gaussian curvature K_{II} a similar one to Brioschi's formula for the Gaussian curvature obtained on Ψ replacing the components of the first fundamental form E, F, G by those of the second fundamental form P, Q, W as [8]

$$K_{II}=\frac{1}{(PW-Q^2)^2}\left\{\begin{vmatrix} -\frac{1}{2}P_{uu}+Q_{uv}-\frac{1}{2}W_{uu} & \frac{1}{2}P_u & Q_u-\frac{1}{2}P_v \\ Q_v-\frac{1}{2}W_u & P & Q \\ \frac{1}{2}W_v & Q & W \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}P_v & \frac{1}{2}W_u \\ \frac{1}{2}P_v & P & Q \\ \frac{1}{2}W_u & Q & W \end{vmatrix}\right\}. \tag{3.10}$$

A surface in the 3 - dimensional Minkowski space E_1^3 is called a timelike surface if the induced metric on the surface is a Lorentz metric and is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, i.e., the normal vector on the timelike(spacelike) surface is a spacelike(timelike) vector [5] , [10] , [12].

In the following, we investigate the timelike tube surface Ψ in E_1^3 satisfying the Jacobi equation $\Omega(U,V)=0, U\neq V$, of the curvatures K, H and K_{II} of Ψ and we formulate the main results in the next theorems.

Let Ψ be a timelike tube surface in 3 - dimensional Minkowski space E_1^3 given in (3.4). So, from equation (3.3), partial differentiation of Ψ with respect to u and v are as follows

$$\begin{aligned} \Psi_u &= \mathbf{e}_1(u)+r\ [\mathbf{e}_2(u)\tau(u)\ \sinh v+\cosh v\ (-\kappa(u)\mathbf{e}_1(u)+\tau(u)\mathbf{e}_3(u))]\,, \\ \Psi_v &= r\ (\mathbf{e}_3(u)\cosh\ v+\mathbf{e}_2(u)\sinh v\ ). \end{aligned} \tag{3.11}$$

Therefore, we find the components of the first fundamental form of Ψ to be

$$E=-2r\ \kappa(u)\cosh v+r^2\kappa^2(u)\cosh^2 v-r^2\tau^2(u), \qquad F=-r^2\tau(u), \qquad G=-r^2. \tag{3.12}$$

Considering equation (3.2), from equation (3.5) we write the unit surface normal as

$$\sigma=(\mathbf{e}_2(u)\cosh v\ +\mathbf{e}_3(u)\sinh v\ ). \tag{3.13}$$

The second order partial differentials of Ψ are found

$$\begin{aligned} \Psi_{uu} &= (-r\ \kappa'\cosh v-r\ \kappa\ \tau\sinh v)\mathbf{e}_1+\big(\kappa(1-r\ \kappa\cosh v)+r\ \tau'\sinh v+r\ \tau^2\cosh v\big)\ \mathbf{e}_2 \\ &\quad +(r\ \tau^2\sinh v+r\ \tau'\cosh v)\ \mathbf{e}_3, \end{aligned}$$

$$\Psi_{uv}=-r\ \kappa\sinh v\ \mathbf{e}_1+r\ \tau\cosh v\ \mathbf{e}_2+r\ \tau\sinh v\ \mathbf{e}_3,$$

$$\Psi_{vv}=r\ (\mathbf{e}_2\cosh v+\mathbf{e}_3\sinh v).$$

From equation (3.13) and the last equations we find the second fundamental form coefficients as follow

$P = -\kappa(u)\cosh v + r\ \kappa^2(u)\cosh^2 v\ - r\ \tau^2(u), \qquad Q = r\ \tau(u), \qquad W = r.$

Making use of the data described above, the Gaussian curvature K, the mean value curvature H and the second Gaussian curvature K_{II} are given respectively as follows

$$K = \frac{\kappa(u)\cosh v}{r\ (-1 + r\ \kappa(u)\cosh v)}, \tag{3.14}$$

$$H = \frac{(1 - 2r\ \kappa(u)\cosh v)}{2r\ (-1 + r\ \kappa(u)\cosh v)}, \tag{3.15}$$

$$K_{II} = -\frac{(3 + \cosh 2v - 12r\ \kappa(u)\cosh^3 v + 8r^2\kappa^2(u)\cosh^4 v)\mathrm{sech}^2 v}{8r\ (-1 + r\ \kappa(u)\cosh v)^2}. \tag{3.16}$$

Theorem 3.1 Let Ψ be a timelike tube surface in Minkowski 3-space E_1^3 defined in (3.4) satisfying the Jacobi condition

$$\Omega(K, H) = 0, \tag{3.17}$$

for the Gaussian curvature K and the mean curvature H of Ψ.Then,

(i) Ψ is a Weingarten surface.

(ii) The Gaussian curvature K and the mean curvature H of Ψ are tied by the relation

$$K = \frac{2\kappa(u)\cosh v}{1 - 2r\ \kappa(u)\cosh v} H \tag{3.18}$$

Proof. Let Ψ be a timelike tube surface in E_1^3 parametrized by (3.4) and satisfying (3.17).Then, we have

$$\frac{\partial K}{\partial u}\frac{\partial H}{\partial v} - \frac{\partial K}{\partial v}\frac{\partial H}{\partial u} = 0. \tag{3.19}$$

Differentiating K given by (3.14) and H given by (3.15) with respect to u and v respectively, to obtain

$$\frac{\partial K}{\partial u} = -\frac{\acute{\kappa}(u)\cosh v}{r\ (-1 + r\ \kappa(u)\cosh v)^2}, \qquad \frac{\partial K}{\partial v} = -\frac{\kappa(u)\sinh v}{r\ (-1 + r\ \kappa(u)\cosh v)^2}, \tag{3.20}$$

$$\frac{\partial H}{\partial u}=\frac{r\ \acute{\kappa}\,(u)\cosh v}{2r\ (-1+r\ \kappa(u)\cosh v)^2},\qquad \frac{\partial H}{\partial v}=\frac{r\ \kappa(u)\sinh v}{2r\ (-1+r\ \kappa(u)\cosh v)^2}. \tag{3.21}$$

From (3.20) and (3.21), equation (3.19) is satisfied, and then, the surface Ψ is a Weingarten surface.

Furthermore, from (3.14) and (3.15) we get the required relation

$$K=\frac{2\kappa(u)\cosh v}{1-2r\ \kappa(u)\cosh v}H.$$

Theorem 3.2 Let Ψ be a timelike tube surface with non- degenerate second fundamental form in 3 - dimensional Minkowski space E_1^3 satisfying the Jacobi equation

$$\Omega(K_{II},K)=0, \tag{3.22}$$

for the second Gaussian curvature K_{II} and the Gaussian curvature K . Then, the surface Ψ is a Weingarten surface.

Proof. Let Ψ be a timelike tube surface in E_1^3 satisfying (3.22). Thus, after a derivation of K_{II} given by (3.16) with respect to u, followed by a derivation with respect to v, we get

$$(K_{II})_u=\frac{r^3(-\cosh 2v+r\ \kappa(u)\cosh^3 v)\ \acute{\kappa}\,(u)\ \mathrm{sech}\ v}{2(r^2(-1+r\ \kappa(u)\cosh v)^2)^{\frac{3}{2}}}, \tag{3.23}$$

$$(K_{II})_v=\frac{r^2\,\mathrm{sec\,h}^2 v\ (-1+r^2\kappa^2(u)\cosh^4 v-2r\ \kappa(u)\cosh v\ \ \sinh^2 v)\tanh v}{2(r^2(-1+r\ \kappa(u)\cosh v)^2)^{\frac{3}{2}}}. \tag{3.24}$$

Then, by (3.20) and (3.23), (3.24), the condition $\left|\frac{\partial(K_{II},K)}{\partial(u,v)}\right|=0$ that must be satisfied for the Weingarten surface Ψ, leads to

$$(K_{II})_u\ K_v-(K_{II})_v\ K_u=\frac{r^3(r\ \kappa(u)-\mathrm{sech}v)\ \acute{\kappa}\,(u)\tanh v}{2(r^2(-1+r\ \kappa(u)\cosh v)^2)^{\frac{5}{2}}}=0. \tag{3.25}$$

The obtained expression is equivalent with

$$r\ \kappa(u)\ \acute{\kappa}\,(u)\ \ \sinh v\ \ \cosh v-\acute{\kappa}\,(u)\sinh v=0.$$

The above equation is equal to zero for every value of v , so the coefficients of $\sinh v$ $\cosh v$ and $\sinh v$ must be zero. In this case, we find that $\acute{\kappa}=0$.

Similarly, we consider a timelike tube surface defined by (3.4) with non- degenerate second fundamental form in E_1^3 is (K_{II}, H)-Weingarten surface. Then, by using the equations (3.21) and (3.23), (3.24), we have

$$(K_{II})_u \ H_v - (K_{II})_v \ H_u = -\frac{\kappa' \text{sech } v \ \tanh v}{4r \ (-1 + r \ \kappa \cosh v)^4} = 0, \tag{3.26}$$

that is

$$\kappa' \sinh v = 0.$$

Thus, we obtain $\kappa' = 0$.

Now, to examine **the linear Weingarten property** of the timelike tube surface Ψ defined along the spacelike curve $\delta(u)$, we consider the following definition

Definition 3.1 A surface M is said to be a linear Weingarten surface if its Gaussian curvature K and mean curvature H satisfy the relation

$$aK + bH = c \tag{3.27}$$

on M for real numbers a , b and c (not all zero).
Let us analyze the following theorems.

Theorem 3.3 Suppose that Ψ is a linear Weingarten timelike tube surface in E_1^3 satisfying (3.27).Then Ψ is an open part of a circular cylinder.

Proof. Consider the parametrization (3.4). With K and H given by (3.14) and (3.15) respectively, we rewrite (3.27) to get

$$\frac{a \ \kappa(u) \cosh v}{r \ (-1 + r \ \kappa(u) \cosh v)} + \frac{b(1 - 2r \ \kappa(u) \cosh v))}{2r \ (-1 + r \ \kappa(u) \cosh v)} = c.$$

The previous equation can be expressed in a simpler form

$$(2a \ \kappa(u) - 2b \ r \ \kappa(u) - 2cr^2 \kappa(u)) \cosh v + (b + 2cr) = 0.$$

According to the definition of the linear independent of vectors, we obtain

$$2a \ \kappa - 2br \ \kappa - 2cr^2 \kappa = 0, \qquad b + 2c \ r = 0.$$

From which

$$\kappa(a + c \ r^2) = 0.$$

Therefore $\kappa = 0$ and $(a + c\, r^2) \neq 0$

because, if $(a + c\, r^2) = 0$ then $a = -c\, r^2$ and from the above $b = -2c\, r$, it leads to if $c = 0$, so $a = b = 0$ which contradicts the fact that $(a, b, c) \neq (0, 0, 0)$. Thus, the surface Ψ is *an open part of a circular cylinder.*

Theorem 3.4 If Ψ is a timelike tube surface with non-degenerate second fundamental form in E_1^3 where its Gaussian curvature and second Gaussian curvature are in linear relation, then Ψ is a non-linear Weingarten surface.

Proof. We assume that K and K_{II} of Ψ are in the form

$$aK + bK_{II} = c. \tag{3.28}$$

By (3.14) and (3.16), equation (3.28) can be written as follows

$$\frac{8a\ \kappa \cosh v}{r} - \frac{b\ (\ 3\ +\ \cosh 2v\ - 12r\ \kappa \cosh^3 v\ +\ 8r^2\kappa^2 \cosh^4 v\ ) \operatorname{sech}^2 v}{r\ (-1 + r\ \kappa \cosh\ \ v\ )} - 8c\ (-1 + r\ \kappa \cosh v) = 0.$$

The coefficients of the last equation must be zero. Thus, we have

$$8ar\kappa^2 - 8br^2\kappa^2 - 8r^3\kappa^2 c = 0, \qquad 8a\ \kappa - 12br\ \kappa - 16r^2 c\ \kappa = 0,$$

$$2b + 8rc = 0, \qquad 2b = 0.$$

From these equations, we obtain $b = 0,\ c = 0$ and $\kappa = 0$. So, the second fundamental form of the surface Ψ would vanish identically. Therefore, the surface Ψ is non-linear Weingarten surface.

Theorem 3.5 A timelike tube surface Ψ in E_1^3 with non-degenerate second fundamental form is linear Weingarten surface if the surface's curvatures H and K_{II} are written in a linear form.

Proof. Let us write the relation between H and K_{II} as follows

$$aH + bK_{II} = c. \tag{3.29}$$

Inserting (3.15) and (3.16) in (3.29), we get

$$\frac{-4a+12(a+b)r\ \kappa \cosh v-8(a+b)\ r^2\kappa^2 \cosh^2 v-b\ (3+\cosh 2v)\ \mathrm{sech}^2 v}{8r(-1+r\ \kappa \cosh v)^2}-c=0.$$

Also, all the coefficients in the above (algebraic) expression must be zero and consequently

$c \neq 0$, $a=2rc$, $b=-4rc$ and $\kappa=0$. So, the surface Ψ is a linear Weingarten surface.

We end this section with the following two theorems.

Theorem 3.6 The timelike tube surface of a spacelike curve with a timelike binormal vector in the Minkowski 3-space E_1^3 is a Weingarten surface.

Theorem 3.7 **(**N*on-existence theorem***)** There is no timelike tube surface, along a spacelike curve with a timelike binormal vector, of linear Weingarten type case in which $aK+bK_{II}=c$ in the three-dimensional Minkowski space E_1^3. In this case, the surface is parabolic.

4 Timelike tube surface for a spacelike curve with a spacelike binormal

Consider **a spacelike curve** $\beta(u)$ **with a spacelike binormal** $\mathbf{e}_3$ in Minkowski 3-space E_1^3, parametrized by its arc length u . Let $\mathbf{e}_1(u)$ be its tangent vector, i.e., $\mathbf{e}_1(u)=\acute{\beta}(u)=\frac{d}{du}\beta(u)$. The arc length parametrization of the curve makes $\mathbf{e}_1(u)$ a unit vector, i.e., $\parallel \mathbf{e}_1(u) \parallel = 1$, therefore its derivative is orthogonal to $\mathbf{e}_1$ .The principal normal vector $\mathbf{e}_2$ is defined as $\mathbf{e}_2=\frac{\acute{\mathbf{e}_1}}{\|\acute{\mathbf{e}_1}\|}$. The binormal vector $\mathbf{e}_3=\mathbf{e}_1\wedge\mathbf{e}_2$. The three vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ make an orthonormal frame field along the spacelike curve $\beta(u)$. The Frenet equations for this frame are given by,

$$\acute{\mathbf{e}}_1=\kappa(u)\mathbf{e}_2(u), \qquad \acute{\mathbf{e}}_2=\kappa(u)\mathbf{e}_1(u)+\tau(u)\mathbf{e}_3(u), \qquad \acute{\mathbf{e}}_3=\tau(u)\mathbf{e}_2(u), \tag{4.1}$$

with the following conditions,

$$\begin{aligned}\langle\mathbf{e}_1,\mathbf{e}_1\rangle &= \langle\mathbf{e}_3,\mathbf{e}_3\rangle=1,\\ \langle\mathbf{e}_2,\mathbf{e}_2\rangle &= -1,\\ \langle\mathbf{e}_1,\mathbf{e}_2\rangle &= \langle\mathbf{e}_2,\mathbf{e}_3\rangle=\langle\mathbf{e}_3,\mathbf{e}_1\rangle=0.\end{aligned} \tag{4.2}$$

The functions $\kappa(u)$ and $\tau(u)$ are respectively, the curve's curvature and torsion .

To examine the **Weingarten property** as well as the **linear Weingarten property** for the timelike tube surface $\Phi(u, v)$ along the **spacelike curve** $\beta(u)$ with a **spacelike binormal** $\mathbf{e}_3(u)$, we will consider the following basic equations:

Let Φ is parametrized by

$$\Phi(u, v) = \beta(u) + r\ (\mathbf{e}_2(u) \sinh v\ + \mathbf{e}_3(u) \cosh v\ ). \tag{4.3}$$

The components of the first and second fundamental forms of Φ are respectively

$$E = 1 + 2r\kappa(u) \sinh v + r^2\kappa^2(u) \sinh^2 v - r^2\tau^2(u), \qquad F = -r^2\tau(u), \qquad G = -r^2, \tag{4.4}$$

$$P = -(\kappa(u) \sinh v + r\ \kappa^2(u) \sinh^2 v - r\ \tau^2(u)), \qquad Q = r\ \tau(u), \qquad W = r. \tag{4.5}$$

The unit surface normal of the timelike tube surface Φ is

$$\sigma = (\mathbf{e}_3(u) \cosh v\ + \mathbf{e}_2(u) \sinh v\ ). \tag{4.6}$$

The curvatures of Φ are as follow

$$K = \frac{\kappa(u) \sinh v}{r + r^2\ \kappa(u) \sinh v}, \tag{4.7}$$

$$H = \frac{-1 - 2r\ \ \kappa(u) \sinh v}{2r\ (1 + r\ \ \kappa(u) \sinh v)}, \tag{4.8}$$

$$K_{II} = \frac{-3 - 3\coth^2 v + \operatorname{csch}^2 v - 2r\ (1 + 7\cosh 2v)\ \operatorname{csch}\ v\ \kappa(u) - 4r^2(1 + 3\cosh 2v)\ \kappa^2(u)}{8r\ (1 + r\ \ \kappa(u) \sinh v)^2}. \tag{4.9}$$

The straightforward calculations similar to the procedure which we have done for the above timelike tube surface Ψ, we have the following theorems:

Theorem 4.1 The timelike tube surface of a spacelike curve with a spacelike binormal vector in the Minkowski 3-space E_1^3 is a Weingarten surface.

Theorem 4.2 There is no timelike tube surface around a spacelike curve with a spacelike binormal vector, of linear Weingarten type case in which $aH + bK_{II} = c$ in Minkowski 3-space E_1^3, and the surface is then of constant mean curvature.

References

[1] H. S. Abdel-Aziz, Weingarten Tube Surfaces in 3-Dimensional Minkowski Space, Accepted for Pub. in Int. J. of Mathematics and Computation, No. S11 (12) (2011).

[2] F. Dillen, W. Goemans and I.Van de Woestyne, Translation surfaces of Weingarten type in 3-space, Bull.Transilvania Univ. Brasov, 50 (2008), 109-122.

[3] F. Dillen and W. Kühnel, Ruled Weingarten surfaces in Minkowski 3-space, Manuscripta Math, 98 (1999), 307-320.

[4] F. Dillen and W. Sodsiri, Ruled surfaces of Weingarten type in Minkowski 3-space, J.Geom., 83 (2005), 10-21.

[5] K. Duggal and A. Bejancu, Lightlike submanifolds of semi-Riemannian manifolds and applications, Mathematics and its Applications, Kluver Academic Publishers, Dordrecht,1996.

[6] W. Goemans and I. Van de Woestyne, Translation surfaces with the second fundamental form with vanishing Gaussian curvature in Euclidean and Minkowski 3-space, proceeding PADGE, 2007.

[7] R. López, Special Weingarten surfaces foliated by circles, Monatsh Math,154 (4) (2008), 289-302.

[8] M. Munteanu and A. Nistor, Polynomial translation Weingarten surfaces in 3-dimensional Euclidean space, Differential Geometry, Proceedings of the VIII international colloquium, Santiago de compostela, Spain, World Scientific, Hackensack, (2009), 316-320.

[9] M. Munteanu and A. Nistor, On the geometry of the second fundamental form of translation surface in E_1^3, Houston J.Math (in print), preprint available at http: // arXiv.org/abs/08123166, 2010.

[10] B. O'Neill, Semi-Riemannian geometry with applications to relativity, Academic press, NewYork,1983.

[11] J. Suk Ro and D. Won Yoon, Tubes of Weingarten types in a Euclidean 3-space, J. Chungcheong Mathematical Society, 22(3)(2009), 359-366.

[12] T. Weinsteim, An introduction to Lorentz surfaces, de Gruyter Exposition in Math, Walter de Gruyter, Berlin, 23 (1996).

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