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On Linear Dependence to the Initial State for a Class of Nonstationary Cryptodeterministic Processes

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Abstract:A one-dimensional version of the Koksma-Hlawka inequality is used to obtain autocovariance bounds for a class of nonstationary stochastic processes generated by the evolution of discrete-time chaotic dynamical systems from a random initial state

Keywords:chaos; covariance bounds; cryptodeterminism; Koksma-Hlawka inequality; linear dependence; nonlinearity; nonstationarity

1 Introduction

Let $(X_n)_{n\in\mathbb{N}}$ denote a scalar stochastic process with values in $[a,b] \subset \mathbb{R}$ $(a < b)$ satisfying $X_n = T(X_{n-1})$ for every $n \geq 1$, where $T : [a,b] \to [a,b]$ is a nonlinear measurable map and X_0 is a random initial state. Such processes are often called *cryptodeterministic*, a term which seems to have been proposed originally by E. T. Whittaker [7] to express the fact that randomness enters the dynamics only through the initial state of the process. Beside their exotic character, such stochastic processes have attracted the interest of scientists for two essential reasons; on one side, as emphasized by Hall and Wolff [1], they provide reduced scale models which can capture some aspects of the dynamics of complex chaotic dynamical systems and give qualitative insight about their behaviour; on the other side, they arise quite naturally in telecommunications engineering, most notably in the synthesis of random signals with specific correlational properties, as shown by Kohda, Lawrance *et al.* in various papers [3, 5, 6].

An important dynamical characteristic of $(X_n)_{n\in\mathbb{N}}$ is the linear dependence of the state at time n on the initial state X_0, which is usually quantified by the autocovariance function

$$\gamma_0(j) = cov(X_0, X_j) = \mathbb{E}[X_0X_j] - \mathbb{E}[X_0]\mathbb{E}[X_j] \quad (j \geq 0)$$

where $\mathbb{E}[\cdot]$ denotes the mathematical expectation operator.

As a matter of fact, although a cryptodeterministic process $(X_n)_{n\in\mathbb{N}}$ is structurally completely dependent on the initial state X_0, there are various examples of cryptodeterministic processes which show only short range *linear* dependence on X_0, that is $\sum_{j\geq 0} |\gamma_0(j)| < \infty$.

Before considering some particular cases, let us note that, since $(X_n)_{n\in\mathbb{N}}$ satisfies the first-order Markov property, it is strictly stationary if and only if X_1 and X_0 are identically distributed, that is, if and only if the probability distribution of X_0 is invariant by T.

For instance, let us consider the cryptodeterministic process defined recursively from the initial condition $X_0 \sim \beta(1/2, 1/2)$ and the evolution map

$$T : [0,1] \ni x \mapsto 4x(1-x) \in [0,1].$$

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It is a standard result that the probability distribution $\beta(1/2, 1/2)$ is invariant by T. Hence, this process is strictly stationary, and a direct application of theorem 2.1 from Hall and Wolff [1] shows that $\gamma_0(j) = 0 \quad (j \geq 1)$.

Another example is provided by the cryptodeterministic process $(X_n)_{n\in\mathbb{N}}$ defined recursively from $X_0 \sim \mathcal{U}([0,1])$ and the evolution map

$$T : [0,1] \ni x \mapsto r\,x \mod 1 \in [0,1]$$

where $r \in \{2, 3, \dots\}$. Straightforward computations show that the probability distribution $\mathcal{U}([0,1])$ is invariant by T. Thus, $(X_n)_{n\in\mathbb{N}}$ is strictly stationary. Moreover, for any $j \geq 1$,

$$\gamma_0(j) = cov(X_0, X_j) = \sum_{i=0}^{r^j-1} \int_{\beta_i} (r^j u - i) u du - \left(\int_0^1 u du \right)^2$$

where $\beta_i = [ir^{-j}, (i+1)r^{-j})$ for $i = 0, \dots, r^j - 1$. Hence,

$$\gamma_0(j) = r^{-2j} \sum_{i=0}^{r^j-1} \frac{3i+2}{6} - \frac{1}{4} = \frac{r^{-j}}{12}.$$

Other examples of cryptodeterministic processes can be found, for instance, in the work of Kohda, Tsuneda and Lawrance [3] on Chebyshev maps. It is a remarkable fact that, in these examples, as in the ones previously exposed, the autocovariances with the initial state can be explicitly computed, for it is not always the case that explicit formulas can be obtained, due to computational difficulties.

In this paper, this problem is dealt with for a specific class of chaotic maps, without requiring the strict stationarity of the generated cryptodeterministic process, that is, without requiring that the initial distribution be invariant. It is shown that, even if explicit computations are intractable, it is possible to use a one-dimensional version of the well-known Koksma-Hlawka inequality [2, 4] to obtain an upper bound on the absolute autocovariances, which is of the same order in the stationary and the nonstationary case.

2 A general covariance inequality

Before defining the class of chaotic maps that will be dealt with in the sequel, let us introduce some definitions and notations.
Let h be a continuously differentiable real-valued map defined on $[a,b] \subset \mathbb{R}$ $(a < b)$. The quantity

$$V_{[a,b]}(h) = \int_a^b |h'(x)| dx.$$

is called the total variation or the variation of h on $[a,b]$. In this paper, this notion will be used in connection with the following inequality, which can be seen as a simplified version of the Koksma-Hlawka inequality [2, 4].

Proposition 2.1 Let h be continuously differentiable on $[0,1]$, and let $x_1 < \cdots < x_n \in [0,1]$. Then,

$$\left| \int_0^1 h(x) dx - \frac{1}{n} \sum_{i=1}^n h(x_i) \right| \leq V_{[0,1]}(h) \cdot D^*_\infty(x_1, \dots, x_n)$$

where $D^*_\infty(x_1, \dots, x_n)$ is the star-discrepancy of $\{x_1, \dots, x_n\}$, that is

$$D^*_\infty(x_1, \dots, x_n) = \sup_{x\in[0,1]} \left| x - \frac{1}{n} \sum_{i=1}^n 1_{[0,x]}(x_i) \right|$$

Proof. We can write, using integration by parts

$$\begin{aligned}\int_0^1 h(x)dx - \frac{1}{n}\sum_{i=1}^n h(x_i) &= h(1) - \int_0^1 xh'(x)dx - \frac{1}{n}\sum_{i=1}^n (h(1) - \int_{x_i}^1 h'(x)dx) \\ &= -\int_0^1 xh'(x)dx + \frac{1}{n}\sum_{i=1}^n \int_0^1 1_{[x_i,1]}(x)h'(x)dx \\ &= -\int_0^1 \left(x - \frac{1}{n}\sum_{i=1}^n 1_{[x_i,1]}(x)\right) h'(x)dx\end{aligned}$$

hence,

$$\left|\int_0^1 h(x)dx - \frac{1}{n}\sum_{i=1}^n h(x_i)\right| \le \sup_{x\in[0,1]} \left|x - \frac{1}{n}\sum_{i=1}^n 1_{[0,x]}(x_i)\right| \cdot \int_0^1 |h'(x)|dx$$

which is the desired inequality. □

Now, for every integer $r > 1$, define the map

$$S_r : [0,1] \ni x \mapsto r\,x \mod 1 \in [0,1].$$

A map T is called ϕ-conjugated to S_r if there exists an interval $[a,b] \subset \mathbb{R}$ $(a < b)$ and a diffeomorphism ϕ from $[a,b]$ to $[0,1]$ such that $S_r \circ \phi = \phi \circ T$. We will say that T and S_r are positively ϕ-conjugated if ϕ is increasing. Conjugacy is a fundamental notion of dynamical systems theory since it can be used to reduce the study of computationally complex systems to the study of simpler ones, with essentially equivalent dynamical properties. To see this, assume that T is ϕ-conjugated to S_r, and let (y_n) be a trajectory of S_r, that is $y_n = S_r(y_{n-1})$. Then $x_n = \phi^{-1}(y_n)$ satisfies $x_n = T(x_{n-1})$. Thus, (x_n) is a trajectory of T and each of the sequences (x_n) and (y_n) can be recovered from the other.

Now, let us consider a map $T : [a,b] \to [a,b]$ which is positively ϕ-conjugated to some S_r, and let $(X_n)_{n\in\mathbb{N}}$ be the cryptodeterministic process defined by the recurrence equation $X_{n+1} = T(X_n)$ for every $n \ge 0$, and the initial state $X_0 \sim \mu$, where μ is a probability distribution on the borelian σ-algebra of $[a,b]$, which is assumed to be absolutely continuous with respect to the Lebesgue measure on $[a,b]$. Let us denote by f a version of its probability density function, defined everywhere on $[a,b]$. Additionally, let us define $(U_n)_{n\in\mathbb{N}}$ as the cryptodeterministic process generated from the recurrence equation $U_{n+1} = S_r(U_n)$ for every $n \ge 0$, and the initial state $U_0 = \phi(X_0)$. Then, for every $n \ge 0$,

$$X_n = \phi^{-1} \circ S_r^n \circ \phi(X_0) = \phi^{-1}(U_n).$$

Moreover, $U_0 \sim \mu\phi^{-1}$. This is a probability measure which is absolutely continuous with respect to the Lebesgue measure on $[0,1]$, and a version of its probability density function is given by $f \circ \phi^{-1}(u)/\phi' \circ \phi^{-1}(u)$.

Theorem 2.1 *If there is $w \in C^1([0,1])$ such that*

$$w(u) = \frac{f \circ \phi^{-1}(u)}{\phi' \circ \phi^{-1}(u)} \quad (\lambda - a.e)$$

where λ denotes the Lebesgue measure on $[0,1]$, then

$$\begin{aligned}|cov(X_0, X_j)| \le\ & r^{-j} \cdot V_{[0,1]}(w\phi^{-1}) \cdot \int_0^1 |\phi^{-1}(v)| \max(v, 1-v)dv \\ &+ r^{-j} \cdot V_{[0,1]}(w) \cdot \left|\int_0^1 \phi^{-1}(v)w(v)dv\right| \cdot \int_0^1 |\phi^{-1}(v)| \max(v, 1-v)dv.\end{aligned}$$

Proof. First, let us note that for every $j \geq 0$,

$$\begin{aligned} \mathbb{E}[X_0 X_j] &= \mathbb{E}[\phi^{-1}(U_0)\phi^{-1}(U_j)] = \int_0^1 \phi^{-1}(u)\phi^{-1}(S_r^j u)w(u)du \\ &= \sum_{i=0}^{r^j-1} \int_{\beta_i} \phi^{-1}(u)\phi^{-1}(r^j u - i)w(u)du, \end{aligned}$$

where $\beta_i = [ir^{-j}, (i+1)r^{-j})$. Accordingly,

$$\begin{aligned} cov(X_0, X_j) &= \mathbb{E}[X_0 X_j] - \mathbb{E}[X_0]\mathbb{E}[X_j] = cov(\phi^{-1}U_0, \phi^{-1}U_j) \\ &= \sum_{i=0}^{r^j-1} \int_{\beta_i} \phi^{-1}(u)\phi^{-1}(r^j u - i)w(u)du \\ &\quad - \left(\int_0^1 \phi^{-1}(u)w(u)du\right)\left(\sum_{i=0}^{r^j-1} \int_{\beta_i} \phi^{-1}(r^j u - i)w(u)du\right) \\ &= \sum_{i=0}^{r^j-1} \int_{\beta_i} \phi^{-1}(r^j u - i)w(u)\left(\phi^{-1}(u) - \int_0^1 \phi^{-1}(t)w(t)dt\right)du \\ &= \sum_{i=0}^{r^j-1} \int_0^1 \phi^{-1}(v)w(\frac{v+i}{r^j})(\phi^{-1}(\frac{v+i}{r^j}) - \int_0^1 \phi^{-1}(t)w(t)dt)r^{-j}dv \end{aligned}$$

where $v = r^j u - i$. Now, set $W(\phi^{-1}) = \int_0^1 \phi^{-1}(t)w(t)dt$. Then,

$$\begin{aligned} cov(X_0, X_j) &= \sum_{i=0}^{r^j-1} \int_0^1 \phi^{-1}(v)w(\frac{v+i}{r^j})\left(\phi^{-1}(\frac{v+i}{r^j}) - W(\phi^{-1})\right)r^{-j}dv \\ &= r^{-j}\int_0^1 \phi^{-1}(v)\sum_{i=0}^{r^j-1}\left\{w(\frac{v+i}{r^j})\phi^{-1}(\frac{v+i}{r^j}) - w(\frac{v+i}{r^j})W(\phi^{-1})\right\}dv \\ &= \int_0^1 \phi^{-1}(v)\left(\frac{1}{r^j}\sum_{i=0}^{r^j-1} w(\frac{v+i}{r^j})\phi^{-1}(\frac{v+i}{r^j}) - \int_0^1 \phi^{-1}(t)w(t)dt\right)dv \\ &\quad - W(\phi^{-1})\int_0^1 \phi^{-1}(v)\left(\frac{1}{r^j}\sum_{i=0}^{r^j-1} w(\frac{v+i}{r^j}) - \int_0^1 w(t)dt\right)dv \end{aligned}$$

since, by definition, $\int_0^1 w(t)dt = 1$. Hence,

$$\begin{aligned} |cov(X_0, X_j)| &\leq \int_0^1 |\phi^{-1}(v)| \cdot \left|\frac{1}{r^j}\sum_{i=0}^{r^j-1} w(\frac{v+i}{r^j})\phi^{-1}(\frac{v+i}{r^j}) - \int_0^1 \phi^{-1}(t)w(t)dt\right|dv \\ &\quad + |W(\phi^{-1})| \cdot \int_0^1 |\phi^{-1}(v)| \cdot \left|\frac{1}{r^j}\sum_{i=0}^{r^j-1} w(\frac{v+i}{r^j}) - \int_0^1 w(t)dt\right|dv \end{aligned}$$

But, according to proposition 2.1, we have, by setting $v_i = \frac{v+i}{r^j}$ for $i = 0, \ldots, r^j - 1$,

$$\left|\frac{1}{r^j}\sum_{i=0}^{r^j-1} w(\frac{v+i}{r^j})\phi^{-1}(\frac{v+i}{r^j}) - \int_0^1 \phi^{-1}(t)w(t)dt\right| \leq V_{[0,1]}(w\phi^{-1}) \cdot D_\infty^*(v_0, \ldots, v_{r^j-1})$$

and

$$\left|\frac{1}{r^j}\sum_{i=0}^{r^j-1} w(\frac{v+i}{r^j}) - \int_0^1 w(t)dt\right| \leq V_{[0,1]}(w) \cdot D_\infty^*(v_0, \ldots, v_{r^j-1})$$

where the discrepancy $D^*_\infty(v_0,\ldots,v_{r^j-1})$ is given by

$$\begin{aligned} D^*_\infty(v_0,\ldots,v_{r^j-1}) &= \sup_{x\in[0,1]}\left|x-\frac{1}{r^j}\sum_{i=0}^{r^j-1}1_{[0,x]}(v_i)\right| \\ &= \sup_{x\in[0,1]}\left|x-\frac{1}{r^j}\sum_{i=0}^{r^j-1}1_{[0,x]}(\frac{v+i}{r^j})\right|. \end{aligned}$$

But

$$\frac{v+i_0-1}{r^j}\leq x<\frac{v+i_0}{r^j}$$

if and only if

$$xr^j-v<i_0\leq xr^j-v+1$$

hence

$$\frac{v-1}{r^j}\leq x-\frac{1}{r^j}\sum_{i=0}^{r^j-1}1_{[0,x]}(\frac{v+i}{r^j})=x-\frac{i_0}{r^j}<\frac{v}{r^j}$$

and

$$\left|x-\frac{1}{r^j}\sum_{i=0}^{r^j-1}1_{[0,x]}(\frac{v+i}{r^j})\right|\leq\frac{\max(v,1-v)}{r^j}.$$

Thus,

$$D^*_\infty(v_0,\ldots,v_{r^j-1})\leq\frac{\max(v,1-v)}{r^j}$$

and

$$\begin{aligned} |cov(X_0,X_j)| \leq\ & r^{-j}\cdot V_{[0,1]}(w\phi^{-1})\cdot\int_0^1|\phi^{-1}(v)|\cdot\max(v,1-v)dv \\ &+r^{-j}\cdot V_{[0,1]}(w)\cdot\left|\int_0^1\phi^{-1}(v)w(v)dv\right|\cdot\int_0^1|\phi^{-1}(v)|\cdot\max(v,1-v)dv \end{aligned}$$

which completes the proof. □

The previous inequality holds for example if the following properties are satisfied

(i) f is continuously differentiable on $[a,b]$

(ii) ϕ is twice continuously differentiable on $[a,b]$

(iii) $\inf_{[a,b]}\phi'>0$.

Then, we can take

$$w(u)=\frac{f\circ\phi^{-1}(u)}{\phi'\circ\phi^{-1}(u)}.$$

The quantities $V_{[0,1]}(w)$, $V_{[0,1]}(w\phi^{-1})$ and the other integrals appearing in the previous inequality can be approximated numerically, or computed analytically in some cases.

3 The strictly stationary case

In the stationary case, $\mu T^{-1}\equiv\mu$, thus $\mu\phi^{-1}S_r^{-1}\equiv\mu\phi^{-1}$. Since the uniform distribution on $[0,1]$ is the unique absolutely continuous invariant probability distribution for every r-adic map, we have necessarily

$$\mu\phi^{-1}\equiv\mathcal{U}([0,1])$$

and in this case, the probability distribution function of μ is necessarily ϕ; hence, its probability density function is ϕ'.

Thus, $w\equiv 1$ and $V_{[0,1]}(w)=0$. Then, one can deduce the following result.

Theorem 3.1 *Under the previous assumptions and notations, if (X_n) is strictly stationary, then*

$$|cov(X_0, X_j)| \leq r^{-j} \cdot (b-a) \cdot \int_0^1 |\phi^{-1}(v)| \cdot \max(v, 1-v) dv.$$

Proof. The result is a straightforward consequence of theorem 2.1. Note that since ϕ is increasing, it is also the case of ϕ^{-1}, hence $V(\phi^{-1}) = b - a$. □

In the case of the cryptodeterministic process obtained by iterating the r-adic map from $X_0 \sim \mathcal{U}([0,1])$, we have seen that

$$\gamma_0(j) = \frac{r^{-j}}{12}, \qquad j \geq 1.$$

Let us compare this result to the bound obtained in our inequality.

We have here $\phi^{-1} = Id_{[0,1]}$, hence

$$\begin{aligned} |cov(X_0, X_j)| &\leq r^{-j} \cdot (1-0) \cdot \int_0^1 v \cdot \max(v, 1-v) dv \\ &\leq r^{-j} \cdot (\int_0^{1/2} v(1-v)dv + \int_{1/2}^1 v^2 dv) = \frac{3\, r^{-j}}{8} \end{aligned}$$

Hence, our bound is $36/8 = 4.5$ times the true value of the covariance and, thus, has the correct order of magnitude.

If we denote by $B_j(w)$ the bound obtained in the general case, we can denote by $B_j(1)$ the bound obtained in the strictly stationary case, and we see that

$$\frac{B_j(w)}{B_j(1)} = \frac{V_{[0,1]}(w\phi^{-1}) + V_{[0,1]}(w) \cdot \left| \int_0^1 \phi^{-1}(v) w(v) dv \right|}{b-a}.$$

Thus, the ratio between the two bound depends only on the conjugacy map and the probability density function of the initial condition. Both bounds can be computed analytically in some cases, and approximated numerically if a direct computation is not feasible.

In the special case of the r-adic maps, the inequalities above show that if the stationary case is taken as reference, then the rate of convergence of $\gamma_0(j)$ to 0 cannot be decreased by a random perturbation of the initial state which has a smooth probability density function, which is an interesting stability result.

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