ERG RELATION ON HYPER K-ALGEBRAS

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ABSTRACT. In this manuscript at first by considering the notion of equivalence, regular and good, (ERG), relations on a hyper K-algebra H, we characterize it on a hyper K-algebra H in which H is an (H-absorbing hyper K-ideal)-decomposable. Then we investigate the quotient hyper K-algebra related to an ERG relation on H. Moreover we show that the Theorem 4.1 and Corollary 4.3 of [2] are not true in general.

1 Introduction

The study of BCK-algebra was initiated by Imai and Iséki [5] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. The hyper structure theory (called also multi algebras) was introduced in 1934 by Marty [7] at the 8th congress of Scandinavian Mathematicians. Hyper structures have many applications to several sectors of both pure and applied sciences. Borzooei, et.al [6, 3] applied the hyper structure to BCK-algebras and introduced the concept of hyper BCK-algebra and hyper K-algebra in which each of them is a generalization of BCK-algebra. Also in [10, 1], they have investigated the quotient related to an equivalence, regular and good, ERG, relation on hyper K-algebra H . Now we use an ERG relation on a hyper K-algebra H, in which H is an (H-absorbing hyper K-ideal)-decomposable, and obtain some related results.

2 Preliminaries

Let H be a non-empty set, the set of all non-empty subset of H is denoted by $\mathcal{P}^*(H)$. A *hyperoperation* on H is a map $\circ : H \times H \to \mathcal{P}^*(H)$, where $(a, b) \to a \circ b, \forall a, b \in H$. A set H, endowed with a hyperoperation, "$\circ$", is called a *hyperstructure.* If $A, B \subseteq H$, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$ of H.

Definition 2.1. [3, 6] Let H be a non-empty set containing a constant "0" and "$\circ$" be a hyperoperation on H. Then H is called a *hyper K-algebra (hyper BCK-algebra)* if it satisfies K1-K5(HK1-HK4).

K1: $(x \circ z) \circ (y \circ z) < x \circ y,$ HK1: $(x \circ z) \circ (y \circ z) \ll x \circ y,$
K2: $(x \circ y) \circ z = (x \circ z) \circ y,$ HK2: $(x \circ y) \circ z = (x \circ z) \circ y,$
K3: $x < x,$ HK3: $x \circ H \ll x,$
K4: $x < y, \ y < x,$ then $x = y,$ HK4: $x \ll y, \ y \ll x,$ then $x = y,$
K5: $0 < x,$

for all $x, y, z \in H$, where $x < y(x \ll y)$ means $0 \in x \circ y$. Moreover for any $A, B \subseteq H$, $A < B$ if $\exists a \in A, \exists b \in B$ such that $a < b$ and $A \ll B$ if $\forall a \in A, \exists b \in B$ such that $a \ll b$.

For briefly the readers could see some definitions and results about hyper K-algebra and hyper BCK-algebra in [3, 6].

Note: Form now on, H is a hyper K-algebra unless otherwise is stated. Also if $I \subset H$ we set $I^{'} = H \setminus I$, $I^* = I^{'} \cup \{0\}$.

Theorem 2.2. *[3] Let S be a non-empty subset of $(H, \circ, 0)$. Then S is a hyper K-subalgebra of H if and only if $x \circ y \subseteq S$ for all $x, y \in S$.*

Definition 2.3. [4] An element $b \in H$ is called a *left(right) scalar* if $|b \circ x| = 1 (|x \circ b| = 1)$ for all $x \in H$. An element is called a *scalar* if it is a left and a right scalar.

Definition 2.4. [8, 3, 11] A non-empty subset I of H is said to be *closed* if $x < y$ and $y \in I$ imply that $x \in I$, and it is said to be *hyper K-ideal* of H if $x \circ y < I$ and $y \in I$ imply that $x \in I$ for all $x, y \in H$.

Theorem 2.5. *[8] Any hyper K-ideal of H is a closed set.*

Theorem 2.6. *[9] Let $(H_i, \circ_i, 0), i \in \Omega$ be a family of hyper K-algebras such that $H_i \cap H_j = \{0\}, i \neq j \in \Omega$, 0 be a left scalar in each H_i, $i \in \Omega$, $H = \cup_{i \in \Omega} H$ and "$\circ$" on H is defined as follows:*

$$x \circ y := \left\{ \begin{array}{ll} x \circ_i y & \text{if } x, y \in H_i, \\ \{x\} & \text{if } x \in H_i, y \notin H_i \end{array} \right. \quad \forall x, y \in H.$$

Then $(H, \circ, 0)$ is hyper K-algebra and denoted by $H = \oplus_{i \in \Omega} H_i$(hyper K-algebra).

Theorem 2.7. *[9] Let $(H, \circ, 0)$ be a hyper BCK-algebra. Then $H = \oplus_{i \in \Omega} H_i$(hyper BCK-algebra) if and only if $H = \oplus_{i \in \Omega} H_i$(hyper BCK-ideal).*

Definition 2.8. [8] Let I and S be non-empty subsets of H. Then we say that I is S-absorbing if $x \in I$ and $y \in S$ then $x \circ y \subseteq I$. If $S = I^{'}$ or $S = I^*$ we say that I is *C-absorbing* or C^*-*absorbing*, respectively.

Theorem 2.9. *[8] Let H be a hyper BCK-algebra and I be a hyper BCK-ideal of H. Then I is an H-absorbing.*

Definition 2.10. [8] A hyper K-algebra H is called *(P)-decomposable* if there exists a non-trivial family $\{A_i\}_{i \in \Lambda}$ of subsets of H with P-property such that:
(i) $H \neq \{A_i\}$ for all $i \in \Lambda$, (ii) $H = \cup_{i \in \Lambda} A_i$, (iii) $A_i \cap A_j = \{0\}, i \neq j$.
In this case, we write $H = \oplus_{i \in \Lambda} A_i$(P) and say that $\{A_i\}_{i \in \Lambda}$ is a *(P)-decomposition* for H. Also if each A_i, $i \in \Lambda$, is S-absorbing we write $H \overset{\text{S}}{=} \oplus_{i \in \Lambda} A_i$(P). Moreover we say that this decomposition is *closed union*, in short (P)-CUD, if $\cup_{i \in \Delta} A_i$ has P-property for any non-empty subset Δ of Λ. If there exists a (P)-CUD for H, then we say that H is a (P)-CUD.

Theorem 2.11. *[8] Let $H \overset{\text{C}^*}{=} \oplus_{i \in \Lambda} A_i$(hyper K-ideal) . Then H is (hyper K-ideal)-CUD and $H \overset{\text{C}^*}{=} I \oplus I^*$(hyper K-ideal) where, $I = \cup_{i \in \Delta} A_i$ for any non-empty subset Δ of Λ.*

Theorem 2.12. *[8] Let $H \overset{\text{C}^*}{=} \oplus_{i \in \Lambda} A_i$(closed set). Then A_i is a hyper K-ideal for all $i \in \Lambda$.*

Theorem 2.13. *[8] Let $H \overset{\text{H}}{=} A \oplus B$. Then 0 is a left scalar element.*

Lemma 2.14. *[8] Let I be hyper K-ideal of H. (i) If $x \circ y < I$, then $x \circ y \cap I \neq \phi$. (ii) If $x \circ y < I$ and $x \circ y \subseteq I^*$, then $x < y$.*

Definition 2.15. [10] Let $\sim$ be an equivalence relation on H and $A, B \subseteq H$. Then
(i) $A \sim B$ if and only if $\exists\, a \in A$ and $\exists\, b \in B$ such that $a \sim b$,
(ii) $A \approx B$ if and only if for all $a \in A$, $\exists\, b \in B$ such that $a \sim b$, and for all $b \in B$, $\exists\, a \in A$ such that $a \sim b$,
(iii) $\sim$ is called regular to the right(left) if $a \sim b$ implies that $a \circ c \approx b \circ c (c \circ a \approx c \circ b)$, for any $a, b, c \in H$. A regular equivalence to the right and to the left is called regular,
(iv) $\sim$ is called a congruence relation on H if $a \sim b$ and $x \sim y$ then $a \circ x \approx b \circ y$,
(v) $\sim$ is called good, if $a \circ b \sim \{0\}$ and $b \circ a \sim \{0\}$ implies $a \sim b$ for any, $a, b, c \in H$.

Note: Let $\sim$ be an equivalence, regular and good relation on a hyper K-algebra H. For short, we call this relation, " ERG relation". Also we denote the equivalence class x by C_x and use I instead of the C_0, unless otherwise is stated.

Theorem 2.16. *[2] An equivalence $\sim$ is a regular relation if and only if it is a congruence relation.*

Proposition 2.17. *[10] If $\sim$ is an ERG relation on H, then I is a hyper K-ideal of H, and $(H/\sim, *, C_0)$ is a hyper K-algebra, where $H/\sim = \{C_x : x \in H\}$ and*

$$* : H/\sim \times H/\sim \to H/\sim$$
$$(C_x, C_y) \to \{C_t | t \in x \circ y\}$$

Theorem 2.18. *[10] (First isomorphism theorem)Let $f : H_1 \to H_2$ be a homomorphism, i.e. f is a map such that, $f(0) = 0$ and $f(x \circ y) = f(x) \circ f(y)$, for all $x, y \in H_1$. Then $H_1/Kerf \cong Im(f)$, where $Kerf = \{x \in H_1 : f(x) = 0\}$.*

3   Characterization of ERG relation on a CUD- hyper K-algebra.

In this section we investigate the characteristics of an ERG relation, $\sim$, on a hyper K-algebra. We show that any such relation is as form:

$$x \sim y \Rightarrow x \circ y < I \text{ and } y \circ x < I, \text{ where } I = C_0.$$

Moreover if $H \overset{\mathrm{H}}{=} \oplus_{i \in \Lambda} A_i$(hyper K-ideal), then this relation change into

$$x, y \in I \text{ or } x = y \notin I,$$

and under some conditions this relation is unique. Finally, by Remark 3.19 we shall show that Theorem 4.1 and Corollary 4.3 of [2] is not true in general.

In the beginning, we give some examples which show that, the notion of regular and good equivalence relation are independent. In order to get this fact, consider two following examples.

Example 3.1. Let hyper K-algebra $(H, \circ, 0)$ and equivalence relation $\sim$ on $H = \{0, 1, 2\}$ be as follows:

$\circ$	0	1	2
0	{0}	{0,1}	{0,1,2}
1	{1}	{0,1}	{1,2}
2	{1,2}	{0,2}	{0,1,2}

$$\sim = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2)\}.$$

Then by some manipulations we can get that, the relation $\sim$ is a regular relation on $(H, \circ, 0)$, but it is not a good relation. Because $2 \circ 1 = \{0, 2\} \sim \{0\}$, $1 \circ 2 = \{1, 2\} \sim \{0\}$, and $1 \nsim 2$.

Example 3.2. Consider the following hyper K-algebra and equivalence relation $\sim$ as preceding example on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	{0,1,2}	{0}	{0}
1	{1}	{0,1,2}	{1}
2	{2}	{2}	{0,1,2}

Then $\sim$ is a good relation. Because from $1 \circ 0 = \{1\} \sim \{0\}$ and $0 \circ 1 = \{0\} \sim \{0\}$ we have $0 \sim 1$. But it is not regular relation. Since from $0 \sim 1$ we con not imply that $\{0, 1, 2\} = 0 \circ 0 \approx 0 \circ 1 = \{1\}$.

Lemma 3.3. *Let $\sim$ be an equivalence relation on H and $x \circ y \sim \{0\}$. Then $x \circ y \cap I \neq \phi$ and $x \circ y < I$.*

Proof. By Definition 2.15(i) there is an element $u \in x \circ y$ in which $u \sim 0$. So $u \in I$, i.e., $x \circ y \cap I \neq \phi$. This fact implies that $x \circ y < I$.

Theorem 3.4. *Let $\sim$ be an ERG relation on H. Then $x \sim y$ if and only if $x \circ y < I$ and $y \circ x < I$.*

Proof. ($\Rightarrow$) Let $x \sim y$. Since $\sim$ is regular, we have $x \circ x \approx x \circ y$ and $x \circ x \approx y \circ x$. As $0 \in x \circ x$, we conclude that $x \circ y \sim \{0\}$ and $y \circ x \sim \{0\}$. Hence by using Lemma 3.3 we imply that $x \circ y < I$ and $y \circ x < I$.
($\Leftarrow$) Suppose $x \circ y < I$ and $y \circ x < I$. Therefore there exist $u \in x \circ y$ and $v \in I$ such that $u < v$. Since by Proposition 2.17, I is hyper K-ideal, we get that $u \in I$, that is, $x \circ y \sim \{0\}$. By similar discussion, from $y \circ x < I$ we conclude that $y \circ x \sim \{0\}$. Since $\sim$ is a good relation we get that $x \sim y$.

Lemma 3.5. *Let $\sim$ be an equivalence relation on H and $A \subseteq I$. If $x \circ y \approx A$, then $x \circ y \subseteq I$.*

Proof. By Definition 2.15(ii) for any $u \in x \circ y$ there is an element $v \in A$ in which $u \sim v$. Since $v \sim 0$ and $\sim$ is transitive, we conclude that $u \sim 0$. Hence $u \in I$ and $x \circ y \subseteq I$.

Theorem 3.6. *Let $\sim$ be an equivalence and regular relation on H. If $0 \circ 0 \subseteq I$, then I is a hyper K-subalgebra of H.*

Proof. By using Theorem 2.2 we have to show that for any $x, y \in H$, $x \circ y \subseteq I$. Since $x \sim \{0\}$ and $y \sim \{0\}$ by regularity of $\sim$ and hypothesis we have $x \circ y \approx 0 \circ 0 \subseteq I$. Hence by using Lemma 3.5 we get that $x \circ y \subseteq I$.

Theorem 3.7. *Let $\sim$ be an equivalence and regular relation on H. If $0 \circ y \subseteq I$ for any $y \in H$, then I is H-absorbing.*

Proof. We have to show that for any $y \in H$ and $x \in I$, $x \circ y \subseteq I$. Since $x \sim \{0\}$ by regularity of $\sim$ we have $x \circ y \approx 0 \circ y$ for any $y \in H$. By Lemma 3.5 we conclude that $x \circ y \subseteq I$.

Corollary 3.8. *Let $\sim$ be an equivalence and regular relation on H. If 0 is a left scalar in H, then I is H-absorbing.*

Proof. Since $0 \circ y = \{0\}$ for any $y \in H$, the proof follows from Theorem 3.7.

The following example shows that if $0 \circ y \not\subseteq I$, then Theorem 3.7 is not true.

Example 3.9. Let $H = \{0, 1, 2, 3\}$ with a following structure.

$\circ$	0	1	2	3
0	{0,1}	{0,1}	{01,2,3}	{0,1,2,3}
1	{1}	{0,1}	{0,1,2,3}	{0,1,2,3}
2	{2}	{2}	{0,1,2,3}	{2}
3	{3}	{3}	{3}	{0,1,2,3}

Then $(H, \circ, 0)$ is a hyper K-algebra and the set $\theta = \{(0,0), (0,1), (1,0), (1,1), (2,2), (3,3)\}$ is an ERG relation on H, but the class $C_0 = I = \{0,1\}$ is not H-absorbing. Because $1 \circ 2 = \{0, 1, 2, 3\} \not\subset I$

Lemma 3.10. *Let A be a non-empty closed set of H. Then $0 \in A$.*

Proof. Suppose $x \in A$. By K5 we have $0 < x$. Since A is closed set, it implies that $0 \in A$.

Theorem 3.11. *Let $H \stackrel{\mathrm{H}}{=} A \oplus B$(closed set). Then $a \circ b < A$ and $b \circ a < A$ if and only if $a, b \in A$ or $a = b \notin A$.*

Proof. ($\Rightarrow$) Let $a \circ b < A$, $b \circ a < A$ and $a, b \notin A$, hence we have the following cases:
(i): $a \in A, b \notin A$, or $b \in A, a \notin A$,
(ii): $a \notin A, b \notin A$.
We show that (i) does not hold. Suppose $a \in A, b \notin A$, then $b \circ a \subseteq B$ as B is H-absorbing. On the other hand, since $b \circ a < A$ by using Lemma 2.14 we have $b \circ a \cap A \neq \phi$. Hence form these we conclude that $b \circ a \cap (A \cap B = \{0\}) \neq \phi$, so $0 \in b \circ a$ or $b < a$. Since A is a closed set and $a \in A$ we get that $b \in A$ which is a contradiction. By Similar discussion we can show that the case "$b \in A, a \notin A$" dose not hold.
Now suppose (ii) holds, we show that $a = b$. Since B is H-absorbing, we have $a \circ b \subseteq B$ and $b \circ a \subseteq B$, also by hypothesis $a \circ b < A$ and $b \circ a < A$. As same argument in case(i) we conclude that $0 \in a \circ b$ and $0 \in b \circ a$. So by K4 we get that $a = b$.
($\Leftarrow$) Suppose $a, b \in A$ or $a = b \notin A$. If $a, b \in A$, since A is H-absorbing we have $a \circ b \subseteq A$ and $b \circ a \subseteq A$, hence $a \circ b < A$ and $b \circ a < A$. Whenever $a = b \notin A$, since $0 \in a \circ a$ and by Lemma 3.10, $0 \in A$ we get that $a \circ a < A$. The proof is completed.

Corollary 3.12. *Let $H \stackrel{\mathrm{H}}{=} A \oplus B$(hyper K-ideal). Then $a \circ b < A$ and $b \circ a < A$ if and only if $a, b \in A$ or $a = b \notin A$.*

Proof. The proof follows from Theorems 2.5 and 3.11.

Theorem 3.13. *Let $\sim$ be an ERG relation on H, $0 \circ y \subseteq I$ (or 0 is a scaler) for all $y \in H$ and I^* be an H-absorbing hyper K-ideal. Then $a \sim b$ if and only if $a, b \in I$ or $a = b \notin I$.*

Proof. By Corollary 3.8, $I = C_0$ is H-absorbing. So we have $H \stackrel{\mathrm{H}}{=} I \oplus I^*$(hyper K-ideal), and the proof follows from Theorem 3.4 and Corollary 3.12.

Theorem 3.14. *Let $H \stackrel{\mathrm{H}}{=} A \oplus B$(closed set), and*

$$u \sim v \text{ if and only if } u, v \in A \text{ or } u = v \notin A.$$

Then
(i) $\sim$ is equivalence relation on H such that $C_0 = A$ and $|H/\sim| = |B|$.
(ii) $\sim$ is good relation.
(iii) $\sim$ is congruence(regular) relation if and only if $b \circ x = b \circ y$ where $x, y \in A$ and $b \notin A$.

Proof. (i) It is clear that $\sim$ is an equivalence relation on H and $C_0 = A$. Now we show that $|H/\sim| = |B|$. It is sufficient to show that $\varphi : B \to H/\sim$ where $\varphi(x) = C_x$ is bijection. Since whenever $x = y$, then $C_x = C_y$ for all $x, y \in B$ we get that φ is well-defined. Suppose $\varphi(x) = \varphi(y), \forall x, y \in B$, this yields $C_x = C_y$ and $x \sim y$. If $x = 0$ or $y = 0$, then from $x \sim y$ we get that $x, y \in A$. Hence $x, y \in A \cap B = \{0\}$. Therefore $x = y = 0$. If $x, y \notin A$ form definition of $\sim$ we conclude that $x = y$. Hence φ is one-to-one. On the other hand suppose $C_x \in H/\sim$. If $x \in A$ then $C_x = C_0$ and $\varphi(0) = C_x$, otherwise $x \in B \setminus \{0\}$ therefor $\varphi(x) = C_x$, that is, φ is onto. Thus $|H/\sim| = |B|$.

(ii) Suppose $a \circ b \sim \{0\}$ and $b \circ a \sim \{0\}$, from these and Lemma 3.3 we conclude that

$$(3.1) \qquad a \circ b \cap A \neq \phi \text{ and } b \circ a \cap A \neq \phi.$$

Now by considering the following cases we show that $a \sim b$.
Case1. $a, b \in A$,
Case2. $a \in A$ and $b \notin A$(or $b \in A$ and $a \notin A$),
Case3. $a, b \notin A$.
If Case1 holds, then $a \sim b$. If Case2 holds and $a \in A$ and $b \notin A$, then we have $b \circ a \subseteq B$, because B is H-absorbing. By using relation (3.1) and Lemma 2.14 we get that $b < a$. Since A is closed set we conclude that $b \in A$, which is a contradiction. Similarly, if $b \in A$ and $a \notin A$ holds, then we get the same contradiction. If Case3 holds, then since B is H-absorbing we have

$$(3.2) \qquad a \circ b \subseteq B \text{ and } b \circ a \subseteq B.$$

Hence by using relations (3.1), (3.2) and $A \cap B = \{0\}$ we get that $0 \in a \circ b$ and $0 \in b \circ a$, that is, $a = b$ or $a \sim b$. Hence $\sim$ is good relation.

(iii)($\Leftarrow$) Let $a \sim b$ and $x \sim y$. We consider the following cases to show that $\sim$ is congruence relation, i.e., $a \circ x \approx b \circ y$ and $x \circ a \approx y \circ b$.
Case1. If $a, b, x, y \in A$, then since A is H-absorbing we have $a \circ x \subseteq A(x \circ a \subseteq A)$ and $b \circ y \subseteq A(y \circ b \subseteq A)$. Hence $a \circ x \approx b \circ y(x \circ a \approx y \circ b)$.
Case 2. If $a = b \notin A$ and $x, y \in A$, then by hypothesis $a \circ x = b \circ y$. Hence $a \circ x \approx b \circ y$. On the other hand, Since A is H-absorbing we have $x \circ a \subseteq A$ and $y \circ b \subseteq A$, hence $x \circ a \approx y \circ b$.
Case 3. If $a = b \notin A$ and $x = y \notin A$ we have $a \circ x = b \circ y$ and $x \circ a = y \circ b$. So $a \circ x \approx b \circ y$ and $x \circ a \approx y \circ b$.
In any case we have $a \circ x \approx b \circ y$ and $x \circ a \approx y \circ b$, where $a \sim b$ and $x \sim y$. Therefore $\sim$ is congruence relation and by Theorem 2.16 it is also a regular relation.
($\Rightarrow$) Let $\sim$ be congruence relation, we have to show that $b \circ x = b \circ y$, where $x, y \in A$ and $b \notin A$. Suppose $x, y \in A$ and $b \notin A$. The former yields $x \sim y$. Since $b \sim b$ and $\sim$ is congruence relation we get that $b \circ x \approx b \circ y$. On the other hand if $b \circ x \cap A \neq \phi$ and $b \circ y \cap A \neq \phi$, then since $x, y \in A$ and by Theorem 2.12 A is hyper K-ideal, these yield $b \in A$, which is a contradiction. So $b \circ x \subseteq A'$ and similarly $b \circ y \subseteq A'$. Now, let $t \in b \circ x$. Then from $b \circ x \approx b \circ y$, we conclude that there exist $s \in b \circ y$ such that $t \sim s$. Therefore $t = s \notin A$, hence $b \circ x \subseteq b \circ y$. Similarly, we conclude $b \circ y \subseteq b \circ x$. These imply $b \circ y = b \circ x$, and the proof is completed.

Example 3.15. Consider the following hyper K-algebra:

∘	0	1	2	3	4	5
0	{0}	{0}	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}	{1}	{0}
2	{2}	{2}	{0,1,2}	{2}	{2}	{0}
3	{3,4}	{3,4}	{3,4}	{0,3,4}	{0,4}	{3}
4	{4}	{4}	{4}	{4}	{0,4}	{4}
5	{5}	{5}	{5}	{5}	{5}	{0,5}.

It is clear that $H = \{0,1,2\} \oplus \{0,3,4,5\}$(hyper K-ideal). We set $A = \{0,1,2\}$ and $B = \{0,3,4,5\}$. It is easy to check that A and B are H-absorbing and $b \circ x = b \circ y$ for all $b \notin A$ and $x, y \in A$. Let

$$\sim = \{(0,0),(1,0),(0,1),(1,1),(0,2),(2,0),(1,2),(2,1),(2,2),(3,3),(4,4),(5,5)\}.$$

Then by Theorem 3.14 $\sim$ is an ERG relation on H. Hence the quotient hyper K-algebra, $(H/\sim, C_0, *)$, is as follows:

*	C_0	C_3	C_4	C_5
C_0	{C_0}	{C_0}	{C_0}	{C_0}
C_3	{C_3, C_4}	{C_0, C_3, C_4}	{C_0, C_4}	{C_3}
C_4	{C_4}	{C_4}	{C_0, C_4}	{C_4}
C_4	{C_5}	{C_5}	{C_5}	{C_0, C_5}.

Now we give a hyper K-algebra H, such that $H = A \oplus B$(hyper K-ideal) and at least one of A or B are not H-absorbing, while there is an ERG relation different from the relation given in Theorem 3.14 on H.

Example 3.16. Let $H = \{0,1,2,3\}$ with the following structure:

∘	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{2,3}	{0,2}	{1,2}
3	{3}	{3}	{0,3}	{0,2,3}.

Then $(H, \circ, 0)$ is a hyper K-algebra and

$$\chi = \{(0,0),(0,1),(1,0),(1,1),(2,2),(3,3),(2,3),(3,2)\}$$

is an ERG relation. It is clear that $C_0 = \{0,1\}$, $C_2 = \{2,3\}$ and $H = A \oplus B$, where $A = \{0,1\}$ and $B = \{0,2,3\}$, moreover B is not H-absorbing. Because $2 \circ 3 = \{1,2\} \nsubseteq \{0,2,3\}$. Also $H/I = \{C_0, C_2\}$ has the following structure:

*	C_0	C_2
C_0	{C_0}	{C_0}
C_2	{C_2}	{C_0, C_2}.

Theorem 3.17. Let $H \overset{\mathrm{H}}{=} A \oplus B$(closed set) and $b \circ x = b \circ y$ where $x, y \in A$, and $b \notin A$. Then the relation:

(3.3) $$u \sim v \Longleftrightarrow u, v \in A \text{ or } u = v \notin A$$

is the only ERG relation on H such that $C_0 = A$ and $H/\sim$ is a hyper K-algebra of order $|B|$.

Proof. By Theorem 3.14, $\sim$ is an ERG relation on H, $|H/\sim| = |B|$ and $C_0 = A$. Hence by Proposition 2.17, $H/\sim$ is a hyper K-algebra of order $|B|$. Finally we show that this ERG relation is unique. If Θ is another ERG relation on H with $C_0 = A$, then by Theorems 3.4 and 3.11 $x\Theta y$ if and only if $x, y \in A$ or $x = y \notin A$, i.e., $\Theta = \sim$.

Theorem 3.18. *Let* $H \stackrel{\mathrm{H}}{=} A \oplus B$(closed set), and $b \circ x = b \circ y$ where $x, y \in A$ and $b \notin A$. *Then* $H/A \cong B$.

Proof. By Theorem 3.17 and hypothesis the relation (3.3) is the only ERG relation on H such that that $C_0 = A$ and $H/\sim$ is a hyper K-algebra of order $|B|$. Now we define $f : H \to H$ as follows:

$$f(x) = \begin{cases} 0 & x \in A, \\ x & \text{otherwise.} \end{cases}$$

Since $A \cap B = \{0\}$, f is well-defined. We show that f is a homomorphism. To do this we consider the following cases:

Case1. If $x \in A$, then $x \circ y \subseteq A$ for all $y \in H$, since A is H-absorbing. Hence $f(x \circ y) = 0$ and $f(x) \circ f(y) = 0 \circ f(y) = 0$, Since by Theorem 2.13, 0 is a left scalar. Therefore $f(x \circ y) = f(x) \circ f(y)$.

Case2. If $x, y \in B$, then as above $x \circ y \subseteq B$ and $f(x \circ y) = x \circ y = f(x) \circ f(y)$.

Case3. If $x \in B \setminus \{0\}$, $y \in A$, Since $0 \in A$ then by hypothesis $x \circ y = x \circ 0$. Moreover $x \circ y \subseteq B$, because B is H-absorbing. Thus by definition of f, $f(x \circ y) = x \circ y = x \circ 0 = f(x) \circ f(y)$. Hence for any cases we have $f(x \circ y) = f(x) \circ f(y)$. These imply that f is homomorphism and $Ker f = A$. Consequently, by Theorem 2.18, $H/A \cong Im(f) = B$.

Now we show that Theorem 4.1 and Corollary 4.3 of [2], are not true in general.

Remark *3.19.* Theorem 4.1. and Corollary 4.3 of [2], are as follows:
"*Theorem 4.1: Let H(hyper BCK-algebra) be decomposable with decomposition $H = A \oplus B$. Then there exist a regular(i.e., it is a good relation according to the our definition) congruence relation Θ on H and a hyper BCK-algebra X of order 2 such that $H/\Theta \cong X$*".
" *Corollary 4.3: Let H be decomposable with decomposition $H = A \oplus B$ and $b \circ x = b \circ y$ for all $b \in B$ and $x, y \in A$. Then $|B| = 2$.*"

In the proof of Theorem 4.1 of [2], they have claimed that the relation

$$(3.4) \qquad\qquad x\Theta y \Longleftrightarrow x, y \in A \text{ or } x, y \in B - \{0\}$$

is a congruence relation and also in Corollary 4.3 they have claimed that $|B| = 2$, but now, we show that these are not true in general. To see this, consider the following example.

Example 3.20. Consider the following hyper BCK-algebra:

$\circ$	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0,1}	{1}	{1}
2	{2}	{2}	{0,2}	{2}
3	{3}	{3}	{3}	{0,3}

It is easy to check $H = \{0, 1\} \oplus \{0, 2, 3\}$, where $A = \{0, 1\}$ and $B = \{0, 2, 3\}$. By definition of Θ in relation (3.4) we have $2\Theta 2$, $3\Theta 2$ but $2 \circ 3 = \{2\} \not\approx 2 \circ 2 = \{0, 2\}$, since $\{0, 2\} \not\subset A$ and $\{0, 2\} \not\subset B - \{0\}$, i.e., Θ is not a congruence relation. Also we see that $|B| \neq 2$.

On the other hand, since any hyper BCK-ideal is H-absorbing, so if $H = A \oplus B$(hyper BCK-ideal), then we have $H \stackrel{\mathrm{H}}{=} A \oplus B$(hyper BCK-ideal). Therefore by considering Theorem 3.17, the only ERG relation on H with $C_0 = A$ is exactly as relation (3.3). Hence the relation have been introduced by relation (3.4) is not a congruence relation.

Now we give the correct version of Corollary 4.3. of [2].

Theorem 3.21. *Let H be a BCK-algebra and $H = A \oplus B$(hyper BCK-ideal). Then $H/A \cong B$.*

Proof. By Theorems 2.9, A and B are H-absorbing . Moreover by Theorem 2.7 we get that $H = A \oplus B$(hyper BCK-algebra). Hence by Theorem 2.6 we have $b \circ x = \{b\} = b \circ y$ for any $b \notin A$ and $x, y \in A$. Therefore by Theorem 3.18 the proof is completed.

Corollary 3.22. *Let H be a BCK-algebra and $H = \oplus_{i \in \Lambda} A_i$(hyper BCK-ideal). Then $H/A \cong B$ where $A = \cup_{i \in \Delta} A_i$ and $B = \cup_{i \in \Lambda \setminus \Delta} A_i$ for any non-empty subset Δ of Λ.*

Proof. By Theorem 2.11 we have $H = A \oplus B$(hyper BCK-ideal). Hence by using Theorem 3.21, $H/A \cong B$.

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