On Progressively Type-II Censored Two-Parameter Rayleigh Distribution

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Abstract

Recently, Rayleigh distribution has received considerable attention in the statistical literature. In this paper, we consider the point and interval estimation of the functions of the unknown parameters of a two-parameter Rayleigh distribution. First, we obtain the maximum likelihood estimators (MLEs) of the unknown parameters. The MLEs cannot be obtained in explicit forms, and we propose to use the maximization of the profile log-likelihood function to compute the MLEs. We further consider the Bayesian inference of the unknown parameters. The Bayes estimates and the associated credible intervals cannot be obtained in closed forms. We use the importance sampling technique to approximate (compute) the Bayes estimates and the associated credible intervals. For comparison purposes we have also used the exact method to compute the Bayes estimates and the corresponding credible intervals. Monte Carlo simulations are performed to compare the performances of the proposed method, and one data set has been analyzed for illustrative purposes. We further consider the Bayes prediction problem based on the observed samples, and provide the appropriate predictive intervals. A data example has been provided for illustrative purposes.

Key Words and Phrases: Progressive Type-II censoring; maximum likelihood estimators; importance sampling; scale parameter; location parameter; Bayes estimators; simulation consistent estimators.

AMS Subject Classifications: MSC (2000): 62F15, 62F10, 62F99.

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1 Introduction

The Weibull distribution is one of the most popular distributions in analyzing lifetime data. It can have increasing, decreasing and constant hazard functions depending on the shape parameter. Rayleigh distribution can be obtained as a special case of the Weibull distribution, where the shape parameter is taken to be 2.

Rayleigh distribution was introduced by Lord Rayleigh (1880), in connection with a problem in acoustics, way before Weibull distribution was introduced by W. Weibull (1951). Rayleigh distribution has a nice connection with other distributions like chi-square and extreme value distributions. Estimations, predictions, and inferential issues of one-parameter Rayleigh distribution haven been extensively studied by different authors, see for example Johnson, Kotz and Balakrishnan (1994), Dey and Das (2007), Dey (2009), Howlader and Hossain (1995), and the references therein.

Very recently, Khan, Provost and Singh (2010) considered the two-parameter Rayleigh distribution and considered the predictive inference based on doubly censored samples. The two-parameter Rayleigh distribution, which has one scale and one location parameter has the following probability density function (PDF);

$$f(x;\lambda,\mu) = 2\lambda(x-\mu)e^{-\lambda(x-\mu)^2}; \quad x > \mu, \quad \lambda > 0. \tag{1}$$

Here $\lambda > 0$ and $0 < \mu < \infty$ are the scale and location parameters respectively. Due to the presence of the location parameter, the two-parameter Rayleigh distribution can be used more effectively to analyze real life data than one-parameter Rayleigh distribution.

In reliability or life testing experiments, the data are often censored. Among the different censoring schemes, Type-I and Type-II are the most popular censoring schemes. Unfortunately, in any of these censoring schemes, it is not possible to withdraw live items during the experiment. In this paper, we consider a generalization of the classical Type-II censoring scheme, where it is possible to withdraw live items during the experiment, and it is known as progressive Type-II censoring scheme.

Although, progressive censoring scheme was introduced long ago in the statistical literature, in recent years the progressive censoring scheme has received considerable attention in the statistical literature, see for instance the book by Balakrishnan and Aggrawalla (2000), and an excellent review article by Balakrishnan (2007). For some recent references, the readers are referred to Kundu (2008), Pradhan and Kundu (2009) or Pradhan and Kundu (2011) and see the references cited therein.

Recently, Huang and Wu (2008), Kim and Han (2009) and Wu and Huang (2010) considered the statistical inferences and some related issues for one-parameter Rayleigh distribution when the data are progressively censored. In this paper we consider the point and interval estimation of the functions of the unknown parameters of a two-parameter Rayleigh distribution when the data are progressively Type-II censored. We first consider the maximum likelihood estimators (MLEs) of the unknown parameters. The MLEs of the unknown parameters cannot be obtained in closed form, and we propose to use the maximization of the profile log-likelihood function to compute the MLEs of the unknown parameters.

Next, we consider the Bayesian inference of the unknown parameters under the assumptions of independent gamma and uniform priors on the scale and the location parameters respectively. It is observed that the Bayes estimates and the associated credible intervals cannot be obtained in closed form. We use the importance sampling procedure to approximate (compute) the Bayes estimates and the associated credible intervals. For comparison purposes we have also used the exact method to compute the Bayes estimates and the corresponding credible intervals. Monte Carlo simulations are performed to see the effectiveness of the proposed method, and a data analysis is performed for illustration. It is observed that the importance sampling method works quite satisfactorily in this case. The main advantage of the importance sampling method is that the same generated samples can be used to construct the Bayes estimates of the different functions of the parameters, and also to construct the associated credible intervals. It is not required to perform different integrations for different estimators.

Another important problem in life testing experiments is the prediction of the unknown observable belonging to a future sample, based on the current available sample, known in literature as informative sample. For different applications and for relevant references, the readers are referred to Al-Hussaini (1999), Basak, Basak and Balakrishnan (2006) or Kundu and Howlader (2010). In this paper, we consider the prediction problem in terms of the estimation of the predictive density of an unobserved observation based on the observed sample. We also construct a predictive interval for a future observation using the importance sampling procedure. An illustrative example has been provided.

The rest of the paper is organized as follows. In Section 2, we briefly discuss about the basic properties of the two-parameter Rayleigh distribution, and also progressive Type-II censoring scheme. In Section 3, we provide the MLEs of the unknown parameters. Bayes estimators and the associated credible intervals are provided in Section 4. Monte Carlo simulation results and the analysis of a data set have been provided in Section 5. Prediction problem has been considered in Section 6, and finally conclusions appear in Section 7.

2 Preliminaries

2.1 Rayleigh Distribution: A Brief Review

The two-parameter Rayleigh distribution as defined in (1) is always unimodal and it has the following cumulative distribution function (CDF)

$$F(x; \lambda, \mu) = 1 - e^{-\lambda(x-\mu)^2}; \qquad x > \mu. \tag{2}$$

The hazard function of the two-parameter Rayleigh distribution is an increasing function. All the moments of a two-parameter Rayleigh distribution exists. But if the random variable X follows two-parameter Rayleigh distribution as in (2), then the , then the elements of the Fisher information matrix may not be finite.

2.2 Progressive Type-II Censoring Scheme

Progressive Type-II censoring scheme can be described as follows: Suppose n units are placed on a life test and the experimenter decides before hand the quantity m, the number of failures to be observed. Now at the time of the first failure, R_1 of the remaining $n-1$ surviving units are randomly removed from the experiment. At the time of the second failure, R_2 of the remaining $n - R_1 - 1$ units are randomly removed from the experiment. Finally, at the time of the m-th failure, all the remaining surviving units $R_m = n - m - R_1 - \cdots - R_{m-1}$ are removed from the experiment.

Therefore, a progressive Type-II censoring scheme consists of m, and $R_1, \cdots, R_m$, such that $R_1 + \cdots + R_m = n - m$. The m failure times obtained from a progressive Type-II censoring scheme will be denoted by $t_1, \cdots, t_m$.

3 Maximum Likelihood Estimators

Based on the observed sample $t_1 < \cdots < t_m$ from a progressive Type-II censoring scheme, $(R_1, \cdots, R_m)$, the likelihood function can be written as

$$L(\lambda, \mu) = c \prod_{i=1}^{m} f(t_i; \lambda, \mu) \left[1 - F(t_i; \lambda, \mu)\right]^{R_i}; \quad \lambda > 0, 0 < \mu < t_1, \tag{3}$$

where $c = n(n-1-R_1) \cdots (n - R_1 - \cdots - R_{m-1} - m + 1)$, and $f(\cdot)$ and $F(\cdot)$ are same as defined before in (1) and (2) respectively. Therefore ignoring the additive constant, the log-likelihood function can be written as

$$l(\mu,\lambda) = m\ln\lambda + \sum_{i=1}^{m}\ln(t_i-\mu) - \lambda\sum_{i=1}^{m}(R_i+1)(t_i-\mu)^2; \quad \text{if} \quad \mu < t_1 \tag{4}$$

and 0 otherwise. The MLEs of the unknown parameters can be obtained by maximizing the log-likelihood function (4) with respect to the unknown parameters. We have the following result:

THEOREM 1: For $m \geq 2$, the MLEs of μ and λ for $(\mu,\lambda) \in [0,t_1)\times(0,\infty)$ exists and they are unique.

PROOF: See in the Appendix. ∎

To compute the MLEs, consider the two normal equations;

$$\frac{\partial l(\lambda,\mu)}{\partial\lambda} = \frac{m}{\lambda} - \sum_{i=1}^{m}(R_i+1)(t_i-\mu)^2 = 0 \tag{5}$$

$$\frac{\partial l(\lambda,\mu)}{\partial\mu} = -\sum_{i=1}^{m}(t_i-\mu)^{-1} + 2\lambda\sum_{i=1}^{m}(R_i+1)(t_i-\mu) = 0. \tag{6}$$

From (5), we obtain the MLE of λ, as a function of μ, say $\widehat{\lambda}(\mu)$, as

$$\widehat{\lambda}(\mu) = \frac{m}{\sum_{i=1}^{m}(R_i+1)(t_i-\mu)^2}. \tag{7}$$

Note that for known μ, (7) is the MLE of λ. Now when μ is also unknown, substituting, $\widehat{\lambda}(\mu)$ in (4), we obtain the profile log-likelihood function of μ as

$$g(\mu) = l(\widehat{\lambda}(\mu),\mu) = m\ln m - m\ln\left(\sum_{i=1}^{m}(R_i+1)(t_i-\mu)^2\right) + \sum_{i=1}^{m}\ln(t_i-\mu) - m. \tag{8}$$

Note that, for $m = 1$, the profile log-likelihood function (8) is an increasing function of μ, hence the MLE of μ is t_1. In this case the MLE of λ is not finite. For $m > 1$, due to Theorem 1, the MLEs of μ and λ exist and they are unique. In this case the MLE of μ can be obtained by maximizing (8) with respect to μ for $\mu \in [0,t_1)$. Once the MLE of μ, say $\widehat{\mu}_{MLE}$ is obtained, the MLE of λ, $\widehat{\lambda}_{MLE} = \widehat{\lambda}(\widehat{\mu}_{MLE})$ can be easily obtained.

Note that although the MLEs of the unknown parameters can be obtained quite conveniently, the exact distribution of the MLEs is not possible to obtain. Due to this reason the construction of the exact confidence intervals is also very difficult. One has to rely on the different bootstrap procedures for this purpose. Alternatively it is possible to use Bayesian procedure to compute Bayes estimates and to construct the associated credible intervals.

4 Bayesian Inference

In this section we develop the Bayesian inference of the unknown parameters of the two-parameter Rayleigh distribution when the data are progressively Type-II censored. We mainly discuss the Bayes estimates and the associated credible intervals of the unknown parameter(s). Although we have developed the estimates using squared error loss function, any other loss function can be easily incorporated.

Note that if the location parameter is known, then the scale parameter has a gamma conjugate prior. On the other hand, if both the parameters are unknown, the joint conjugate prior does not exist. Moreover, even for complete sample case, all the elements of the expected Fisher information matrix are not finite. Therefore the Jeffrey's prior also does not exist for this case. We consider the following priors on λ and μ and they are fairly general.

Since for a given μ, λ has a conjugate gamma prior, we assume the same prior on λ as follows;

$$\pi_1(\lambda|a,b) \propto \lambda^{a-1}e^{-b\lambda}; \quad \lambda > 0, a > 0, b > 0, \tag{9}$$

and for μ the following data dependent uniform prior has been considered

$$\pi_2(\mu) \propto 1, \quad 0 < \mu < t_1. \tag{10}$$

Moreover, they are assumed to be independent.

Based on the observed sample the joint posterior density function of λ and μ is

$$\pi(\lambda, \mu | Data) = K\ \lambda^{a+m-1} e^{-\lambda(b+\sum_{i=1}^{m}(R_i+1)(t_i-\mu)^2)} \prod_{i=1}^{m}(t_i - \mu), \quad 0 < \lambda < \infty, 0 < \mu < t_1, \quad (11)$$

here the normalizing constant K is

$$K = \left[\int_0^{t_1}\int_0^{\infty} \lambda^{a+m-1} e^{-\lambda(b+\sum_{i=1}^{m}(R_i+1)(t_i-\mu)^2)} \prod_{i=1}^{m}(t_i - \mu)\mathrm{d}\lambda\mathrm{d}\mu\right]^{-1}. \quad (12)$$

Note that

$$\int_0^{t_1}\int_0^{\infty} \lambda^{a+m-1} e^{-\lambda(b+\sum_{i=1}^{m}(R_i+1)(t_i-\mu)^2)} \prod_{i=1}^{m}(t_i-\mu)\mathrm{d}\lambda\mathrm{d}\mu = \int_0^{t_1} \frac{\Gamma(a+m)\prod_{i=1}^{m}(t_i-\mu)}{[b+\sum_{i=1}^{m}(R_i+1)(t_i-\mu)^2]^{a+m}}\mathrm{d}\mu. \quad (13)$$

Suppose, we want to compute the Bayes estimate of some function of λ and μ, say $\theta(\lambda, \mu)$, then under squared error loss function it should be the posterior mean, *i.e.*

$$\widehat{\theta}_{Bayes} = K \int_0^{t_1}\int_0^{\infty} \theta(\lambda,\mu)\ \lambda^{a+m-1} e^{-\lambda(b+\sum_{i=1}^{m}(R_i+1)(t_i-\mu)^2)} \prod_{i=1}^{m}(t_i - \mu)\mathrm{d}\lambda\mathrm{d}\mu, \quad (14)$$

provided it is finite.

For general $\theta(\lambda, \mu)$, (14) cannot be obtained in explicit form. Numerical integration procedure can be used to compute (14). Alternatively, some approximation like Lindley or some of its variants can be used to approximate (compute) the Bayes estimate. Although it is possible to compute the approximate Bayes estimate by this method, the associated credible interval cannot be obtained by this method.

Alternatively, the importance sampling technique can be used to compute the Bayes estimate and also to construct the associated credible interval. It is well known that with proper importance sampling procedure, simulation consistent Bayes estimates and the associated credible interval can be obtained, see for example Chen *et al.* (2000), Kundu and Pradhan (2009), Pradhan and Kundu (2009), Pradhan and Kundu (2011) and the references cited therein.

For implementing the importance sampling procedure, we re-write (11) as follows;

$$\pi(\lambda,\mu|Data) = \frac{f_1(\lambda|\mu,Data)f_2(\mu|Data)h(\mu)}{\int_0^{t_1}\int_0^{\infty} f_1(\lambda|\mu,Data)f_2(\mu|Data)h(\mu)\mathrm{d}\lambda\mathrm{d}\mu}, \tag{15}$$

where

$$f_1(\lambda|\mu,Data) = \frac{(b+\sum_{i=1}^m(R_i+1)(t_i-\mu)^2)^{m+a}}{\Gamma(m+a)}\lambda^{m+a-1}e^{-\lambda(b+\sum_{i=1}^m(R_i+1)(t_i-\mu)^2)}; \quad \lambda > 0, \tag{16}$$

$$f_2(\mu|Data) = \frac{k \quad 2(\sum_{i=1}^m t_i(R_i+1)-n\mu)}{(n\mu^2-2\mu\sum_{i=1}^m t_i(R_i+1)+b+\sum_{i=1}^m t_i^2(R_i+1))^{m+a}}; \quad 0<\mu<t_1 \tag{17}$$

where

$$\begin{aligned} k &= (m+a-1)\left[(g(t_1))^{m+a-1} - A^{m+a-1}\right] \\ g(t) &= \left(nt^2 - 2t\sum_{i=1}^m t_i(R_i+1) + b + \sum_{i=1}^m t_i^2(R_i+1)\right) \\ A &= b + \sum_{i=1}^m t_i^2(R_i+1), \end{aligned}$$

and

$$h(\mu) = \begin{cases} \frac{\Gamma(m+a)\prod_{i=1}^m(t_i-\mu)}{2k(\sum_{i=1}^m t_i(R_i+1)-\mu)} & \text{for} \quad \mu < t_1 \\ 0 & \text{for} \quad \mu \geq t_1. \end{cases} \tag{18}$$

Note that, $f_1(\lambda|\mu,Data)$ is a gamma density function with the shape and scale parameters as $m+a$ and $b+\sum_{i=1}^m(R_i+1)(t_i-\mu)^2$ respectively. $f_2(\mu|Data)$ is a proper density function and it has the distribution function for $0<\mu<t_1$ as

$$F_2(\mu|Data) = \frac{k}{m+a-1}\left[\frac{1}{(g(\mu))^{m+a-1}} - \frac{1}{A^{m+a-1}}\right], \tag{19}$$

which is easily invertible.

Now we propose the following procedure to compute the Bayes estimate of $\theta(\lambda,\mu)$ and the associated credible interval.

- Step 1: Generate $\mu \sim f_2(\mu|Data)$ using (19).

- Step 2: Generate $\lambda|\mu \sim gamma(m+a, b+\sum_{i=1}^{m}(R_i+1)(t_i-\mu)^2)$

- Step 3: Repeat Step 1 and Step 2, to obtain $(\mu_1, \lambda_1), \cdots, (\mu_N, \lambda_N)$

- Step 4: Then a simulation consistent estimator of (14) can be obtained as

$$\frac{\sum_{i=1}^{N}\theta(\lambda_i, \mu_i)h(\mu_i)}{\sum_{i=1}^{N}h(\mu_i)} \tag{20}$$

Now we would like to construct the highest posterior density credible interval (HPD) of θ using the generated importance sampling procedure, as suggested by Chen *et al.* (2000). Suppose θ_p is such that $P(\theta \le \theta_p|Data) = p$, for $0 < p < 1$. Consider the following function

$$g(\lambda, \mu) = \begin{cases} 1 & \text{if} \quad \theta \le \theta_p \\ \\ 0 & \text{if} \quad \theta > \theta_p. \end{cases} \tag{21}$$

Clearly, $E(g(\lambda, \mu)|Data) = p$. Therefore, a simulation consistent Bayes estimate of θ_p under squared error loss function can be obtained from the generated sample $\{(\mu_1, \lambda_1), \cdots, (\mu_N, \lambda_N)\}$ as follows. Let

$$w_i = \frac{h(\mu_i)}{\sum_{i=1}^{N}h(\mu_i)},$$

and $\theta_1 = \theta(\lambda_1, \mu_1), \cdots, \theta_N = \theta(\lambda_N, \mu_N)$. Rearranging $(\theta_1, w_1), \cdots, (\theta_N, w_N)$ as follows $\{(\theta_{(1)}, w_{[1]}), \cdots, (\theta_{(N)}, w_{[N]})\}$, where $\theta_{(1)} < \cdots < \theta_{(N)}$. Note that, $w_{[i]}$'s are not ordered, they are associated with $\theta_{(i)}$'s. Then a simulation consistent Bayes estimate of θ_p can be obtained as $\widehat{\theta}_p = \theta_{(N_p)}$, where

$$\sum_{i=1}^{N_p} w_{[i]} \le p < \sum_{i=1}^{N_p+1} w_{[i]},$$

see Chen and Shao (1999) or Chen *et al.* (2000).

Now using the above procedure, a 100(1-α)% credible interval can be obtained as

$$(\widehat{\theta}_\delta, \widehat{\theta}_{\delta+1-\alpha}), \quad \text{for} \quad \delta = w_{[1]}, w_{[1]}+w_{[2]}, \cdots, \sum_{i=1}^{N_{1-\alpha}} w_{[i]}. \tag{22}$$

Therefore, a 100(1-α)% HPD credible interval of θ becomes, $((\widehat{\theta}_{\delta^*}, \widehat{\theta}_{\delta^*+1-\alpha})$, where δ^* is such that for all δ,

$$(\widehat{\theta}_{\delta^*+1-\alpha} - \widehat{\theta}_{\delta^*}) \leq (\widehat{\theta}_{\delta+1-\alpha} - \widehat{\theta}_{\delta})$$

5 Numerical Experiments and Data Analysis

5.1 Numerical Experiments

In this section we present some experimental results to observe the behavior of the proposed method for different sample sizes, different effective sample sizes, different priors, and for different sampling schemes. We have considered different sample sizes; $n = 20, 25, 30$, different effective sample sizes; $m = 10$, 15, and eleven ([a] - [k]) different sampling schemes. Among the eleven schemes, schemes [a], [c], [f], [h], [j] are the usual Type-II censoring schemes. Here all the remaining $n - m$ surviving units are removed at the m-th failure time point. The censoring schemes [b], [d], [g], [i], [k] are just the opposite of the Type-II censoring schemes. Here $n - m$ surviving units are removed at the first failure time point. It may be mentioned that for fixed n and m, the expected duration of the experiment is maximum for the opposite of Type-II censoring scheme, and minimum for Type-II censoring scheme. For any other censoring scheme, the expected duration lies in between these two extremes.

In all cases we have used $\mu = 1$ and $\lambda = 1$. For a given n, m and a sampling scheme, using the algorithm proposed by Balakrishnan and Sandhu [10], we have generated a sample for a given censoring scheme. We compute the MLEs of the unknown parameters based on the method proposed in Section 3. We also compute the Bayes estimates of the unknown parameter(s) based on the importance sampling procedure proposed in Section 4. It is assumed that λ has non-informative prior, *i.e.* $a = b = 0$. For comparison purposes we have also computed the Bayes estimates and the associated credible intervals based on direct integrations. In computing the Bayes estimates we have used 5000 importance samples, and we compute the average bias and the corresponding standard error of the estimates based on 1000 replications. The results are presented in Tables 1 to 6.

Table 1: The average bias (AB), the standard error (SE) for the MLE of μ are presented for different sample sizes and different sampling schemes.

n	m	Scheme	No.	AB	SE
20	10	(9*0,10)	[a]	0.0747	0.1152
20	10	(10, 9*0)	[b]	0.0637	0.1122
25	10	(9*0, 15)	[c]	0.0687	0.1040
25	10	(15, 9*0)	[d]	0.0556	0.0993
25	10	(5, 5, 5, 7*0)	[e]	0.0571	0.0993
25	15	(14*0, 10)	[f]	0.0591	0.1038
25	15	(10, 14*0)	[g]	0.0533	0.0977
30	10	(9*0, 20)	[h]	0.0606	0.0979
30	10	(20, 9*0)	[i]	0.0485	0.0910
30	15	(14*0, 15)	[j]	0.0502	0.0934
30	15	(15, 14*0)	[k]	0.0476	0.0898

Table 2: The average bias (AB), the standard error (SE) for the MLE of λ are presented for different sample sizes and different sampling schemes.

n	m	Scheme	No.	AB	SE
20	10	(9*0,10)	[a]	0.5469	0.9574
20	10	(10, 9*0)	[b]	0.3658	0.6942
25	10	(9*0, 15)	[c]	0.5532	0.9679
25	10	(15, 9*0)	[d]	0.3423	0.6595
25	10	(5, 5, 5, 7*0)	[e]	0.3743	0.7102
25	15	(14*0, 10)	[f]	0.3098	0.5717
25	15	(10, 14*0)	[g]	0.2558	0.5059
30	10	(9*0, 20)	[h]	0.5344	1.0231
30	10	(20, 9*0)	[i]	0.3284	0.6250
30	15	(14*0, 15)	[j]	0.3099	0.5656
30	15	(15, 14*0)	[k]	0.2299	0.4636

Table 3: The average bias (AB), the standard error (SE) for the Bayes estimate (when both are unknown) of μ when $\mu = 1$ are presented for different sample sizes and different sampling schemes. In the Table "Bayes1" represents the Bayes estimates using the importance sampling procedure, whereas "Bayes2" represents the Bayes estimates using the direct computation procedure.

				Bayes1		Bayes2	
n	m	Scheme	No.	AB	SE	AB	SE
20	10	(9*0,10)	[a]	0.0139	0.1409	0.5226	0.4551
20	10	(10, 9*0)	[b]	0.0289	0.1396	0.4616	0.2453
25	10	(9*0, 15)	[c]	0.0150	0.1269	0.4876	0.4085
25	10	(15, 9*0)	[d]	0.0291	0.1203	0.4092	0.2576
25	10	(5, 5, 5, 7*0)	[e]	0.0199	0.1222	0.4331	0.4279
25	15	(14*0, 10)	[f]	0.0409	0.1161	0.4323	0.3764
25	15	(10, 14*0)	[g]	0.0075	0.1127	0.3210	0.2236
30	10	(9*0, 20)	[h]	0.0142	0.1183	0.3812	0.3312
30	10	(20, 9*0)	[i]	0.0290	0.1049	0.3134	0.2459
30	15	(14*0, 15)	[j]	0.0335	0.1038	0.2789	0.2129
30	15	(15, 14*0)	[k]	0.0013	0.1034	0.2645	0.2106

Table 4: The average bias (AB), the standard error (SE) for the Bayes estimate (when both are unknown) of λ when $\lambda = 1$ are presented for different sample sizes and different sampling schemes. In the Table "Bayes1" represents the Bayes estimates using the importance sampling procedure, whereas "Bayes2" represents the Bayes estimates using the direct computation procedure.

				Bayes1		Bayes2	
n	m	Scheme	No.	AB	SE	AB	SE
20	10	(9*0,10)	[a]	0.3667	0.8780	0.7624	1.2340
20	10	(10, 9*0)	[b]	0.1712	0.6059	0.2548	0.8121
25	10	(9*0, 15)	[c]	0.3738	0.8941	0.6614	0.9059
25	10	(15, 9*0)	[d]	0.1460	0.5699	0.2519	0.7953
25	10	(5, 5, 5, 7*0)	[e]	0.1781	0.6194	0.4538	0.7389
25	15	(14*0, 10)	[f]	0.2778	0.5907	0.3945	0.6812
25	15	(10, 14*0)	[g]	0.1660	0.5033	0.2219	0.5109
30	10	(9*0, 20)	[h]	0.3675	0.5678	0.4124	0.6125
30	10	(20, 9*0)	[i]	0.1364	0.5019	0.2245	0.5715
30	15	(14*0, 15)	[j]	0.2791	0.5998	0.3078	0.5823
30	15	(15, 14*0)	[k]	0.1281	0.4512	0.1876	0.5257

Some of the points are quite clear from these experimental results. It is clear that as the effective sample size increases, the SE decrease for all the cases. The performance of the

Table 5: The average credible interval and the coverage percentage for the Bayes estimate (when both are unknown) of μ when $\mu = 1$ are presented for different sample sizes and different sampling schemes.

n	m	Scheme	No.	Interval	Coverage
20	10	(9*0,10)	[a]	(0.4524, 1.5512)	89.1%
20	10	(10, 9*0)	[b]	(0.4243, 1.5406)	90.3%
25	10	(9*0, 15)	[c]	(0.4450, 1.5325)	91.3%
25	10	(15, 9*0)	[d]	(0.4228, 1.5244)	91.5%
25	10	(5, 5, 5, 7*0)	[e]	(0.3589, 1.4250)	92.2%
25	15	(14*0, 10)	[f]	(0.3845, 1.4351)	92.5%
25	15	(10, 14*0)	[g]	(0.3287, 1.4237)	93.2%
30	10	(9*0, 20)	[h]	(0.3531, 1.4206)	93.3%
30	10	(20, 9*0)	[i]	(0.3537, 1.4106)	93.4%
30	15	(14*0, 15)	[j]	(0.3896, 1.4190)	94.2%
30	15	(15, 14*0)	[k]	(0.3181, 1.4154)	94.4%

Table 6: The average credible interval and the coverage percentage for the Bayes estimate (when both are unknown) of λ when $\lambda = 1$ are presented for different sample sizes and different sampling schemes.

n	m	Scheme	No.	Interval	Coverage
20	10	(9*0,10)	[a]	(0.5827, 2.3674)	89.2%
20	10	(10, 9*0)	[b]	(0.4717, 2.0263)	92.5%
25	10	(9*0, 15)	[c]	(0.5879, 2.3787)	93.1%
25	10	(15, 9*0)	[d]	(0.4553, 1.9780)	93.3%
25	10	(5, 5, 5, 7*0)	[e]	(0.4792, 2.0402)	94.2%
25	15	(14*0, 10)	[f]	(0.6410, 1.9549)	94.0%
25	15	(10, 14*0)	[g]	(0.5751, 1.7974)	94.1%
30	10	(9*0, 20)	[h]	(0.5741, 2.3158)	94.2%
30	10	(20, 9*0)	[i]	(0.4366, 1.9642)	94.5%
30	15	(14*0, 15)	[j]	(0.6452, 1.9426)	94.3%
30	15	(15, 14*0)	[k]	(0.5554, 1.7411)	94.7%

MLEs are quite satisfactory. The Bayes estimates using importance sampling method work quite well, and in most of the cases it performs better, in terms of SE, than the corresponding Bayes estimate obtained by direct calculations. Since the computation of the Bayes estimates using importance sampling method is quite straight forward, it can be used for all practical purposes. Moreover, if we want to compute the Bayes estimate or the associated credible interval of any function of the parameters, the same samples can be used.

Table 7: Strength Data

0.562	0.564	0.729	0.802	0.950	1.053	1.111	1.115	1.194	1.208
1.216	1.247	1.256	1.271	1.277	1.305	1.313	1.348	1.390	1.429
1.474	1.490	1.503	1.520	1.522	1.524	1.551	1.551	1.609	1.632
1.632	1.676	1.684	1.685	1.728	1.740	1.761	1.764	1.785	1.804
1.816	1.824	1.836	1.879	1.883	1.892	1.898	1.934	1.947	1.976
2.020	2.023	2.050	2.059	2.068	2.071	2.098	2.130	2.204	2.262
2.317	2.334	2.340	2.346	2.378	2.483	2.683	2.835	2.835	

5.2 Data Analysis

In this section we present a data analysis for illustrative purposes. It is a strength data set originally reported by Badar and Priest (1982). The authors are thankful to Professor R.G. Surles for providing the data, which represent the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. The data set is presented below in Table 7.

Preliminary data analysis indicate that the data are positively skewed. The scaled TTT transform is provided in Figure 1, it's concavity indicates that the sample hazard function is an increasing function, see for example Aarset (1987). Therefore, two-parameter Rayleigh distribution may be used to analyze this data set. Recently, it has been observed by Dey *et al.* (2011) that the two-parameter Rayleigh distribution fits the data quite well.

We have generated progressively censored samples using three different sampling schemes from the full data set with $m = 25$ as follows.

Censoring Scheme 1: (24*0,44): 0.562 0.564 0.729 0.802 0.950 1.053 1.111 1.115 1.194 1.208 1.216 1.247 1.256 1.271 1.277 1.305 1.313 1.348 1.390 1.429 1.474 1.490 1.503 1.520 1.522.

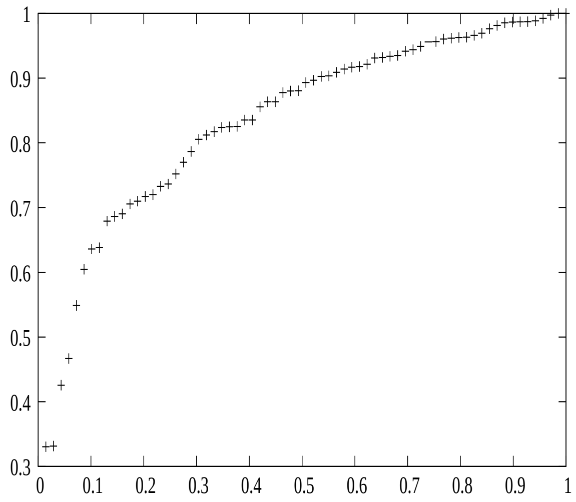

Figure 1: Scaled TTT transform data of the complete sample.

Censoring Scheme 2: (44, 24*0): 0.562 0.564 0.729 0.950 1.053 1.208 1.271 1.277 1.390 1.522 1.551 1.609 1.676 1.816 1.824 1.879 1.898 1.934 1.947 1.976 2.050 2.204 2.262 2.346 2.835

Censoring Scheme 3: (24*1, 20): 0.562 0.564 0.729 0.802 0.950 1.053 1.111 1.115 1.194 1.208 1.247 1.256 1.271 1.277 1.348 1.390 1.429 1.474 1.503 1.520 1.524 1.551 1.551 1.609 1.632

First we obtain the MLEs of μ and λ based on the data obtained in Scheme 1. In Figure 2 we provide the plot of the profile log-likelihood function of μ. It is an unimodal function, and based on that we obtain $\widehat{\mu}_{MLE} = 0.448$, and finally we obtain $\widehat{\lambda} = 0.375$. Similarly, we obtain the MLEs of μ and λ for other schemes too. Although we could not prove it theoretically, it is observed that the profile log-likelihood function is an unimodal function. Therefore, finding the MLE of μ is quite simple, and once the MLEs of μ is obtained the MLE of λ can be obtained immediately.

As we have no prior information of the hyper-parameters of the prior distribution of λ, we assume non-informative prior of λ, *i.e.* $a = b = 0$. We also construct 95% credible intervals

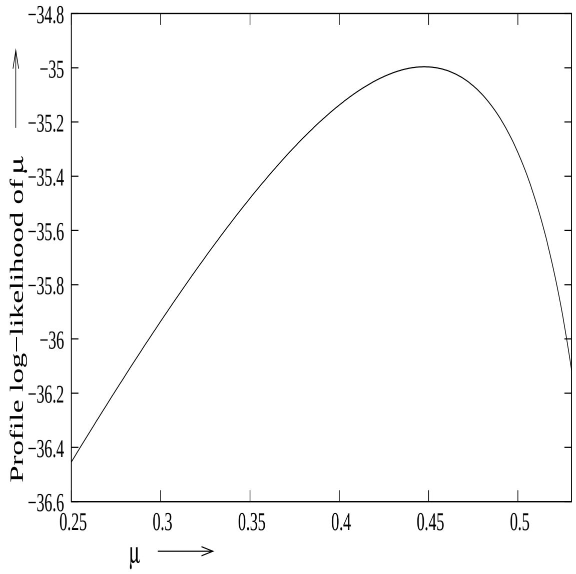

Figure 2: Profile log-likelihood of μ.

for μ and λ. All results are presented in Table 8. In all cases, it is clear that the MLEs and the Bayes estimators with respect to the non-informative priors behave quite similarly. One major advantage of the Bayesian method is that the construction of the credible intervals become quite simple in this case.

Table 8: Data analysis result of strength data. The credible intervals are presented within parenthesis.

Scheme	$\widehat{\mu}_{MLE}$	$\widehat{\lambda}_{MLE}$	$\widehat{\mu}_{Bayes1}$	$\widehat{\lambda}_{Bayes1}$	$\widehat{\mu}_{Bayes2}$	$\widehat{\lambda}_{Bayes2}$
Scheme 1	0.448	0.375	0.476	0.399	0.512	0.428
			(0.430, 0.529)	(0.375, 0.515)	(0.213, 0.725)	(0.102, 0.781)
Scheme 2	0.447	0.591	0.416	0.611	0.472	0.635
			(0.360, 0.512)	(0.476, 0.769)	(0.143, 0.723)	(0.189, 0.912)
Scheme 3	0.455	0.407	0.471	0.463	0.496	0.503
			(0.424, 0.531)	(0.412, 0.589)	(0.195, 0.782)	(0.142, 0.843)

6 Prediction

In this section we consider the prediction of the censored observations based on the current available sample, popularly known as the informative sample. Predicting the censored or future observations based on the informative sample is important in applied statistics. This problem has received considerable attention recently, see the articles by Al-Hussaini (1999), Ali Mousa *et al.* (2005), Balakrishnan *et al.* (2010), Kundu and Raqab (2012) and see the references cited therein.

The main objective is to provide the estimate of the posterior predictive density of the censored observations based on the current type-II censored data, and also to construct predictive intervals of the censored sample, based on an informative sample, see for example Dunsmore (1974) for a nice discussion on it. Prediction of censored observations for progressively censored samples was first considered by Balakrishnan and Rao (1997). They mainly considered the prediction of the future observation. Later Basak *et al.* (2006) considered more general problems under the same setup. We mainly consider the similar prediction problem as of Basak *et al.* (2006), as described below.

Let $X_1, \cdots, X_n$ denote the failure times of n independent units placed on a life-testing experiment. It is assumed that X_i's are random samples from a distribution function with CDF (2). Let $T_1 < \cdots < T_m$ denote the ordered statistics from a progressively Type-II censoring scheme $(R_1, \cdots, R_m)$, observed from $X_1, \cdots, X_n$. It should be noted that instead of the complete sample $X_1, \cdots, X_n$, we observe only $T_1 < \cdots < T_m$. The prediction problem involves predicting life-lengths $T_{j:R_i}$ for $j = 1, \cdots, R_i$ and $i = 1, \cdots, m$ of all censored units in all m stages of censoring. Here $T_{j:R_i}$ denotes the j-th order statistics out of R_i removed units at the i-th stage.

In this case since the CDF $F(\cdot; \lambda, \mu)$ is absolutely continuous, the conditional distribution of $T_{j:R_i}$ given $T_1, \cdots, T_m$ is the conditional distribution of $T_{j:R_i}$ given T_i. Moreover, the PDF of $T_{j:R_i}$ given $T_i = t_i$ is same as the PDF of the j-th order statistics out of R_i units from the PDF

$$g(x; \lambda, \mu) = \frac{f(x; \lambda, \mu)}{1 - F(t_i; \lambda, \mu)} = 2\lambda(x - \mu)e^{-\lambda(x-t_i)(x+t_i-2\mu)}; \quad x > t_i. \tag{23}$$

For predicting $T_{j:R_i}$, we first obtain the posterior predictive density of $T_{j:R_i}$ given $T_i = t_i$. The posterior predictive density of $T_{j:R_i}$ given $T_i = t_i$ is given by

$$\begin{aligned} f^*_{T_{j:R_i}|T_i=t_i}(t|data) &= E_{Posterior}\left[f_{T_{j:R_i}|T_i=t_i}(t|\lambda, \mu)\right] \\ &= \int_0^\infty \int_0^{t_1} f_{T_{j:R_i}|T_i=t_i}(t|\lambda, \mu)\pi(\lambda, \mu|Data)\mathrm{d}\mu\mathrm{d}\lambda. \end{aligned} \tag{24}$$

Here $f_{T_{j:R_i}|T_i=t_i}(t|\lambda, \mu)$ is the conditional PDF of $T_{j:R_i}$ given $T_i = t_i$. Using the Markov property of the conditional order statistics, we have

$$\begin{aligned} f_{T_{j:R_i}|T_i=t_i}(t|\lambda, \mu) &= \frac{2\lambda R_i!}{(j-1)!(R_i-j)!} \times (t-\mu)^2 \times e^{-\lambda(R_i-j+1)(t-\mu)^2+\lambda(t_i-\mu)^2 R_i} \\ &\quad \times \left(e^{-\lambda(t_i-\mu)^2} - e^{-\lambda(t-\mu)^2}\right)^{j-1}; \quad t > t_i, \end{aligned} \tag{25}$$

and $\pi(\lambda, \mu|Data)$ is same as defined in (11). It is immediate that $f^*_{T_{j:R_i}|T_i=t_i}(t|data)$ cannot be expressed in closed form. We propose to use the importance sampling technique as it has been mentioned in Section 4, to estimate (25). Suppose $\{(\lambda_i, \mu_i); i = 1, \cdots, N\}$ is a sample of size N from the posterior distribution function $\pi(\lambda, \mu|Data)$, then a simulation consistent estimator of $f^*_{T_{j:R_i}|T_i=t_i}(t|data)$ becomes;

$$\widehat{f}^*_{T_{j:R_i}|T_i=t_i}(t|data) = \frac{1}{\sum_{k=1}^N h(\mu_k)} \sum_{k=1}^N f_{T_{j:R_i}|T_i=t_i}(t|\lambda_k, \mu_k)h(\mu_k). \tag{26}$$

Along the same line we want to consider the estimation of the predictive distribution function of $T_{j:R_i}$ given $T_i = t_i$. Suppose $F_{j:R_i|T_i=t_i}(t|\lambda, \mu)$ denotes the distribution function of $T_{j:R_i}$ given $T_i = t_i$, *i.e.*

$$\begin{aligned} F_{j:R_i|T_i=t_i}(t|\lambda, \mu) &= \frac{R_i!}{(j-1)!(R_i-j)!}\int_0^t [G(z|\lambda, \mu)]^{j-1}[1 - G(z|\lambda, \mu)]^{R_i-j} g(z|\lambda\mu) \\ &= \frac{R_i!}{(j-1)!(R_i-j)!}\int_0^{G(t|\lambda,\mu)} u^{j-1}(1-u)^{R_i-j}\mathrm{d}u, \end{aligned} \tag{27}$$

and

$$G(t|\lambda,\mu) = \frac{F(t|\lambda,\mu) - F(t_i|\lambda,\mu)}{1 - F(t_i|\lambda,\mu)}; \quad t > t_i.$$

Therefore, the posterior predictive density of $T_{j:R_i}$ given $T_i = t_i$ is given by

$$\begin{aligned} F^*_{T_{j:R_i}|T_i=t_i}(t|data) &= E_{Posterior}\left[F_{T_{j:R_i}|T_i=t_i}(t|\lambda,\mu)\right] \\ &= \int_0^\infty \int_0^{t_1} F_{T_{j:R_i}|T_i=t_i}(t|\lambda,\mu)\pi(\lambda,\mu|Data)\mathrm{d}\mu\mathrm{d}\lambda. \end{aligned} \quad (28)$$

A simulation consistent estimator of $F^*_{j:R_i}(t|Data)$ can be obtained as

$$\widehat{F}^*_{j:R_i|T_i=t_i}(t|Data) = \frac{1}{\sum_{k=1}^N h(\mu_k)} \sum_{k=1}^N F_{j:R_i|T_i=t_i}(t|\lambda_k,\mu_k)h(\mu_k). \quad (29)$$

Another interesting problem is to construct a two-sided predictive interval of $T_{j:R_i}$ based on the observed sample. Now we will briefly discuss how to construct a 100(1-β)% predictive interval of $T_{j:R_i}$. Note that a symmetric 100(1-β)% predictive interval of $T_{j:R_i}$ can be obtained by solving the following two equations for the lower bound L and upper bound U, see for example; Al-Jarallah and Al-Hussaini (2007),

$$\begin{aligned} \frac{1+\beta}{2} &= P\left[T_{j:R_i} > L|Data\right] = 1 - F^*_{T_{j:R_i}|T_i=t_i}(L|Data) \Rightarrow F^*_{T_{j:R_i}|T_i=t_i}(L|Data) = \frac{1}{2} - \frac{\beta}{2}, \\ \frac{1-\beta}{2} &= P\left[T_{j:R_i} > U|Data\right] = 1 - F^*_{T_{j:R_i}|T_i=t_i}(U|Data) \Rightarrow F^*_{T_{j:R_i}|T_i=t_i}(U|Data) = \frac{1}{2} + \frac{\beta}{2}. \end{aligned}$$

It is not possible to obtain the solutions analytically. We need to apply suitable numerical techniques for solving non-linear equations.

For illustrative purposes we have considered the data set obtained by censoring scheme 1, and obtained the posterior predictive density and the corresponding predictive distribution function of the 26th observation in Figure 3.

7 Conclusions

In this paper we consider the statistical inference of the unknown parameters of the two-parameter Rayleigh distribution when the data are progressively Type-II censored. Al-

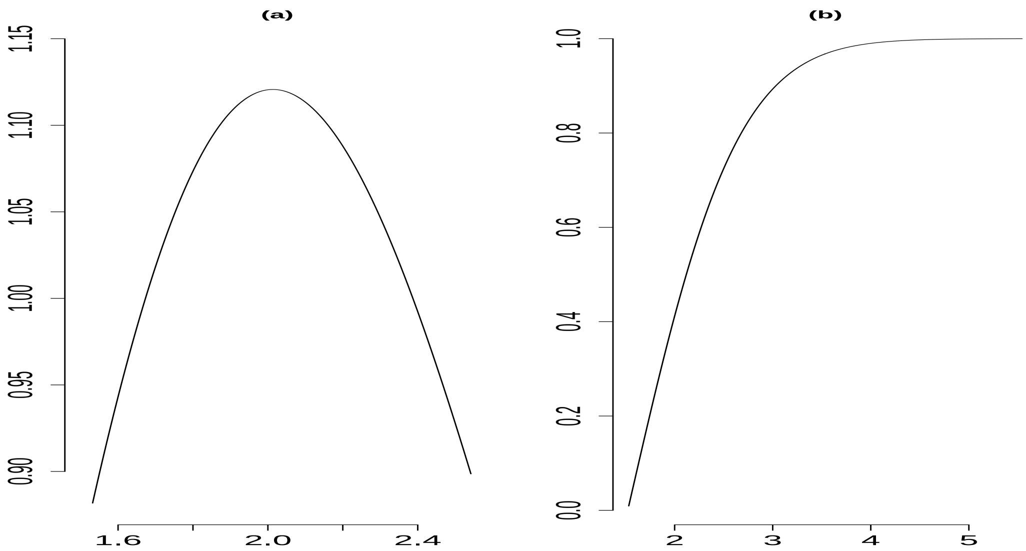

Figure 3: (a) Posterior predictive density, (b) Posterior predictive distribution function of the 26-th observation based on censoring Scheme 1 for the strength data.

though, the MLEs can be obtained by solving a one dimensional optimization problem, the corresponding confidence intervals are not easy to obtain. Due to this reason we consider the Bayesian inference of the unknown parameters and we suggest using importance sampling technique to compute the Bayes estimates and also to construct associated credible intervals. It is observed that Bayes estimates with respect to the non-informative priors behave quite similarly with the corresponding MLEs. We further consider the prediction of the unobserved observation using the observed sample from a predictive density approach. Although the predictive density cannot be obtained in explicit forms, we have proposed to use importance sampling technique to estimate the predictive density function and also the associated distribution function.

Acknowledgements:

The authors are thankful to unknown referees for their constructive comments which had helped to improve the earlier draft of the manuscript considerably.

Appendix

Proof of Theorem 1: We write (4) as

$$l(\mu, \lambda) = m \ln \lambda + \sum_{i=1}^{m} \ln(t_i - \mu) - \lambda \sum_{i=1}^{m} c_i (t_i - \mu)^2, \tag{30}$$

where $c_i = (R_i + 1) \geq 1$ for $i = 1, \ldots, m$. The domain of the log-likelihood function $l(\mu, \lambda)$ is $[0, t_1) \times (0, \infty)$. First we will show that for $(\mu, \lambda) \in (-\infty, t_1) \times (0, \infty)$, the maximum of $l(\mu, \lambda)$ exists and it is unique. The following observations will be useful

$$\frac{\partial^2 l(\mu, \lambda)}{\partial \lambda^2} = -\frac{m}{\lambda^2} < 0, \quad \frac{\partial l(\mu, \lambda)}{\partial \mu^2} = -\sum_{i=1}^{m} \frac{1}{(t_i - \mu)^2} - 2\lambda \sum_{i=1}^{m} c_i < 0.$$

Hence for fixed $\lambda(\mu)$, $l(\mu, \lambda)$ is a strictly concave function of $\mu(\lambda)$. Since $m \geq 2$, for fixed λ

$$\lim_{\mu \to -\infty} l(\mu, \lambda) = -\infty \quad \text{and} \quad \lim_{\mu \to t_1} l(\mu, \lambda) = -\infty \tag{31}$$

and for fixed μ

$$\lim_{\lambda \to 0} l(\mu, \lambda) = -\infty \quad \text{and} \quad \lim_{\lambda \to \infty} l(\mu, \lambda) = -\infty. \tag{32}$$

Therefore, for fixed λ (μ), $l(\mu, \lambda)$ is an unimodal function with respect to μ (λ). Further,

$$\lim_{\substack{\mu \to -\infty \\ \lambda \to 0}} l(\mu, \lambda) = -\infty, \ \lim_{\substack{\mu \to t_1 \\ \lambda \to 0}} l(\mu, \lambda) = -\infty, \ \lim_{\substack{\mu \to -\infty \\ \lambda \to \infty}} l(\mu, \lambda) = -\infty \ \lim_{\substack{\mu \to t_1 \\ \lambda \to \infty}} l(\mu, \lambda) = -\infty. \tag{33}$$

Suppose $(\mu_0, \lambda_0) \in (-\infty, t_1) \times (0, \infty)$, and $l(\mu_0, \lambda_0) = c$. Consider, the following set

$$A = \{(\mu, \lambda) : (\mu, \lambda) \in (-\infty, t_1) \times (0, \infty), l(\mu, \lambda) \geq c\}.$$

A is a closed and bounded set, hence A is compact. Since $l(\mu, \lambda)$ is a continuous function of (μ, λ), hence $l(\mu, \lambda)$ has a maximum for $(\mu, \lambda) \in A$. To show that if $(\mu_1, \lambda_1) \in (-\infty, t_1) \times (0, \infty)$ maximizes $l(\mu, \lambda)$, then (μ_1, λ_1) is unique, observe that for $(\mu, \lambda) \in (-\infty, t_1) \times (0, \infty)$,

$$l(\mu_1, \lambda_1) > l(\mu_1, \lambda) > l(\mu, \lambda). \tag{34}$$

Therefore, if $\mu_1 \geq 0$, the proof is complete. Suppose $\mu_1 < 0$, in this case for $(\mu, \lambda) \in (0, t_1) \times (0, \infty)$

$$l(\mu_1, \lambda_1) > l(0, \lambda_1) > l(\mu, \lambda_1) > l(\mu, \lambda). \tag{35}$$

Hence in this case the MLE of (μ, λ) becomes $(0, \lambda_1)$, and it is unique. ∎

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