# Sierpinski signal generates $1/f^{\alpha}$ spectra

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We investigate the row sum of the binary pattern generated by the Sierpinski automaton: Interpreted as a time series we calculate the power spectrum of this Sierpinski signal analytically and obtain a unique rugged fine structure with underlying power law decay with an exponent of approximately 1.15. Despite the simplicity of the model, it can serve as a model for $1/f^{\alpha}$ spectra in a certain class of experimental and natural systems such as catalytic reactions and mollusc patterns.

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The phenomenon of $1/f^{\alpha}$ noise is found in a widespread variety of systems [1–3]. Usually a noise signal is said to be $1/f^{\alpha}$ (or $1/f$) if its spectrum follows over some decades a power law $S(\omega)\sim\omega^{-\alpha}$ with an exponent near to one. Despite $1/f$ noise being known for a long time, up to now there is no general explanation for universal mechanisms (if they exist). However, on a very general level it is believed that complex systems [4] are able to generate $1/f$ noise. While real complex systems are usually not exactly solvable, extremely simplified models are under investigation. One prominent example is the sandpile-sketching Bak-Tang-Wiesenfeld model [5], a two-dimensional cellular automaton, which itself does not reproduce $1/f$ sufficiently.

In this work we study an even more simplified model introduced in 1984 by Wolfram [6], which is known to be able to exhibit complex behavior: the Sierpinski automaton. Looking not at the generated fractal Sierpinski gasket itself, but on the *row sum*, corresponding to the total *(in)activity* of the whole system, we have a signal as shown in Fig. 1 with increasing mean and increasing spatial size of the corresponding system; thus every physical realization of the system will be finite size limited.

Despite the fact that on first glance they seem to be theoretical toy models only, Sierpinski patterns have been found in nature. Detailed models have explained mollusc patterns by reaction-diffusion models and cellular automata [7,8]; Sierpinski patterns also occur in kink breeding dynamics [9] and have been observed in catalysis [10]. This phenomenon occurs generically for suitable parameter choices in four standard types of nonlinear spatiotemporal dynamics including the Bonhoeffer–van der Pol and the complex Ginzburg-Landau equation [11].

Consequently, catalytic processes can exhibit similar time signals as the Sierpinski sum signal. A comparison of the reaction rate of a catalytic process with the Sierpinski sum signal $X(t)=\Sigma x_i(t)$ has been given in Refs. [12,13]. A single state $x_i(t)$ at a time $t$ can be interpreted to indicate the activity of a local catalysis process, i.e., reaction (activity) when $x_i(t)=0$ and no reaction (inactivity) when $x_i(t)=1$. The authors [12] observe a qualitative similarity between the experimental and theoretical time series. Due to dominating finite size effects $1/f^{\alpha}$ spectra (or long-time correlations) could not be identified in the spectrum of the experimental data [14]. As models of chemical reactions, cellular automata have been studied widely [15], explaining spiral waves and pattern formation in chemical reactions. CO oxidation on Pt(110) and its control by global delayed feedback has been studied and compared to models [16], including the occurrence of patterns similar to Sierpinski structures in the intermittent turbulent phase. Recently, $1/f^{\alpha}$ spectra have been measured directly in a chemical reaction [17] by a superconducting quantum interference device setup that allows for much higher resolution in time, space, and signal-to-noise ratio than the direct gravimetric measurement of the reaction rate in Ref. [12]. The power law extends over more than two decades, indicating the spatiotemporal dynamics of the catalytic reaction exhibits avalanches on all sizes and self-organized critical behavior [17]. Interestingly, the Sierpinski gasket was more recently found in a video feedback system [18]. Apart from observation of the Sierpinski pattern itself, its geometry has been used widely, e.g., for sandpile dynamics [19] and measurements of magnetoresistance on fractal wire networks [20].

*Definition of the model*. The dynamics of the so-called Sierpinski automaton is related to the generation law of the Pascal triangle. This pattern can be generated by the following simple one-dimensional cellular automaton: We consider a linear array of sites (or spins) $x_i(t)$ which can take the values 0 or 1 at discrete time steps $t$. We restrict ourselves to the special initial condition, that for $t=0$ only one spin is different from all others:

$$x_0(0)=1 \quad \text{and} \quad \forall_{i\neq 0} \quad x_i(0)=0. \tag{1}$$

The dynamics is defined by the following next-neighbor interaction:

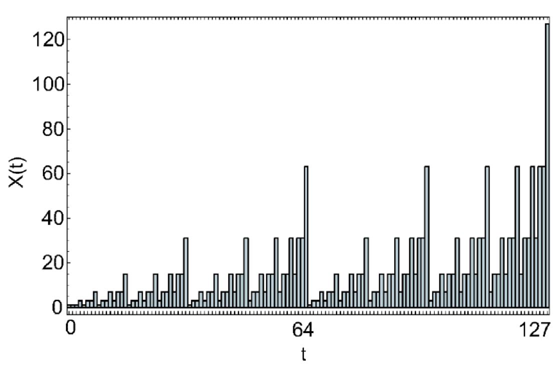


FIG. 1. The self-similar Sierpinski signal $X(t)$ for $T=128$.

 

$$x_i(t+1) = [x_{i-1}(t) + x_{i+1}(t)] \bmod 2, \tag{2}$$

$i \in [-\infty, \infty]$, $x_i(t) \in \{0,1\}$ at discrete time $t$. In the context of catalytic processes [10,12,13] a simplified chemical interpretation of this rule reads: A catalytic process is stopped when too little (i.e., no) or too many (i.e., 2) neighbor sites are active. A catalytic process is initiated (or continued) when only one neighbor site is active. This can originate from a minimal catalysis temperature combined with a local self-limiting reaction rate.

The spatiotemporal evolution is obtained from the rows of the Pascal triangle by applying modulo 2 elementwise,

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array} \quad \xrightarrow{\text{mod } 2} \quad \begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&0&&1&& \\ &1&&1&&1&&1& \\ 1&&0&&0&&0&&1 \end{array}$$

which is also known as the Sierpinski gasket. The Sierpinski gasket is a well-known self-similar structure (with point dimension $\ln 3/\ln 2$ in $x$ for $t \to \infty$) that is obtained by twofold replication of the first four rows to the subsequent four rows, and iteration of this process with the whole triangle.

Instead of considering the fractal pattern itself, we look at a scalar observable that can be compared to experimental time series. Before considering the spectrum, we briefly sketch a direct solution [21] and illustrate the analogy to a formal language approach. The row sum [or total (in)activity] over space at time $t$, defined by

$$X(t) := \sum_i x_i(t), \tag{3}$$

is referred to as the Sierpinski signal. The Sierpinski automaton rule then generates a time series $X(t)$ (Fig. 1) starting from $t=0$ with

$$1,\ 2,\ 2,4,\ 2,4,4,8,\ 2,4,4,8,4,8,8,16,\ \ldots . \tag{4}$$

As we are interested in an analytic expression for Eq. (4), we first note that $X(t)$ can be generated up to $t=2^N-1$ from the start sequence $u_0=(1)$ by $N$ iterations of the sequence replication rule

$$u_n \to u_{n+1} = (u_n, 2u_n). \tag{5}$$

Obviously $X(t)$ takes only powers of 2, so we consider $[\ln X(t)]/\ln 2$, which starts as

$$0,\ 1,\ 1,2,\ 1,2,2,3,\ 1,2,2,3,2,3,3,4,\ \ldots . \tag{6}$$

Strikingly, this appears to be the number of ones in

$$0,01,10,11,100,101,110,111,1000,1001,$$
$$1010,1011,1100,1101,1111,\ldots, \tag{7}$$

i.e., the binary decomposition of the time variable $t$ starting with $t=0$. Therefore the observable is given by

$$X(t) = 2^{\Sigma_\chi(\mathcal{B}(t))}, \tag{8}$$

which is no longer recursively defined. Here $\Sigma_\chi(.)$ denotes the cross sum, i.e., the sum over all digits (in the respective number system), and $\mathcal{B}(.)$ is the operator of the binary decomposition. This analytic solution was already discovered in 1852 by Kummer [21] in a number-theoretic context.

A convenient closed expression can be obtained by expressing time by a number of *spins*

$$t = \sum_{j=0}^{N-1} \sigma_j 2^j, \quad \sigma \in \{0,1\} \tag{9}$$

and expressing $X(\{\sigma_i\})$ from the same configuration as

$$X(t) = X\left(\sum_{j=0}^{N-1} \sigma_j 2^j\right) = 2^{\sum_{j=0}^{N-1} \sigma_j}. \tag{10}$$

By these expressions for $t(\{\sigma_i\})$ and $X(\{\sigma_i\})$, we have a parametric expression parametrized by a set of $N$ spins for all $X(t)$ with times up to $t=2^N-1$.

*Spectrum of the Sierpinski signal*. The periodogram $X(\omega)$ of the time signal now is calculated analytically. The binary time decomposition allows a Fourier transformation of $X(t)$ fairly direct from the definition

$$\begin{aligned} X(\omega) &= \sum_{t=0}^{2^N-1} e^{i\omega t} X(t) = \sum_{\sigma_0} \cdots \sum_{\sigma_{N-1}} e^{i\omega t(\{\sigma_i\})} X[t(\{\sigma_i\})] \\ &= \sum_{\sigma_0} \cdots \sum_{\sigma_{N-1}} \prod_{j=0}^{N-1} \exp[\sigma_j(i\omega 2^j + \ln 2)] \\ &= \prod_{j=0}^{N-1} \sum_{\sigma_j} \exp[\sigma_j(i\omega 2^j + \ln 2)] \\ &= \prod_{j=0}^{N-1} [1 + \exp(i\omega 2^j + \ln 2)], \end{aligned} \tag{11}$$

where all sums over $\sigma_j$ are taken over the two possible values $\sigma_j=0$ and $\sigma_j=1$. We now calculate the periodogram's power spectrum, i.e., $|X(\omega)|^2$. The absolute value of $X(\omega)$ simplifies to a trigonometric product which the logarithm converts into a sum

$$\ln|X(\omega)|^2 = \sum_{j=0}^{N-1} \ln[5 + 4\cos(\omega 2^j)], \tag{12}$$

showing a rugged fine structure as shown in Fig. 2. A rough estimate of the sum in Eq. (12) is obtained approximating the sum by an integral ($y := \omega 2^j$),

$$\ln|X(\omega)|^2 \approx \int_0^{N-1} \ln[5 + 4\cos(\omega 2^j)]dj \tag{13}$$

$$= \frac{1}{\ln 2}\int_\omega^{\omega 2^{N-1}} \frac{\ln[5 + 4\cos y]}{y} dy. \tag{14}$$

As $\ln(5+4x) \approx \ln(5) + \frac{4}{5}x$ for $|x| \ll 1$, we obtain

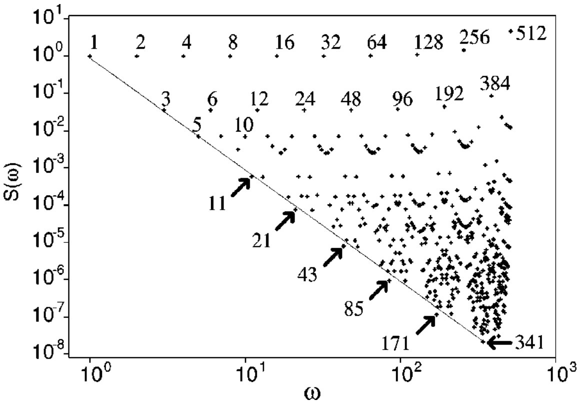


FIG. 2. Power spectrum of $X(t)$ for $T=1024$ time steps up to the Nyquist frequency of $T/2$ (from FFT). The lower envelope is constituted by $\omega(k)=\lfloor(2^k+1)/3\rfloor, k\in\{2,3,\ldots\}$.

$$\ln|X(\omega)|^2 \approx \frac{\ln 5}{\ln 2}\int_{\omega}^{\omega 2^{N-1}} \frac{dy}{y} + \frac{4}{5\ln 2}\int_{\omega}^{\omega 2^{N-1}} \frac{\cos(y)}{y}dy.$$

The integral over the integral cosine is nearly independent of the upper boundary for high values of the boundary. Thus, we can substitute the upper boundary $\omega 2^{N-1}$ by some $N$-dependent constant, say $c_N \gg 1$. Finally, substituting the cosine by one yields immediately a rough approximation of Eq. (12):

$$|X(\omega)|^2 \approx c_N' \omega^{-4/(5\ln 2)} \sim \omega^{-1.15}. \qquad (15)$$

Due to the increase of the mean of $X(t)$, spectral estimation from the periodogram $X(\omega)$ (implying a periodical extension in time domain) has to be discussed carefully. As $\Sigma_{t=0}^{2^N-1} X(t)=3^N$, the average increase is $\langle X(t)\rangle_{\mathrm{L}} = \langle X(t)\rangle_{\{0\cdots 2^N-1\}}=t^{\beta}$ with $\beta=\ln(3/2)/\ln(2)\approx 0.585$. Further, the signal exhibits an increasing variance $\langle X(t)^2\rangle_{\{0\cdots 2^N-1\}}=t^{\gamma}$ with $\gamma=\ln(5/2)/\ln(2)\approx 1.32$. Consequently, we investigate two variants of $X(t)$: A suitable per definitionem mean-free sum signal defined by $Y(t)=X(t)-(1+\beta)t^{\beta}$, and a mean-free signal with nonincreasing variance

$$Z(t) = Y(t)/\langle Y(t)^2\rangle_{\{0\cdots t\}}^{1/2}. \qquad (16)$$

The spectrum of $Y(\omega)$ can be directly obtained from Eq. (11) and the evaluation of the Fourier transform of $t^{\beta}$:

$$Y(\omega) = X(\omega) - (1+\beta)\mathcal{F}(t^{\beta}). \qquad (17)$$

The power spectrum of the periodically extended function $t^{\beta}$ decays (for small values of the frequency) as a power law with an exponent of approximately $-2$. Thus, the decay is much stronger than $X(\omega)$ and the power spectra of $Y(t)$ and $X(t)$ deviate only slightly. Hence, it follows that $|Y(\omega)|^2 \sim |X(\omega)|^2 \sim \omega^{-\alpha}$. Similarly, $|X(\omega)|^2$ also estimates $|Z(\omega)|^2$. We now compare these results with numerically applied discrete Fourier transformation. The power spectrum is fitted (least squares) in Fig. 3 by a power law with exponent $\alpha$ about 1.11, being in good agreement with the analytical result $\alpha \approx 1.15$ of Eq. (15). If one measures a power spectrum experimentally, this may generically be done by observation of

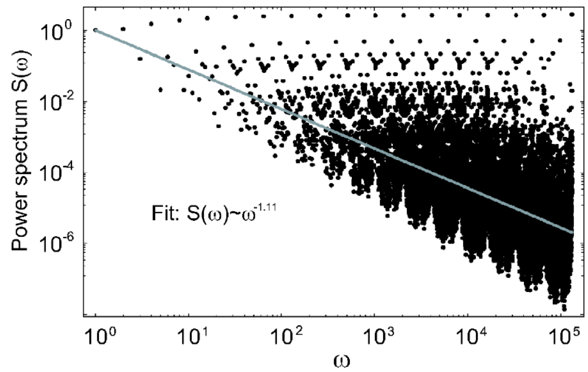


FIG. 3. Power spectrum of the time signal $Z(t)$ up to $T/8$ for $T=2^{20}$ time steps (from FFT) and least-square-fit $\omega^{-1.11}$.

resonances, where the system is coupled with a tuneable oscillator of given frequency and finite bandwidth. Therefore it is quite natural to consider an averaged spectrum. The (incommensurable) averaging procedure applied in Fig. 4 smoothes the peaks at $\omega=2^k$. The peak amplitudes decay as the average spectrum itself. For commensurable averaging (inset of Fig. 4) the peaks at $\omega=2^k$ disappear completely.

Note that the spectra $|X(\omega)|^2$ and $|Y(\omega)|^2$ from FFT for $T=2^{22}$ (not shown) also display $1/f^{\alpha}$ behavior, with exponents $\alpha_X=1.12$ and $\alpha_Y=1.11$, respectively. Moreover, we have used the method of a sliding window that normalizes the fluctuations [28] of the detrended signal $Y(t)$, i.e., $\tilde{Z}(t) = Y(t+l)/\langle Y^2\rangle_{\{t-l+1,t+l\}}^{1/2}$. For different values of the window width $2l$, the power spectra exhibit power law behavior with exponents of about $\alpha\approx 1.1$. Thus $1/f^{\alpha}$ spectra appear to be a robust property of Sierpinski signals.

*Amplitude distribution.* Many systems exhibiting $1/f$ noise possess a Gaussian amplitude distribution [22]. In this paragraph we calculate the amplitude distribution $H_N(2^k)$ of $X(t)$ analytically where $H_N(2^k)$ denotes the frequency occurrence of $X=2^k, k=0,1,\ldots$, for a signal length of $T=2^N$. The number of $2^k$'s in the signal up to $t=2^N-1$ is the number of $2^k$'s plus the number of $2^{k-1}$'s in the signal sequence up to

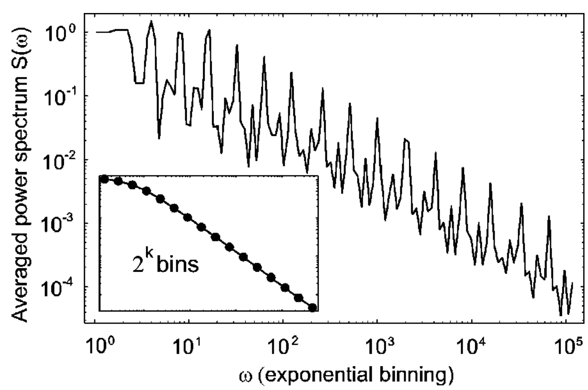


FIG. 4. Averaged power spectrum of $Z(t)$ up to $T/8$ for $T=2^{20}$ using (incommensurable) $1.1^k$-bins, i.e., the $k$th interval is defined by $[\lceil 1.1^k\rfloor, \lceil 1.1^{k+1}\rfloor]$ where the brackets $\lceil\ \rfloor$ denote rounded integer values. The inset shows the same spectrum, averaged using $2^k$ bins, i.e., the $k$th interval is defined by $[2^k, 2^{k+1}-1]$. Both correspond to a constant $\delta\omega/\omega$ ratio.

$t=2^{N-1}-1$, i.e., $H_N(2^k)=H_{N-1}(2^k)+H_{N-1}(2^{k-1})$. For the boundary condition $H_N(1)=H_1(2)=1$ we obtain a sum of binomial coefficients $H_N(2^k)=\Sigma_{j=1}^{N}\binom{j}{k}=\binom{N+1}{k+1}, k\geqslant 1$, which simplifies for $N\gg 1$ to $2^N$ times the Gaussian distribution

$$H_N(2^{k-1}) \approx \frac{2^N}{(\pi N/2)^{1/2}} e^{-\frac{(k-N/2)^2}{N/2}}. \tag{18}$$

Hence, the amplitude distribution of the occurrence of powers of 2 is Gaussian for fixed $N$. The (averaged) amplitude distributions for $Y(t)$ and $Z(t)$ differ from $H_N(2^k)$ but possess a similar shape as $H_N(2^k)$. Note that the variance distributions for $X(t)$, $Y(t)$, and $Z(t)$, are not Gaussian but well defined by Eq. (18).

As a final point, numerical simulations show that the averaged signals are robust against noise, i.e., initial conditions with more than a single 1.

In analogous situations in less simply defined systems, power laws have also been observed in spatial spectra of the scum on fluid surfaces and in the random baker map [23], and in the temporal spectra in dissipative dynamics governed by the Lorenz equations [24] being related to the Thue-Morse sequence [24–27]. In fact the Thue-Morse dynamics $1\rightarrow(1,-1), -1\rightarrow(-1,1)$ itself maps on the string replication rule $u_n\rightarrow[u_n,(-1)u_n]$ for generation of the spatial sequence. As it is not equivalent, but of striking similarity to Eq. (5) for generation of the temporal Sierpinski signal series, the Sierpinski dynamics itself does not follow a replication rule. While the analytic solution of the Thue-Morse sequence is $(-1)^{\Sigma\chi(\mathcal{B}(t))}$ [26], in analogy to Eq. (8), the averaged exponents of the resulting spectra are different.

To conclude, the one-dimensional Sierpinski automaton generates $1/f^{\alpha}$ spectra in the number of active states, and can therefore be considered as one of the simplest models generating $1/f^{\alpha}$ spectra. While the Sierpinski automaton is rather a caricature, the approach of studying the sum signal, or total (in)activity, and its spectrum, can be transferred to more realistic models and compared directly with experiments. Although exact Sierpinski patterns with long-range correlations remain to be experimentally challenging, we conjecture that $1/f^{\alpha}$ spectra in a suitable sum signal can be identified in every experimental setup exhibiting Sierpinski patterns.

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