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Calculation of Cascade Profiles From the Velocity Distribution

The method using the properties of analytical functions is applied to a plane, steady, inviscid, everywhere subsonic flow. From data fixed a priori concerning the external flow and some details of the profile, the hodograph is obtained as an analytical function whose real part is known on a contour. The set of imposed conditions being in general superabundant, the proposed Mach number distribution is corrected by means of a function whose form is fixed a priori, or rejected altogether. The problem is treated on a graphic display console connected with a computer, which provides also the profile corresponding to the calculated hodograph.

I Introduction

The method which is proposed here is due for the main part to R. Legendre [1].[1] It gives the possibility of obtaining the geometric shape (profile) of blades pertaining to a regular cascade. The method used is an inverse one, that is to say, instead of calculating the flow on a given profile, we try to find a profile fulfilling two different classes of conditions:

1 Strict conditions: velocity direction at infinity upstream and downstream, upstream Mach number, profile angle at the leading and at the trailing edges.

2 Nonstrict conditions: Mach number distributions at the upper and lower surfaces must be as near as possible to distributions proposed by the user.

If, instead of the condition 2, one wants the velocity distributions to be strictly equal to a given one, the problem has usually no solution, the set of constraints being superabundant. At the end of the calculation, the user will obtain a profile fulfilling strictly conditions 1, the velocity distribution differing from the proposed distribution by a certain correction.

If the data given by the user were close to a possible solution the correction will be small and the user can accept them; if it is not the case, he has to propose new ones until a compromise is obtained.

For more convenience, the method works on an interactive graphic display connected to a computer.

II Basic Hypothesis

The described method applies to plane, inviscid, steady, irrotational, everywhere subsonic flow.

The fluid is supposed to be compressible but schematized by a fictitious fluid, so-called "Chaplygin fluid." In such an approximation the arc of the isentropic curve which represents the set of physical conditions reached in the flow is approximated by a straight line in the pressure-density plane; we can write:

$$P/P_0 = c - B\rho/\rho_0$$

where B and C are constants and where the reference point is set at the stagnation point of the leading edge.

III Physical Plane Description

Fig. 1 shows a part of the notations used for the physical plane. The parameters pertaining to infinity upstream are characterized

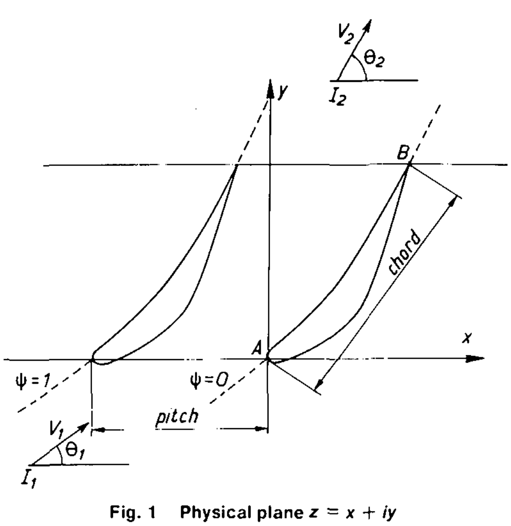

Fig. 1 Physical plane $z = x + iy$

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[1] Numbers in brackets designate References at end of paper.

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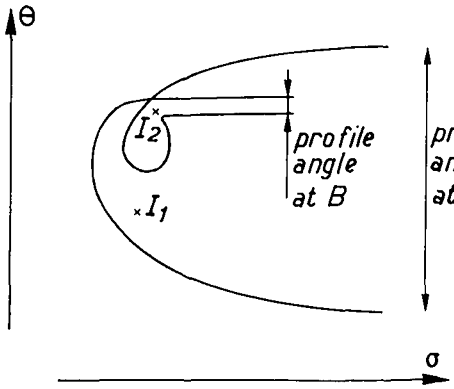

Fig. 2 Hodograph plane $\lambda = \sigma + i\theta$

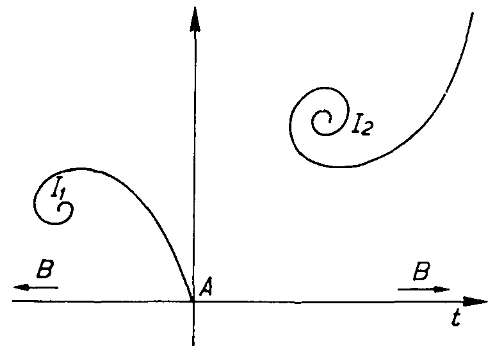

Fig. 3 μ plane

by subscript 1 and those related to infinity downstream by the subscript 2.

The user provides Mach number M_1, angles θ_1, and θ_2 fixing the direction of velocity and profile angles at the leading edge (α_1) and at the trailing edge (α_2).

The pitch is conventionally set to one. The abscissae axis is chosen parallel to the cascade plane.

IV Complex Potential

In the particular case of incompressible fluid, it is well known that a complex analytic function of the complex variable $z = x + iy$ exists; it is called *complex potential* and is such that:

$$F(z) = \phi + i\,\Psi$$

where ϕ is the velocity potential and ψ is the stream function. In the case of a so-called Chaplygin fluid it is possible to show that a complex potential exists but it is now an analytic function of the complex variable:

$$\lambda = \sigma + i\,\theta \quad \text{so that } F(\lambda) = \phi + i\,\Psi \qquad (1)$$

where θ is the angle of the velocity direction with the abscissae axis in the flow and

$$ch\sigma = 1/M$$

where M is the local Mach number.

The following equation which relates the physical plane to the λ (or hodograph plane) is also established:

$$A\ dz = [sh\ \sigma\ d\phi + i\ ch\ \sigma\, d\ \Psi]\ \exp\ (i\theta) \qquad (2)$$

where A is a scale factor.

The whole problem is then to calculate $F(\lambda)$. Later on, there are no difficulties to obtain the velocity field and the corresponding physical coordinates.

Instead of calculating directly $F(\lambda)$, it is easier to introduce an intermediate complex variable μ and to separate the problem into two parts: determination of $F(\mu)$, determination of $\lambda(\mu)$, these two functions being analytic.

Function $\lambda(\mu)$ may be considered as establishing a conformal mapping of the hodograph plane $\lambda = \sigma + i\theta$ (Fig. 2) on the μ plane. To define that mapping more precisely, we set that the upper half-plane μ corresponds to the useful part of the flow, the stagnation point on the leading edge being obtained for $\mu = 0$ and the stagnation point of the trailing edge for $\mu = \pm\infty$, the hodograph itself being mapped on the real axis (Fig. 3).

V Complex Potential Derivation

In the physical plane (z plane) the flow may be regarded as resulting of a vortex-source at upstream-infinity discharging in a vortex-sink at downstream-infinity.

Let $D\phi_1$ and $D\phi_2$ be the circulation of these two singular points.

The situation is not different in the μ plane; let μ_1 and μ_2 be the coordinates of these singular points (Fig. 3). To be sure that the real axis be a streamline, it is sufficient to put two new singular points, mirror images of μ_1 and μ_2 in the real axis. Let $\bar{\mu}_1$ and $\bar{\mu}_2$ be these two images.

The potential may then be written:

$$\begin{aligned} 2\pi\ F(\mu) = {} & (1 - i\ D\phi_1)\ \text{Log}\ (1 - \frac{\mu}{\bar{\mu}_1}) \\ & + (1 + iD\Phi_1)\ \text{Log}\ (1 - \frac{\mu}{\mu_1}) \\ & - (1 - iD\Phi_2)\ \text{Log}\ (1 - \frac{\mu}{\mu_2}) - (1 + iD\Phi_2)\ \text{Log}\ (1 - \frac{\mu}{\bar{\mu}_2}) \end{aligned} \qquad (3)$$

if we require that the stream velocity be zero at the leading edge ($\mu = 0$) and at the trailing edge ($\mu = \pm\infty$) and if we write

$$\mu_1 = \exp\ [\ \pi(h_1' + ih_1'')\ ] \quad \text{and} \quad \mu_2 = \exp\ [\pi(h_2' + ih_2'')]$$

it is possible to express h''_1 and h''_2 as functions of:

$$\sigma_1,\ \sigma_2,\ \theta_1,\ \theta_2 \text{ and } dh^1 = h_2^1 - h_1^1$$

As the length scale in plane μ has not been defined up to now, it is possible to impose $h'_1 = 0$. Only one arbitrary parameter remains: dh'; it is related to the chord of the profile, but the relation is quite intricate. In order to fix a length scale in the physical plane, we shall take the cascade pitch as unity. By means of relation (2) we obtain:

$$D\Phi_1 = -\ \text{coth}\,\sigma_1\ \text{cotg}\ \theta_1 \quad D\Phi_2 = -\ \text{coth}\,\sigma_2\ \text{cotg}\ \theta_2 \qquad (4)$$

$$A = \frac{\text{ch}\sigma_1}{\sin\ \theta_1} = \frac{\text{ch}\sigma_2}{\sin\ \theta_2} \qquad (5)$$

Equation (5) is one of the formulations of conservation of mass written between upstream and downstream infinity.

Nomenclature

x, y = coordinates in the physical plane
z = $x + iy$
M = Mach number
σ = variable defined by $ch\sigma = 1/M$
θ = angle of the velocity with the abscissae axis
λ = $\sigma + i\theta$
μ = auxiliary complex variable
a = sound velocity
F = complex potential $F = \Phi + i\Psi$
α_1 = profile leading edge angle
α_2 = profile trailing edge angle

Subscripts

1 = upstream conditions
2 = downstream conditions

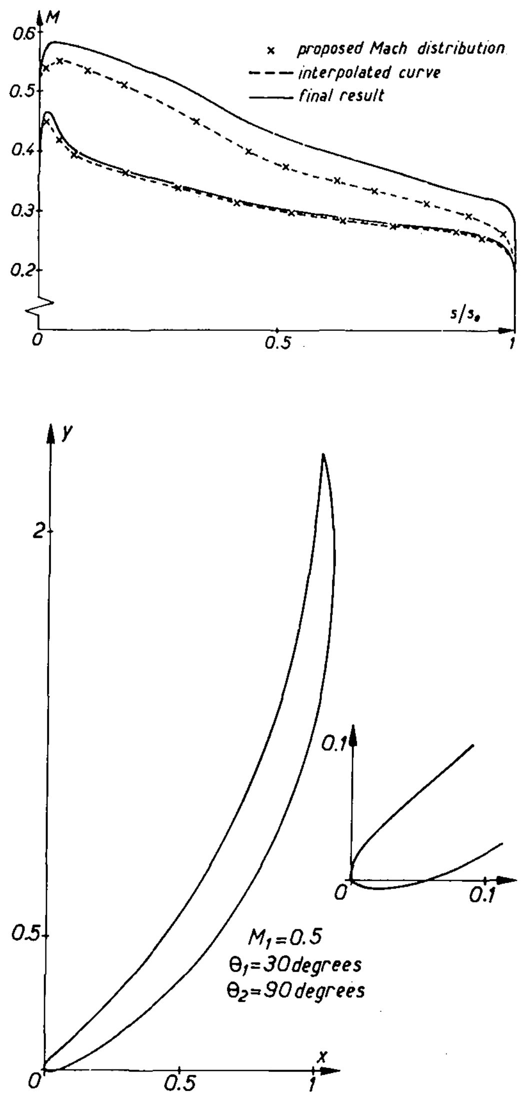

Fig. 4 Typical case
(a) Mach numbers distribution; *(b)* profile

Complex potential $F(\mu)$ is now completely defined.

VI Derivation of $\lambda(\mu)$

Let us suppose that the user knows M as a function of μ, first on the lower surface ($\mu > 0$ real), second on the upper surface ($\mu < 0$ real)—we shall see later how this result can be obtained; $\lambda(\mu)$ derivation may be regarded as the determination of a function, analytic in the upper half plane μ and, the real part of which being known on the real axis; that problem is known as a Dirichlet problem, the solution of which is:

$$\lambda(\mu) = \frac{i}{\pi}\int_{-\infty}^{+\infty} \text{Log}\ (t - \mu)\ d\sigma(t) + Cst \qquad (6)$$

where t is μ taken along the real axis.

There are many difficulties to obtain a numerical computation of that integral:

—σ is infinite (i.e., M → 0) when μ reaches zero or infinity: singularities extraction;

—computation of the constant;

—expression of the profile closing condition.

6.1 Singularities Extraction. It is possible to show that $\lambda(\mu)$ behaves like:

$$-\frac{\alpha_1}{\pi}\ \text{Log}\ \mu \qquad \text{for } \mu \to 0$$

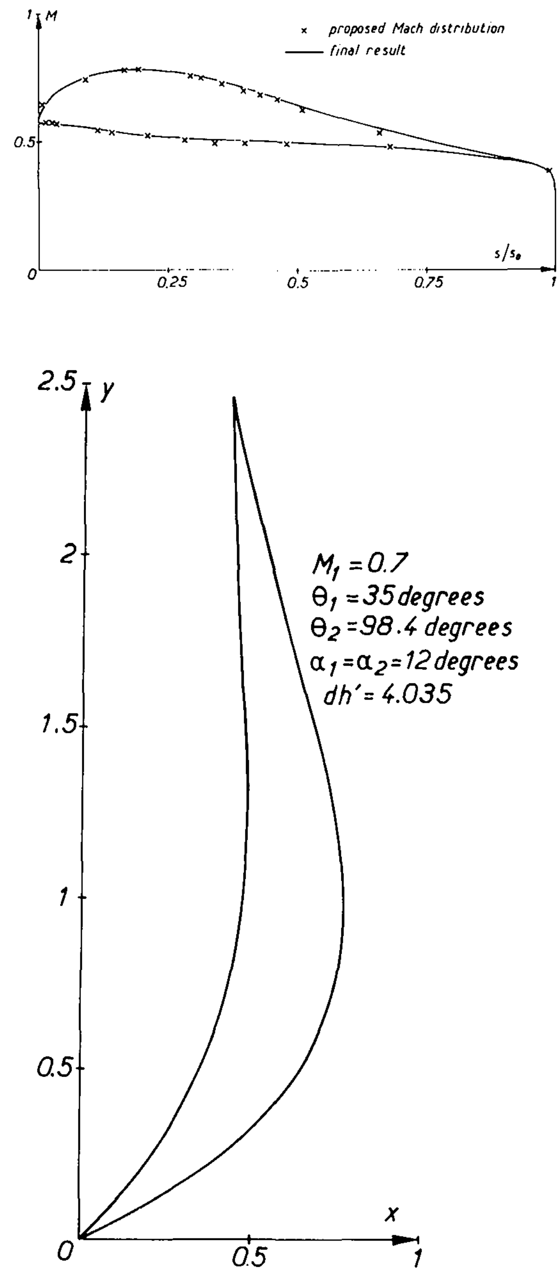

Fig. 5 First example
(a) Mach numbers distribution; *(b)* profile

and

$$+\frac{\alpha_2}{\pi}\ \text{Log}\ \mu \qquad \text{for } \mu \to \pm\infty$$

This fact guarantees that the streamline angle changes by α_1, as μ goes from $0 - \epsilon$ to $0 + \epsilon$; α_1 is then the leading edge angle. Likewise, α_2 is the trailing edge angle.

Let us set:

$$\sigma(t) = \sigma^1(t) - \frac{\alpha_1}{\pi}\ \text{Log}\ |t| + \frac{\alpha_1 + \alpha_2}{2\pi}\ \text{Log}(t^2 + a^2) \qquad (7)$$

We then have

$$\lambda(\mu) = -\frac{\alpha_1}{\pi}\ \text{Log}\ \mu + \frac{\alpha_1 + \alpha_2}{\pi}\ \text{Log}(\mu + ia)$$

$$+ \frac{i}{\pi}\int_{-\infty}^{+\infty} \text{Log}\ (t - \mu) d\sigma^1(t) + Cst \qquad (8)$$

where σ' is now finite when μ tends to zero or infinity and where

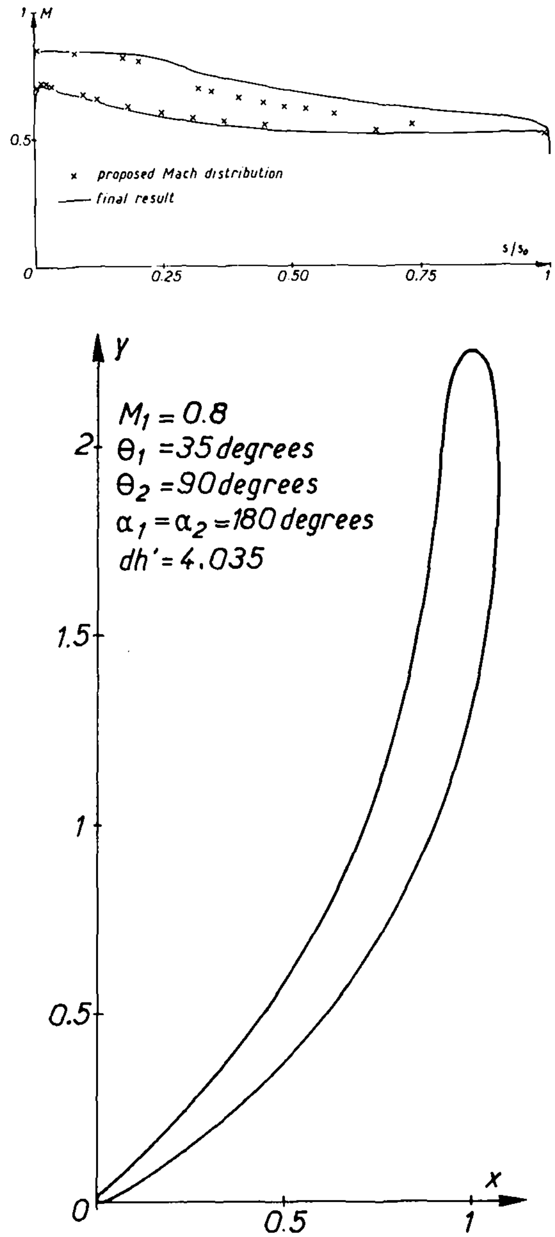

Fig. 6 Second example *(a)* Mach numbers distribution; *(b)* profile

a is an arbitrary positive real.

Using the residue theorem, the following relation also holds:

$$\lambda(\mu) = -\frac{\alpha_1}{\pi}\,\mathrm{Log}\,\mu + \frac{\alpha_1 + \alpha_2}{\pi}\,\mathrm{Log}\,(\mu + ia) + \frac{1}{i\pi}\int_{-\infty}^{+\infty}\frac{\sigma^1(t)\,dt}{t-\mu} + Cst \quad (8')$$

6.2 Computation of the Constant. We have computed μ_1 previously (and also μ_2); moreover λ_1 is known: $\lambda_1 = \sigma_1 + i\theta_1$

Let us write

$$G(\mu) = -\int_{-\infty}^{+\infty}\frac{\sigma^1(t)}{t-\mu}\,dt$$

If we express that $\lambda(\mu_1) = \lambda_1$, then it comes:

$$\lambda(\mu) = \lambda_1 - \frac{\alpha_1}{\pi}\,\mathrm{Log}\,\frac{\mu}{\mu_1} + \frac{\alpha_1 + \alpha_2}{\pi}\,\mathrm{Log}\,\frac{\mu + ia}{\mu_1 + ia} + \frac{i}{\pi}\left[G(\mu) - G(\mu_1)\right] \quad (9)$$

6.3 Profile Closing Condition. If the Mach number distribution $M(t)$ (or $\sigma(t)$) is chosen arbitrarily there is no reason for the relation $\lambda(\mu 2) = \lambda_2$ to be satisfied, whereas λ_2 and μ_2 are known. This results in a profile which does not close. We shall then add to $\lambda(\mu)$, as given by expression (9), a correcting function $\delta\lambda(\mu)$, analytic in the upper-half plane and sufficiently well behaved on the real axis not to disturb $\lambda(\mu)$ in the neighbourhood of the leading or trailing edge.

From a practical point of view it is convenient to choose that function in the following form:

$$\delta\lambda(\mu) = \alpha F_1(\mu) + \beta F_2(\mu) \quad (10)$$

where α and β are real and are obtained through resolution of:

$$\lambda_2 = \lambda(\mu_2) + \alpha[F_1(\mu_2) - F_1(\mu_1)] + \beta[F_2(\mu_2) - F_2(\mu_1)] \quad (11)$$

i.e, through resolution of a system of two linear equations with two real unknowns.

When α and β are obtained, we have:

$$\lambda = \lambda_1 - \frac{\alpha_1}{\pi}\log\frac{\mu}{\mu_1} + \frac{\alpha_1 + \alpha_2}{\pi}\log\frac{\mu + ia}{\mu_1 + ia} + \frac{i}{\pi}[G(\mu) - G(\mu_1)] + \delta\lambda(\mu) - \delta\lambda(\mu_1) \quad (12)$$

VII How to Choose the Correcting Function

This choice is difficult. After a large number of tests we presently use the function:

$$\delta\lambda = i\alpha\log\frac{\mu - \mu_3}{\mu + i\mu_5} + i\beta\log\frac{\mu - \mu_4}{\mu + i\mu_5}$$

in which:

$$\mu_5 = 2|\mu_2|, \quad \mu_3 = \frac{1-i}{2\sqrt{2}} \quad \mu_4 = -\frac{1+i}{2\sqrt{2}} \quad (13)$$

The real part of that function goes to zero when μ goes to infinity (real), and is discontinuous at the origin (the discontinuity is $(\alpha + \beta)\mu$). This may be approximately regarded as two different shifts on the lower and upper surfaces.

VIII Profile Calculation

We have already seen that in any case we have:

$$A\,dz = [\mathrm{sh}\,\sigma\,d\,\Phi + i\,\mathrm{ch}\,\sigma\,d\,\Psi]\exp\,i\theta,$$

with (5) (2)

$$A = \frac{\mathrm{ch}\,\sigma_1}{\sin\theta_1} = \frac{\mathrm{ch}\,\sigma_2}{\sin\theta_2}$$

Along the profile, $d\psi = 0$ and μ is real. It comes:

$$A\,dz = \exp\,(i\theta)\,\mathrm{sh}\,\sigma\frac{dF}{d\mu}\,d\mu \quad (14)$$

With the origin of coordinates at the stagnation point of the leading edge, we integrate the foregoing function of $\mu = t$ from 0 to $+\infty$ to obtain the lower surface and from 0 to $-\infty$ to obtain the upper surface.

IX Different Stages of a Typical Computation

First of all the user has to provide the following quantities:

$$M, \text{ or } \sigma_1,\ \theta_1,\ \theta_2,\ \alpha_1,\ \alpha_2,\ dh^1;$$

M_2 and σ_2, μ_1 and μ_2, $D\phi_1$, and $D\phi_2$ are then obtained. Using equation (3), there is no difficulty to compute the complex potential $F(\mu)$.

The user has now to give the Mach number distributions he wishes on the lower and upper surfaces. As a matter of fact what is needed is σ and it may be obtained either from $M(ch\sigma = 1/M)$ or from $V(sh\sigma = a_0/V)$.

Suppose that he knows these distributions as a function of the reduced curvilinear abscissa ($s = 0$ at the leading edge stagnation point and $s = 1$ at the trailing edge upper or lower surface). However this curvilinear abscissa cannot be directly introduced in the computation. It has been noticed that in most cases the behavior of the quantity ϕ/ϕ_0 (where ϕ is the velocity potential with $\phi = 0$ at the leading edge and $\phi = \phi_0$ at the trailing edge, different on the lower and upper surfaces) looks sufficiently like curvilinear abscissa behavior.

It is then asked to the user to give M(t) but M is presented on the screen of the graphic display unit as a function of ϕ/ϕ_0. He has to adjust the values of t until he obtains a sufficient resemblance with the curve M(s/s_0) he wishes.

As a matter of fact, only ten to twenty points are needed on the upper and on the lower surface, as the neighborhoods of the stagnation points are provided by the program itself. When the points are given, one hundred points are interpolated by means of cubic spline functions so as to allow a sufficient accuracy in the numerical computation of $G(\mu)$.

If the user is satisfied with those curves the computation may then be carried to its end: profile and corrected Mach numbers distribution computation.

The actual Mach number distribution as a function of reduced curvilinear abscissa is presented on the graphic display unit as well as the initially given distribution.

If the result is unsatisfactory, the user has to move a few points until it suits him.

The chord has also to be adjusted for it is known only at the very end of the computation. Parameter dh' must be used for that purpose.

Fig. 4 shows on one hand the points proposed by the user and the interpolated curves going through these points and necessary for the numerical computation of F in (12). On the other hand the final curve obtained has been drawn. A rather large discrepancy between the two curves corresponding to the upper surface may be seen.

Fig. 4(b) shows the profile obtained as well as a detail of the leading edge where it appears that the introduced discontinuity has no noticeable effect on its shape.

Upstream Mach number was 0.5, θ_1 was 30 deg and θ_2 90 deg, the leading edge was rounded ($\alpha_1 = \pi$) and the trailing edge had an angle $\alpha_2 = 15$ deg.

X Examples of Results

Fig. 5 shows a first example of results, in a case where the agreement between the wished Mach number distribution and the curves finally obtained is very good. The profile is sharp at both ends (12 deg angle) for an upstream Mach number of 0.7; chord pitch ratio is 2.5.

Fig. 6 shows a second example. The profile is rounded at both ends. Upstream Mach number is 0.8 and deflexion ($\theta_2 - \theta_1$) is 55 deg.

Contrary to the former example, a Mach number distribution on the upper surface showing a quite steep descent after maximum velocity was requested. This particular result was only partially fulfilled.

XI Comparison With a Direct Method

The described method is exact as far as principles are concerned. Practically, the only possible inaccuracy may come from numerical calculation of integral:

$$G(\mu) = -\int_{-\infty}^{+\infty} \frac{\sigma'(l)}{l - \mu} dl$$

There is no difficulty to obtain a sufficient accuracy. The result may be checked when looking at the precision with which the upper and lower part of the profile close at the trailing edge stagnation point. Some parts of the calculation have to be made with double-precision arithmetic (when t is near μ). Usually the result is obtained with an accuracy better than 1 percent.

Finally the only point which has to be checked is the quality of Chaplygin approximation for the fluid.

For flows in which maximum Mach number is quite low the approximation will be good but as this Mach number raises and approaches unity the results will depart from those obtained with exact compressibility calculations.

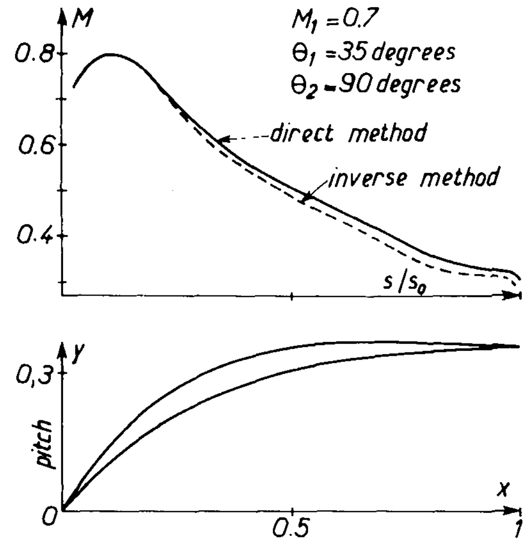

Fig. 7 Comparison with a direct method

Comparison was made with a direct method[3] (finite differences method) using perfect gas approximation with exact compressibility law.

Of course, that comparison is possible only if one first calculates with the inverse method. A profile corresponding to a given Mach number distribution is obtained. The direct method is then used to compute the flow on that profile. Finally the Mach number (or velocity or pressure) distributions are compared.

This was first done for an upstream Mach number 0.5 with a profile very similar to Fig. 7. No visible differences are noticed.

Fig. 7 shows Mach number distribution on the upper part of the profile (as a function of reduced curvilinear abscissa s/s_0) and the profile itself for an upstream Mach number 0.7. The agreement is quite good mainly on the lifting part of the profile.

XII Conclusion

The method proposed in this paper brings a noticeable improvement as far as user point of view is concerned, compared with a former method [2] in which $\lambda(\mu)$ was chosen inside a set of curves depending on four real parameters: in that case these parameters had no physical meaning and the only way to choose them was by trial and error. On the other hand, to impose a particular set of curves was of course a restriction.

In the new method, what is asked from the user has a physical meaning, and the set of possible solutions is much larger.

Computer time is about 30 sec for one execution of the program but it has usually to be run many times before a compromise is obtained. The computer used was an IRIS 80 of CII (Compagnie Internationale pour l'Informatique).

Annex

Complex Potential Derivation. A complex analytic function is completely defined by its singularities. From an hydrodynamic point of view, it is equivalent to say that the flow is completely defined by its source or sinks, vortices, double-sources and by the singularities corresponding to the solid walls.

In the physical plane, the flow results from a source placed at upstream infinity discharging in a sink at downstream infinity with a vortex superimpose in each of these points.

The potential may be obtained through addition of the potentials of the singularities already mentioned to which new ones, placed inside the contour, must be added in order to satisfy to velocity tangency on the profile.

It is well known that:

$$F(z) = \frac{K}{2\pi} \log\left(1 - \frac{z}{a}\right) = \varphi + i\psi$$

is the expression of the potential of a source placed at point a in complex plane.

The source output (integral of ψ on a closed contour around a) is equal to k.

In the same manner

$$F(z) = -\frac{i\Gamma}{2\pi} \log\left(1 - \frac{z}{a}\right) = \varphi + i\psi$$

is the potential of a vortex placed in a, the circulation (integral of φ on a closed contour around a) of which is Γ.

If one names μ_1 and μ_2 the images in the μ plane of upstream and downstream infinity in the physical plane, $D\Phi_1$ and $D\Phi_2$ the circulations around these points, it is easy to write an expression of the corresponding potential with the supplementary condition that the real axis (image of the profile) is a stream line.

One obtains equation (3)

$$2\pi F(\mu) = (1 - iD\Phi_1) \log\left(1 - \frac{\mu}{\mu_1}\right) + (1 + iD\Phi_1) \log\left(1 - \frac{\mu}{\bar{\mu}_1}\right)$$

$$- (1 - iD\Phi_2) \log\left(1 - \frac{\mu}{\mu_2}\right) - (1 + iD\Phi_2) \log\left(2 - \frac{\mu}{\bar{\mu}_2}\right)$$

where $\bar{\mu}_1$ and $\bar{\mu}_2$ are conjugate to μ_1 and μ_2.

At the present time neither μ_1 and μ_2 nor $D\Phi_1$ and $D\Phi_2$ are known.

We shall calculate μ_1 and μ_2 when writing that the flow velocity is zero at the two stagnation points (leading edge corresponding to $\mu = 0$ and trailing edge corresponding to $\mu = \pm\infty$ on the real axis).

In a point of the flow, the velocity is given by:

$$V = u - iv = \frac{dF(\mu)}{d\mu}$$

After deriving and setting the terms in μ in order, it comes:

$$2\pi \frac{dF}{d\mu} = \frac{C_0 + C_1\mu + C_2\mu^2}{D}$$

where C_0, C_1, and C_2 are real. Consequently there are, generally speaking, two points on the real axis where the velocity is zero. In order to put these points at $\mu = 0$ and $\mu = \pm\infty$ we must have $C_0 = 0$ and $C_2 = 0$.

The solution of these two equations gives μ_1 and μ_2.

The method we choose was to write:

$$\mu_1 = \exp\left[\pi(h_1' + ih_1'')\right] \quad \mu_2 = \exp\left[\pi(h_2' + ih_2'')\right]$$

The length scale in plane μ was not already fixed; this can be done by setting $h_1' = 0$ (we then write $h_2' - h_1' = dh' = h_2'$.

Then comes:

$$\cos \pi h_1'' + D\Phi_2 \sin \pi h_1'' = \exp(\pi dh')[\cos \pi h_2'' + D\Phi_2 \sin \pi h_2'']$$

$$\cos \pi h_2'' - D\Phi_2 \sin \pi h_2'' = \exp(\pi dh')[\cos \pi h_1'' - D\Phi_1 \sin \pi h_1'']$$

whose solution is:

$$tg\pi(1 - h_1'') = \frac{sh(\pi dh') \sin \nu_1}{ch(\pi dh') \cos \nu_1 - \cos \nu_2}$$

$$tg\pi h_2'' = \frac{sh(\pi dh') \sin \nu_2}{ch(\pi dh') \cos \nu_2 - \cos \nu_1}$$

where it has been set:

$$\nu_1 = Arctg(th\sigma_1 tg\theta_1) \quad \nu_2 = Arctg(th\sigma_2 tg\theta_2)$$

This gives μ_1 and μ_2 providing dh' has been choosen, which corresponds to the choice of the profile chord (the pitch being conventionally fixed to unity).

We then have to calculate $D\Phi_1$ and $D\Phi_2$.

Starting from equation (2)

$$Adz = [sh\sigma d\Phi + ich\sigma d\psi] \exp(i\theta)$$

in which A is a real scale factor in the physical plane; we shall calculate this factor through writing that the pitch is unity.

When integrating (2) at upstream infinity with with $\Delta z = -1$ and $\Delta\psi = +1$, it comes:

$$-A = (sh\sigma_1 D\Phi_1 + ich\sigma_1)(\cos \theta_1 + i \sin \theta_1)$$

As A is real we have:

$$0 = \sin \theta_1 sh\sigma_1 D\Phi_1 + \cos \theta_1 ch\sigma_1 \Rightarrow D\Phi_1 = -\cot g\theta_1 \coth \sigma_1$$

and

$$-A = \cos \theta_1 sh\sigma_1 D\Phi_1 - \sin \theta_1 ch\sigma_1 \Rightarrow A = \frac{ch\sigma_2}{\sin \theta_1}$$

The same integration at downstream infinity gives the two relations:

$$D\Phi_2 = -\cot g\theta_2 \coth \sigma_2 \quad A = \frac{ch\sigma_2}{\sin \theta_2}$$

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