Transverse Diffusion in Saturated Isotropic Granular Media

GEOLOGICAL SURVEY PROFESSIONAL PAPER 411-B

Ogden—TRANSVERSE DIFFUSION IN SATURATED ISOTROPIC GRANULAR MEDIA—Geological Survey Professional Paper 411-B

Transverse Diffusion in Saturated Isotropic Granular Media

By AKIO OGATA

FLUID MOVEMENT IN EARTH MATERIALS

GEOLOGICAL SURVEY PROFESSIONAL PAPER 411-B

UNITED STATES GOVERNMENT PRINTING OFFICE, WASHINGTON : 1961

UNITED STATES DEPARTMENT OF THE INTERIOR

STEWART L. UDALL, *Secretary*

GEOLOGICAL SURVEY

Thomas B. Nolan, *Director*

For sale by the Superintendent of Documents, U.S. Government Printing Office
Washington 25, D.C. - Price 15 cents (paper cover)

CONTENTS

ILLUSTRATIONS

TRANSVERSE DIFFUSION IN SATURATED ISOTROPIC GRANULAR MEDIA

By Akio Ogata

ABSTRACT

An analytical method of determining the ionic diffusion transverse to the direction of flow in granular media is presented. The basic assumption that the frontal zone created by longitudinal dispersion is fully developed and stationary before the transverse diffusion can cause spreading is made to simplify the mathematical analysis. Solutions were obtained in terms of both a series of Bessel's functions and a hypergeometric series. Numerical computations were made and results are presented in graphical form.

INTRODUCTION

The necessity for disposal of radioactive and other wastes and for more quantitative knowledge of microscopic flow in porous media has focused interest on dispersion phenomena occurring in saturated flow through porous media. Various laboratory techniques have been developed for determining the dispersion coefficient appearing in the analytical solutions. Emphasis, however, has been placed on studies of longitudinal dispersion—for example, those of Beran (1957) and Day (1956) and by the writer (Dispersion in porous media, doctoral dissertation, Northwestern Univ., 1958). The data available on transverse diffusion are still extremely meager, owing to the difficulty in modeling transverse diffusion and measuring the concentration distribution once a model is constructed.

The Geological Survey in its ground-water research office at Phoenix, Ariz., has developed methods for measuring transverse diffusion, by means of radioactive isotopes and commercially available counters (Skibitzke and others, 1960). However, the studies to date have been directed toward feasibility of the methods, and quantitative data are not available. This paper will consider the mathematical aspects of an approximate description of the phenomena by dealing with a simplified mathematical model.

ACKNOWLEDGMENTS

The problem treated in this report was brought to the attention of the writer by H. E. Skibitzke, mathematician in charge of the Phoenix ground-water research office. The writer is greatly indebted to Mr. Skibitzke, whose initial development and ideas greatly facilitated the preparation of this report.

The author would like to express appreciation also to Prof. Richard Skalak, of Columbia University, for his thorough discussion and review of the paper. Professor Skalak points out that Goldstein, in his paper "Some Two-Dimensional Diffusion Problems with Circular Symmetry" (1932), obtained equation 27. However, his paper was not available to the writer at the time of writing. Professor Skalak points out also that there is a shorter method of derivation of equation 9 by use of the Hankel transform. The author utilized the Laplace transformation because this method has a wider range of applicability and is a more familiar method of approach.

TWO-DIMENSIONAL DISPERSION

The differential equation that describes the two-dimensional dispersion of a contaminant in saturated granular media, where radial symmetry exists, is

$$D_r \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)+D_x\frac{\partial^2 C}{\partial x^2}-u\frac{\partial C}{\partial x}=\frac{\partial C}{\partial t} \qquad (1)$$

where
C=concentration of contaminants in lb/ft^3
r=radial distance, in feet, measured from center of contaminant filament
x=longitudinal distance, in feet, measured in the direction of fluid flow
t=time, in seconds
D_r=radial diffusion coefficient, in ft^2/sec
D_x=longitudinal dispersion coefficient, in ft^2/sec
u=average fluid velocity, in ft/sec

Equation 1 is readily obtained from the law of conservation of mass. Owing to difficulty in solving the general case, it is necessary to make some simplifying assumptions that are physically realistic.

584498—61——2

The physical nature of the dispersion process is shown in figure 1. The system may be divided into two regions, the frontal zone and the cylindrical filament of contaminant upstream from the frontal zone. In the frontal zone the dispersion, or spreading, of the contaminant is due to longitudinal plus transverse dispersion. However, in the region upstream from this frontal zone the spreading is due principally to transverse diffusion and convection because of the virtual nonexistence of a concentration gradient in the direction of flow. Thus, if the frontal zone is not considered, $D_x\frac{\partial^2 C}{\partial x^2}=0$ and equation (1) reduces to

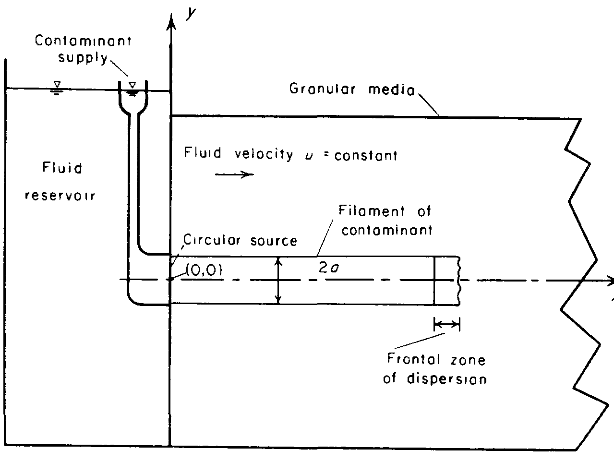

FIGURE 1.—Physical setting of mathematical model.

$$\frac{D_r}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)-u\frac{\partial C}{\partial x}=\frac{\partial C}{\partial t} \tag{2}$$

In discussing the frontal zone, consider a moving coordinate system—that is, let $\xi=x-ut$ and $\tau=t$. Substituting this transformation into equation (1) gives

$$\frac{D_r}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)-D_x\frac{\partial^2 C}{\partial \xi^2}=\frac{\partial C}{\partial \tau}$$

Physically, the above equation indicates that the observer is moving at a velocity, u.

Experimental and analytical development of the dispersion in a one-dimensional system indicates that the length of the frontal zone remains constant, once it is fully developed. Von Rosenberg (1956) indicates that the zone in which the concentration varies by a certain percentage depends on the velocity and the dispersion coefficient. In the system considered here, in which no mass exchange occurs between the liquid and the solid phase, this front will progress through the medium with the velocity of the fluid.

Because of the existence of another interface in the radial direction, the two-dimensional system is not the same as the one-dimensional system. But, if it is assumed that the establishment of this front takes place at a relatively rapid rate in comparison with transverse diffusion, then the time rate of change of concentration depends only on the radial dispersion. Thus, the differential equation describing the process is

$$\frac{D_r}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)=\frac{\partial C}{\partial \tau}$$

Substituting $\xi=x-ut$, $t=\tau$ in equation 2 and resubstituting t for τ gives

$$\frac{D_r}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)=\frac{\partial C}{\partial t} \tag{3}$$

Equation 3 is the fundamental differential equation describing the dispersion process throughout the region for long times. This is the differential equation found in various radial-diffusion or heat-flow problems, whose solutions are known for various boundary conditions.

The boundary conditions for the case to be considered are

$$(1)\quad t=0,\quad C=C_0,\ r<a$$

$$(2)\quad t=0,\quad C=0,\ r>a$$

$$(3)\quad r=0,\ \frac{\partial C}{\partial r}=0 \tag{4}$$

It is evident that an initial-value problem needs to be solved. Because the longitudinal-dispersion coefficient, D_x, no longer appears, the subscript r will be dropped from the radial-diffusion coefficient.

SOLUTION OF THE DIFFERENTIAL EQUATION

Solution of equation 3 may be obtained by employing the Laplace transform; that is, assume that there exists a function

$$\bar{C}(r,p)=\int_0^\infty e^{-pt}C(r,t)dt$$

If equation 3 is multiplied by e^{-pt} and integrated with respect to t within the limits 0 to ∞, equation 3 may be written

$$\frac{D}{r}\frac{d}{dr}\left(r\frac{d\bar{C}}{dr}\right)+C_0-p\bar{C}=0;\ r<a \tag{5a}$$

$$\frac{D}{r}\frac{d}{dr}\left(r\frac{d\bar{C}}{dr}\right)-p\bar{C}=0;\ r>a \tag{5b}$$

It should be noted that by setting $\bar{C}=\bar{\lambda}-\frac{C_0}{p}$ in 5a the two equations become equivalent.

Hence, equation 5 may also be written

$$\frac{d^2\bar{\lambda}}{dr^2}+\frac{1}{r}\frac{d\bar{\lambda}}{dr}-q^2\bar{\lambda}=0;\ r<a$$

$$\frac{d^2\bar{C}}{dr^2}+\frac{1}{r}\frac{d\bar{C}}{dr}-q^2\bar{C}=0;\ q=\sqrt{\frac{p}{D}};\ r>a$$

Above is Bessel's equation, whose solution is as follows (Hildebrand, 1954, p. 167)

$$\bar{C}=AI_0\,(qr)+BK_0\,(qr)$$

where $I_0(\alpha)$=modified Bessel function of the first kind of order zero
$K_0(\alpha)$=modified Bessel function of the second kind of order zero
A,B=arbitrary constants

Since K_0 remains finite as $r\to\infty$, this solution is valid; that is, $A=0$ for $r>a$. In the region $r<a$, I_0 remains finite as $r\to 0$, and also since $\left(\frac{\partial C}{\partial r}\right)_{r=0}$ must be zero, I_0 is the solution, or $B=0$. Hence,

$$\begin{aligned}\bar{C}_1&=\frac{C_0}{p}+AI_0(qr) \qquad r<a\\ \bar{C}_2&=BK_0(qr) \qquad r>a\end{aligned} \tag{6}$$

At $r=a$ $\bar{C}_1=\bar{C}_2$; thus

$$BK_0(qa)=\frac{C_0}{p}+AI_0(qa)$$

In addition, the mass rate of transfer across $r=a$ is given by

$$\left(D\frac{\partial C_1}{\partial r}\right)_{r=a}=\left(D\frac{\partial C_2}{\partial r}\right)_{r=a}$$

giving

$$-BK_1(aq)=AI_1(qa)$$

From the above two equations the coefficients A and B may be determined; that is,

$$-A=\frac{C_0}{p}\,\frac{K_1(qa)}{I_0(qa)K_1(qa)+I_1(qa)K_0(qa)}$$

$$-B=\frac{C_0}{p}\,\frac{I_1(qa)}{K_0(qa)I_1(qa)+K_1(qa)I_0(q)}$$

Substituting values for A and B into equation 6 gives the Laplace solution of equation 5,

$$\begin{aligned}\bar{C}_1&=\frac{C_0}{p}\left[1-\frac{K_1(qa)I_0(qr)}{I_0(qa)K_1(qa)+K_0(qa)I_1(qa)}\right]\\ \bar{C}_2&=-\frac{C_0}{p}\left[\frac{I_1(qa)K_0(qr)}{K_0(qa)I_1(qa)+K_1(qa)I_0(qa)}\right]\end{aligned} \tag{7}$$

To obtain the solution of equation 3 for C as a function of r and t it is necessary to apply the Bromwich inversion theorem,

$$C(r,t)=\frac{1}{2\pi i}\int_{c+i\infty}^{c+i\infty} e^{pt}\bar{C}(r,p)dp$$

The inversion of equation 7, however, has been evaluated by Carslaw and Jaeger (1948, p. 285); the result for this specific case is

$$\begin{aligned}C_1(r,t)&=\frac{2C_0}{\pi}\int_0^\infty e^{-D\alpha^2t}\,\frac{J_1(\alpha a)[J_0(\alpha r)\phi(\alpha)]d\alpha}{\alpha[\phi^2(\alpha)]}\\ C_2(r,t)&=\frac{4C_0}{\pi^2a}\int_0^\infty e^{-D\alpha^2t}\,\frac{J_0(\alpha r)J_1(\alpha a)d\alpha}{\alpha^2[\phi^2(\alpha)]}\end{aligned} \tag{8}$$

where

$$\phi(\alpha)=J_1(a\alpha)Y_0(a\alpha)-J_0(a\alpha)Y_1(a\alpha)$$

Further (Watson, 1948, page 77),

$$J_\nu(z)Y_{\nu+1}(z)-J_{\nu+1}(z)Y_\nu(z)=-\frac{2}{\pi z}$$

Thus, because $\nu=0$,

$$\phi(\alpha)=\frac{2}{\pi a\alpha}$$

Substituting the value for $\phi(\alpha)$, the result is

$$C_1=C_2=C=C_0a\int_0^\infty e^{-D\alpha^2t}\,J_1(\alpha a)J_0(\alpha r)d\alpha \tag{9}$$

where

$J_1(\alpha)$ and $J_0(\alpha)$=Bessel function of the first kind of order one and zero, respectively
a=radius of contaminant filament

By direct substitution, the above can be shown to satisfy the original differential equation. It is now desired that equation 9 be shown to satisfy the boundary conditions. For $t=0$ the solution reduces to the following (Watson, 1948, p. 406):

$$\frac{C}{C_0}=a\int_0^\infty J_1(\alpha a)J_0(\alpha r)d\alpha$$

$$=a\begin{cases}0 & \text{for } r<a\\ \frac{1}{2a} & r=a\\ 1/a & r>a\end{cases} \tag{10}$$

The condition at $r=a$ is due to an averaging process such that the concentration curve becomes a continuous function.

In any numerical integration process difficulties are inherent; thus, if possible, equation 9 should be written in terms of tabulated functions to facilitate computation. However, before going into the general case it may be advantageous to consider special cases for $r=0$ and $r=a$. For these two values of r, equation 9 may be readily integrated, thereby simplifying the analysis considerably. The diffuslon coefficient may be readily obtained from experimental data by using these two expressions.

SOLUTION OF SPECIAL CASES OF $r=0$ AND $r=a$

Consider first the concentration variation along $r=0$. Because $J_0(x)\to 1$ as $x\to 0$ the equation above becomes simply

$$\frac{C}{C_0}=a\int_0^\infty e^{-Dt\alpha^2}J_1(\alpha a)da \tag{11}$$

Integration of the above is given by Watson (1948, p. 394), the result being

$$\frac{C}{C_0}=\frac{\sqrt{\pi}}{2\sqrt{\frac{Dt}{a^2}}}\exp\left(-\frac{a^2}{8Dt}\right)I_{1/2}\left(\frac{a^2}{8Dt}\right) \tag{12}$$

But

$$I_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\,\text{Sinh}\,x$$

Hence, letting $\gamma=\frac{a^2}{8Dt}$, the above may be written

$$\frac{C}{C_0}=2\exp(-\gamma)\,\text{Sinh}\,(\gamma)$$

Further,

$$\text{Sinh}\,(x)=\frac{1}{2}(e^x-e^{-x})$$

Thus, the concentration distribution at $r=0$ is given by

$$\frac{C}{C_0}=1-\exp(-2\gamma) \tag{13}$$

Note that this expression may be obtained directly from the more general solution, equation 24.

However, if the radius of filament a is large enough that the concentration of the centerline does not vary sufficiently in the experimental model, it would be advantageous to choose some other value of r. In this instance the best value would be $r=a$, provided time is large. Substituting for r in equation 9 gives

$$\frac{C}{C_0}=a\int_0^\infty e^{-Dt\alpha^2}J_1(a\alpha)J_0(a\alpha)d\alpha$$

The integration of the above is given by Bateman's Manuscript Project (1953b, p. 50), expressed in terms of a hypergeometric series; that is,

$$\frac{C}{C_0}=\left(\frac{z}{4}\right){}_3F_3\left(1,\frac{3}{2},1:1,2,2;-z\right) \tag{14}$$

where $z=\frac{a^2}{Dt}$. Some properties of the hypergeometric series will be presented in a latter part of the paper.

Equation 14 may be written as a series

$$\frac{C}{C_0}=\frac{1}{4}z\sum_{n=0}^{\infty}(-1)^n\frac{(1)_n\left(\frac{3}{2}\right)_n(1)_n}{n!(1)_n(2)_n(2)_n}$$

or

$$\frac{C}{C_0}=\frac{1}{4}z(A_0-A_1z+A_2z^2-\ldots A_nz^n)$$

where values of A_n are positive coefficients. Also note that

$$(\alpha)_n=\alpha(\alpha+1)(\alpha+2)(\alpha+3)\ldots(\alpha+n-1);\ (\alpha)_0=1$$

Since $(1)_n=n!$; $(2)_n=(n+1)!$ equation 14 further reduces to

$$\frac{C}{C_0}=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\frac{1}{2}\left(\frac{3}{2}\right)_{n-1}}{n!n!}z^n$$

The above expression is the series representation of the confluent hypergeometric function; thus it may be written

$$\frac{C}{C_0}=\frac{1}{2}\left[1-{}_1F_1\left(\frac{1}{2};1;-2\xi\right)\right] \tag{15}$$

where

$$\xi=\frac{z}{2}=\frac{a^2}{2Dt}$$

However, since a relationship exists between the confluent hypergeometric function and the Bessel function, it would be advantageous to write equation 15 in terms of Bessel functions. Bateman (1953a, p. 265) gives the relationship

$$I_\nu(x)=\frac{1}{\Gamma(\nu+1)}\left(\frac{1}{2}x\right)^\nu e^{-x}{}_1F_1\left(\frac{1}{2}+\nu;1+2\nu;2x\right)$$

where $\Gamma(\nu+1)$ is the gamma function. Further, $I_0(x)=I_0(-x)$; thus

$$\frac{C}{C_0}=\frac{1}{2}[1-e^{-\xi}I_0\xi] \tag{16}$$

The concentration distribution along $r=a$ is readily computed by means of equation 16.

The function $I_0(x)$ has been tabulated extensively; thus there is no difficulty in computing equation 16. As stated previously, equations 13 and 16 provide a means of computing the radial-diffusion coefficient from experimental data.

SOME PROPERTIES OF THE HYPERGEOMETRIC SERIES

The hypergeometric function occurs as solution of a linear second-order differential equation, which has at most three singularities at 0, ∞, and 1. In this paper the Pochhammer notation is used. By definition, the generalized hypergeometric series is as follows (Watson, 1948, p. 100; Bateman Manuscript Project, 1953a, p. 56):

$$ {}_pF_q(\alpha_1, \alpha_p, \ldots \alpha_p; \rho_1, \rho_2, \ldots \rho_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n}{n!(\rho_1)_n(\rho_2)_n \ldots (\rho_q)_n} Z^n \quad (17) $$

where

$$ (\alpha)_n = \alpha(\alpha+1)(\alpha+2) \ldots (\alpha+n-1); \; (\alpha)_0 = 1 $$

It should be noted that if $p=2$ and $q=1$ the above reduces to

$$ {}_2F_1(\alpha_1, \alpha_2; \rho_1; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n}{n!(\rho)_n} Z^n \quad (18) $$

Note further that if $\alpha_1=1$ and $\alpha_2=\rho$ in equation 18 this series becomes an elementary geometric series, or

$$ {}_2F_1(1, \rho; \rho; z) = \sum_{n=0} x^n $$

Owing to extreme difficulties in dealing with the generalized hypergeometric function, only the convergence of equation 18 will be stated. For a more general discussion of the hypergeometric series the reader is referred to the Bateman Manuscript Project (1953a, p. 56).

A hypergeometric series of the type given by equation 18 is generally convergent for values of $|z|<1$. Series converge for $x=1$ only if $\rho-\alpha^1-\alpha^2>0$ and for $x=-1$ only if $\rho-\alpha_1-\alpha_2+1>0$ (Hildebrand, 1954, p. 179).

GENERAL SOLUTION FOR COMPUTATION OF CONCENTRATION DISTRIBUTION

As stated previously, although equation 9 is the required solution of the problem, the numerical evaluation of the integral is extremely difficult. Thus, it would be advantageous to rewrite it in such a way as to facilitate computation. Equation 9 may be written in a series of Bessel functions or hypergeometric functions. Because tables of Bessel functions were readily available, this series was used. However, both of the series will be presented.

The product of Bessel functions generally may be written

$$ J_u(\alpha a)J_v(\alpha r) = \sum_{n=0}^{\infty} A_n J_n(\alpha a) J_n(\alpha r) $$

by use of the recurrence formula

$$ J_{\nu-1}(\beta x) + J_{\nu+1}(\beta x) = \frac{2\nu}{\beta x} J_\nu(\beta x) \quad (19) $$

Explicitly,

$$ J_0(\alpha r)J_1(\alpha a) = \sum_{n=0}^{\infty} \frac{\left(\frac{\alpha a}{2}\right)\left(\frac{a}{r}\right)^n}{n+1} J_n(\alpha a)J_n(\alpha r) - \sum_{n=2}^{\infty} \frac{\left(\frac{\alpha a}{2}\right)\left(\frac{a}{r}\right)^{n-2}}{n-1} J_n(\alpha a)J_n(\alpha r) \quad (20) $$

Substituting equation 19 into equation 9 and using the known integral

$$ \int_0^{\infty} e^{-p^2t^2} J_\nu(at)J_\nu(bt)t\,dt = \frac{1}{2p^2} \exp\left(-\frac{a^2+b^2}{4p^2}\right) I_\nu\left(\frac{ab}{2p^2}\right) \quad (21) $$

Equation 9 may be readily integrated. The result of the integration is

$$ \frac{C}{C_0} = \frac{\eta}{2} \exp\left[-\frac{r^2+a^2}{4Dt}\right] \left\{ \xi \sum_{m=0}^{\infty} \frac{(\eta)^m}{m+1} I_m(\xi) - \xi \sum_{m=2} \frac{(\eta)^{m-2}}{m-1} I_m(\xi) \right\} \quad (22) $$

where $\xi = \frac{ra}{2Dt}$ and $\eta = \frac{a}{r}$.

By use of the recurrence relationship for the modified Bessel function,

$$ I_{\nu-1}(\beta x) - 1_{\nu+1}(\beta x) = \frac{2\nu}{\beta x} I_\nu(\beta x) \quad (23) $$

The second term in equation 22 may be written

$$ S_n = \xi \sum_{m=2}^{\infty} \frac{(\eta)^{m-2}}{m-1} I_m(\xi) = \xi \sum_{m=0}^{\infty} \frac{\eta^m}{m+1} I_m(\xi) - 2 \sum_{m=1}^{\infty} (\eta)^{m-1} I_m(\xi) $$

This on substituting into the original equation gives

$$ \frac{C}{C_0} = \exp\left(-\frac{r^2+a^2}{4Dt}\right) \sum_{m=1}^{\infty} \eta^m I_m(\xi) \quad (24) $$

The above result was obtained by Goldstein (1953) in the study of exchange processes in fixed columns. Equation 24 may be written in another form, however:

$$ \frac{C}{C_0} = \exp\left[-\frac{(\eta-1)}{2\eta} \xi\right] \sum_{m=1}^{\infty} \eta^m e^{-\xi} I_m(\xi) \quad (25) $$

Equation 25 also may be written in its alternate form (Goldstein, 1953)

$$ \frac{C}{C_0} = 1 - \exp\left[-\frac{(\eta-1)^2}{2\eta} \xi\right] \sum_{m=0}^{\infty} \frac{1}{\eta^m} e^{-\xi} I_m(\xi) \quad (26) $$

Note that for $\eta=\frac{a}{r}>1, \frac{1}{\eta^m}<1$; thus equation 26 converges more rapidly than 25 in the region $\frac{a}{r}>1$. The converse is true for equation 25; that is, it would be advantageous to use equation 25 for computation in the region $\frac{a}{r}<1$. It should be noted further that equations 13 and 16 may be obtained directly from equations 25 and 26.

As stated previously, equation 9 may be written in terms of the hypergeometric series. There exist the relationships (Watson, 1948, p. 148)

$$J_0(\alpha r)J_1(\alpha a)=\left(\frac{a\alpha}{2}\right)\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{a\alpha}{2}\right)^{2m}}{m!(m+1)!}\,{}_2F_1\left(-m,-1-m;1;\frac{1}{\eta^2}\right)$$

for $\eta^2=\frac{a^2}{r^2}>1$, and

$$J_0(\alpha r)J_1(\alpha a)=\frac{a\alpha}{2}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{r\alpha}{2}\right)^{2m}}{m!m!}\,{}_2F_1(-m,-m;2;\eta^2)$$

for $\eta^2=\frac{a^2}{r^2}<1$.

Substituting the above relationships into equation 9 and using the known integral

$$\int_0^{\infty}e^{-px^2}x^n dx=\frac{[\frac{1}{2}(n-1)]!}{2p^{1/2(n+1)}}$$

the expressions obtained are

$$\frac{C}{C_0}=\sum_{m=0}^{\infty}(-1)^m\frac{{}_2F_1(-m,-1-m;1;\frac{1}{\eta^2})}{(m+1)!}(\lambda)^{n+1};\ \eta^2>1 \quad (27)$$

and

$$\frac{C}{C_0}=\sum_{m=0}^{\infty}(-1)^m(ar^m)^2\frac{{}_2F_1(-m,-m;2;\eta^2)}{m!}\left(\frac{\lambda}{m^2}\right)^{m+1};\ \eta^2<1 \quad (28)$$

where $\lambda=\frac{a^2}{4Dt}$.

Hypergeometric functions appearing as coefficients of the power series are polynomials of m.

Although both equations 27 and 28 and equations 25 and 26 may be readily computed, equations 25 and 26 were used because of their rapid convergence. A plot of $\frac{C}{C_0}$ versus $\frac{a}{4Dt}$ for various values of $\eta=\frac{a}{r}$ is given in figure 2, and a plot of $\frac{C}{C_0}$ versus $\frac{a}{r}$ for various values of $\frac{a^2}{4Dt}$ is given in figure 3.

STEADY-STATE SOLUTION

Owing to the nature of the experiments conducted (Skibitzke and others, 1960), it would be advantageous to assume that a steady state is reached. Accordingly, consider $\frac{\partial C}{\partial t}=0$ in equation 2, which gives

$$D_r\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial C}{\partial r}\right)-u\frac{\partial C}{dx}=0 \quad (29)$$

The boundary conditions become

$$C(r,a)=C_0 \qquad 0<r<a$$

$$C(r,0)=0 \qquad r>a$$

$$\left[\frac{\partial C}{\partial r}(r,x)\right]_{r=0}=0 \qquad x>0$$

The conditions given are the same as those given for the unsteady-state problem. Further, by letting $\tau=\frac{x}{u}$, equations 29 and 3 are identical. Thus, the solution for steady-state condition may be obtained by letting $t=\frac{x}{u}$ in equation 25, 26, 27, or 28.

CONCLUSION

In attacking the mathematical problem of transverse diffusion, an important assumption was made: that the front due to the longitudinal dispersion is established rapidly as compared to the transverse diffusion. If the diffusion transverse to the direction of flow includes mechanical dispersion, this assumption would be erroneous. However, since a homogeneous medium is assumed, the assumption that diffusion is wholly due to molecular agitation seems to be valid. Although there have been no quantitative studies, qualitative study using dye tracers in various types of models seems to bear this out.

Up to this time, investigation has been directed primarily toward evaluating experimental and analytical methods. The approximate method presented leads to what is believed to be a realistic solution that can be computed. When laboratory experiments are completed the resulting data, in conjunction with the solution presented, will furnish a means of determining the magnitude of the coefficient of transverse diffusion.

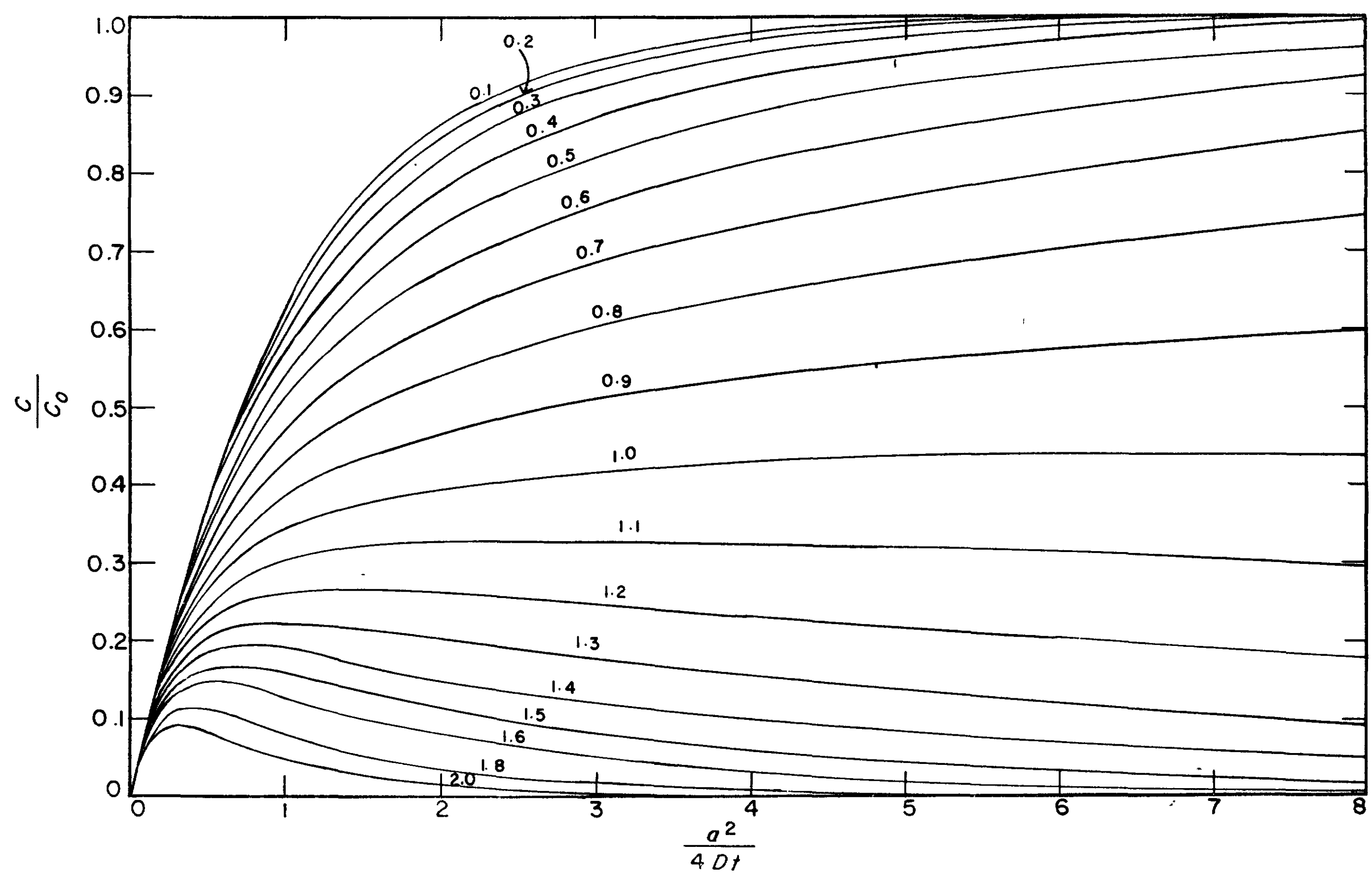

FIGURE 2.—Plot of solution for various values of r/a.

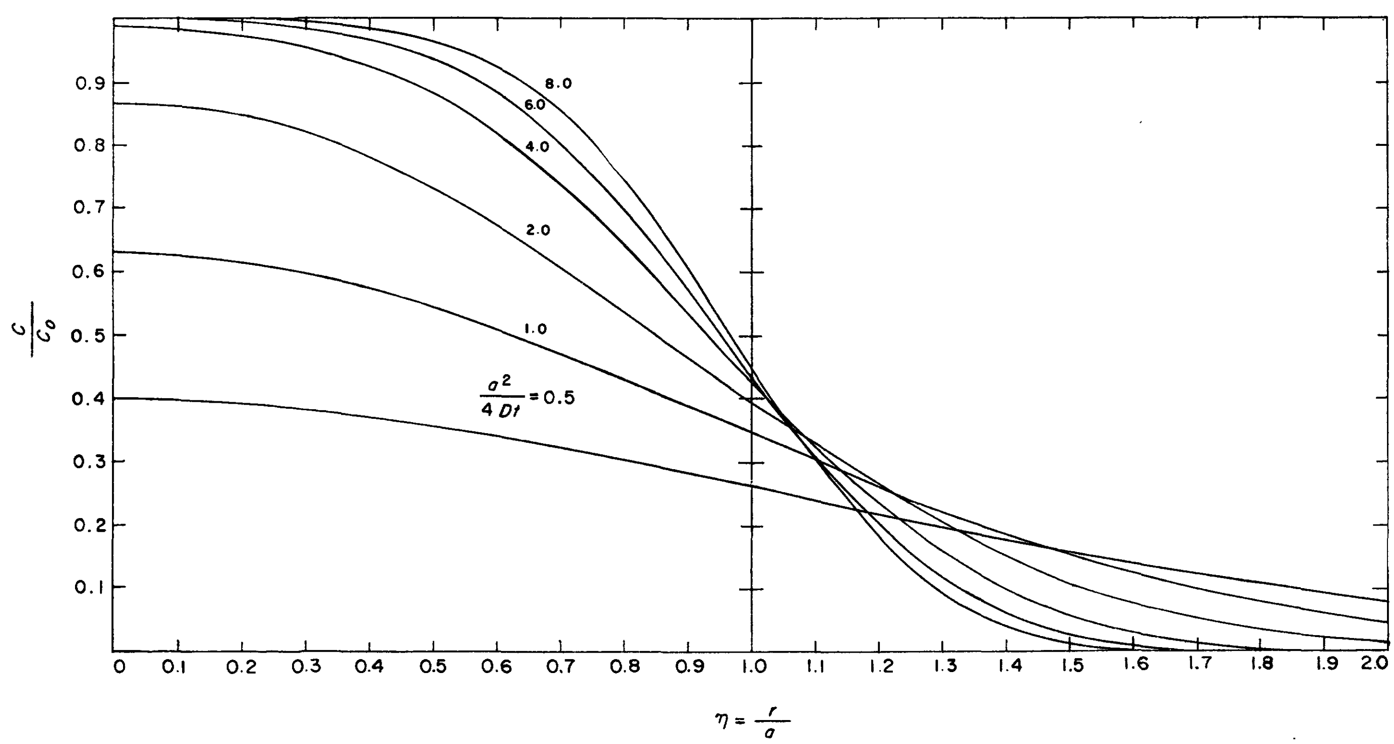

FIGURE 3.—Plot of solution for various values of $a^2/4Dt$.

REFERENCES

Bateman Manuscript Project, 1953a, Higher transcendental functions, v. 1: New York, McGraw-Hill, 302 p.
——— 1953b, Higher transcendental functions, v. 2: New York, McGraw-Hill, 396 p.
Beran, M. J., 1957, Dispersion of soluble matter in flow through granular media: Jour. Chem. Physics, v. 27, no. 1, p. 270–274.
Carslaw, H. S., and Jaeger, J. C., 1948, Conduction of heat in solids: London, Oxford Univ. Press, 386 p.
Day, P. R., 1956, Dispersion of moving salt-water boundary advancing through saturated sand: Am. Geophys. Union Trans., v. 37, no. 5, p. 595–601.
Goldstein, S., 1932, Some two-dimensional diffusion problems with circular symmetry: London Math. Soc. Proc., Ser. 2, 34, p. 51–88.
Goldstein, S., 1953, On the mathematics of exchange processes in fixed columns: Royal Soc. (London) Proc., v. 219, p. 151–185.
Hildebrand, F. B., 1954, Advanced calculus for engineers: New York, Prentice-Hall, 594 p.
Skibitzke, H. E., Chapman, H. T., Robinson, G. M., and McCullough, R. A., 1961, Radio-tracer techniques for the study of flow in saturated porous material: Internat. Jour. Applied Radiation and Isotopes v. 10, no. 1, p. 38–46.
Von Rosenberg, D. U., 1956, Mechanics of steady-state single-phase fluid displacements from porous media: Am. Inst. Chem. Eng. Jour., v. 2, p. 55.
Watson, G. N., 1948, A treatise on the theory of Bessel functions: Cambridge Univ. Press, 804 p.