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A Transient Formulation of Newton's Cooling Law for Spherical Bodies

Newton's law of cooling is shown to underestimate the heat flux between a spherical body (droplet) and a homogeneous gas after this body is suddenly immersed into the gas. This problem is rectified by replacing the gas thermal conductivity by the effective thermal conductivity. The latter reduces to the gas thermal conductivity in the limit of $t \to \infty$, but can be substantially higher in the limit of $t \to 0$. In the case of fuel droplet heating in a medium duty truck Diesel engine the gas thermal conductivity may need to be increased by more than 100 percent at the initial stage of calculations to account for transient effects during the process of droplet heating. [DOI: 10.1115/1.1337650]

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1 Introduction

Newton's law of cooling gives the expression for convective heat flux leaving a solid surface in the form ([1]):

$$\dot{q} = h(T_s - T_{g0}), \tag{1}$$

where h is the convective heat transfer coefficient, T_s is the temperature of the surface of the body, T_{g0} is gas temperature at sufficiently large distances from the body. Although the expression (1) is traditionally called the cooling law, it can be equally applied to the problem of heating of the body.

The value of h can be estimated as ([1]): $h = \mathrm{Nu}k/L$, where Nu is the Nusselt number, k is the gas thermal conductivity, L is the characteristic size. This expression can be considerably simplified in the case of spherical bodies (e.g., droplets, particles), assuming that both Reynolds and Prandtl numbers are small. In this case we can assume that Nu=2 and simplify the expression for h to

$$h = k/r_s, \tag{2}$$

where $r_s = L/2$ is the radius of the sphere.

Strictly speaking, this form of the expression for h is valid in stationary cases only. However, it has been widely used for modeling not only stationary but also transient processes, including the combustion of fuel droplets (e.g., [2–6]). This assumption allows the analysis to be considerably simplified, but its applicability has never been rigorously justified to the best of our knowledge.

The objective of this paper is to investigate the transient heat transfer between a spherical body and surrounding gas in order to clarify the range of applicability of Eqs. (1) and (2). The aim of this paper is to generalize these equations so that they can be applied to both stationary and transient processes. The effects of evaporation and combustion will be ignored. This enables the results to be applied to a heat-up period of fuel droplets—a problem complementary to the unsteady effects in droplet evaporation and combustion discussed by Crespo and Liñan [7].

Basic equations will be derived and discussed in Section 2. In Section 3 our results will be applied to the modeling of a typical problem of heating fuel droplets in a Diesel engine. The main conclusions of the paper are summarized in Section 4.

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2 Basic Equations

Let us assume that a sphere of radius r_s and temperature T_s is immersed into an infinite homogeneous gas at temperature T_{g0}. The temperature of sphere remains constant, but the changes in gas temperature are described by the heat conduction equation in the form ([8]):

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right), \tag{3}$$

where $\kappa = k/(c_{pg}\rho_g)$, c_{pg} is the gas specific heat capacity, ρ_g is the gas density, r is the distance from the center of the sphere, t is time, $T(r_s) = T_s$.

The assumption $T_s = \mathrm{const}$ made to simplify the mathematical formulation of the problem has a number important physical implications. For example, in the case of fuel droplets, their heating by heat transfer from the hot gas can be compensated by their cooling through evaporation (cf. [9,10]).

Equation (3) can be reformulated in terms of new variables

$$V = rT - r_s T_s \quad R = r - r_s$$

and written as

$$\frac{\partial V}{\partial t} = \kappa \frac{\partial^2 V}{\partial R^2}. \tag{4}$$

The boundary and initial conditions for this equation are presented as

$$V(0,t) = 0 \quad V(R,0) = f(R), \tag{5}$$

where

$$f(R) = \begin{cases} RT_{g0} + r_s(T_{g0} - T_s) & \text{when } R > 0 \\ 0 & \text{when } R = 0. \end{cases}$$

The solution of Eq. (4) subject to conditions (5) can be written as ([11]):

$$V(R,t) = \frac{1}{2\sqrt{\pi \kappa t}} \int_0^\infty f(\xi) \left\{ \exp\left[-\frac{(R-\xi)^2}{4\kappa t} \right] - \exp\left[-\frac{(R+\xi)^2}{4\kappa t} \right] \right\} \mathrm{d}\xi. \tag{6}$$

Having substituted the explicit expression for $f(\xi)$ into (6) and returning to variables T and r we obtain:

$$T=T_{g0}+\frac{r_s}{r}(T_s-T_{g0})\left[1-\text{erf}\left(\frac{r-r_s}{2\sqrt{\kappa t}}\right)\right], \tag{7}$$

where

$$\text{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\text{d}t.$$

In the limit $r=r_s$ Eq. (7) gives $T=T_s$. In the limit $t\to 0$, but $r\neq r_s$, this equation gives $T=T_{g0}$ in agreement with conditions (5).

Equation (7) gives the radius of the effective cooling zone around the cold body (droplet) in a hot gas as a function of time. If, for example, we assume that time is large and define the cooling zone as the one where $T=0.8T_{g0}$ then the radius of this zone (r_{cz}) can be obtained as

$$r_{cz}=5r_s(T_{g0}-T_s)/T_{g0}.$$

If $T_{g0}\approx 3T_s$ (realistic situation in Diesel engines: see [12]) then $r_{cz}\approx 10r_s$. This condition needs to be accounted for in computer modeling of heat exchange between droplets and gas in order to avoid grid dependence when coupling Lagrangian droplet tracking with the Eulerian gas phase in CFD calculations. It is usually satisfied in realistic Diesel spray calculations where the drop sizes are of the order of 10 μm and the grid sizes are of the order of 250 μm, although this effect can be important in the case of larger droplets.

Based on (7) the heat flux from the surface of the sphere ($r=r_s$) can be estimated as:

$$\dot{q}=-k\frac{\partial T}{\partial r}=\frac{k(T_s-T_{g0})}{r_s}\left(1+\frac{r_s}{\sqrt{\pi\kappa t}}\right). \tag{8}$$

Comparing Eqs. (1), (2), and (8), it can be seen that the Newton's law of cooling can be used to describe the transient process discussed above, if the gas thermal conductivity k is replaced by the ''effective'' thermal conductivity k_{eff} defined as

$$k_{\text{eff}}=k(1+\zeta), \tag{9}$$

where

$$\zeta=r_s\sqrt{\frac{c_{pg}\rho_g}{\pi kt}}. \tag{10}$$

In the limit $t\to\infty$, $k_{\text{eff}}\to k$ as expected. On the other hand, k_{eff} can be infinitely large when $t\to 0$.

The average value of k_{eff} over the period from $t=t_0$ to $t=t_0+\Delta t$ can be estimated as

$$\overline{k_{\text{eff}}}=k(1+\overline{\zeta}), \tag{11}$$

where

$$\overline{\zeta}=\frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t}\zeta \text{d}t=\frac{2}{\Delta t}r_s\sqrt{\frac{c_{pg}\rho_g}{\pi k}}(\sqrt{t_0+\Delta t}-\sqrt{t_0}). \tag{12}$$

In the case when $t_0=0$ Eq. (12) is simplified to

$$\overline{\zeta}=2r_s\sqrt{\frac{c_{pg}\rho_g}{\pi k\Delta t}}. \tag{13}$$

On the other hand when $\Delta t\ll t_0$ Eq. (12) is simplified to

$$\overline{\zeta}=r_s\sqrt{\frac{c_{pg}\rho_g}{\pi k t_0}}. \tag{14}$$

3 Discussion

Equation (9) is applied to a typical situation of gas cooling around a fuel droplet in a medium truck Diesel engine where droplets are injected at room temperature into air at $T_{g0}=880$ K and pressure 60 bar. Under these conditions $\rho_g=23.8$ kg/m^3, $k=0.061$ W/m·K, $c_{pg}=1120$ J/kg·K. For a typical timestep used in CFD calculations ($\Delta t=10^{-5}$ s) we obtain from Eq. (13) that at the start of calculations $\overline{\zeta}\approx 1.4$. This means that the thermal conductivity of gas used for the estimate of the rate of the initial droplet heating can be significantly higher (more than 100 percent) than commonly believed. This time scale is much less than a typical droplet evaporation time (10^{-3} s). Moreover, the change of droplet temperature over $\Delta t=10^{-5}$ s can be safely ignored. Hence the assumptions of our model regarding the absence of evaporation and the condition that T_s=const are satisfied in this particular case.

4 Conclusions

It is pointed out that the Newton's law of cooling can be applied to the problem of transient cooling or heating of a spherical body (droplet) in a homogeneous gas if one replaces gas thermal conductivity by the effective thermal conductivity. The latter reduces to the gas thermal conductivity in the limit of $t\to\infty$, but can be substantially larger than the gas thermal conductivity in the limit of $t\to 0$. In the case of fuel droplet heating in a medium duty truck Diesel engine gas thermal conductivity may need to be increased by more than 100 percent for the initial stages of calculations to account for transient effects during the process of heating of these droplets.

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