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A note on step-up test in orthogonal saturated designs

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Abstract

Let X_i be the ith order statistic of $\hat{\beta}_j^2$ for $1 \leqslant j \leqslant k$, where $\hat{\beta}_j$'s follow independent normal distributions with respective means β_j and common variance σ^2. In this paper, we provide a stochastic ordering of the random vector $\boldsymbol{T} = (X_2/X_1, 2X_3/(X_1+X_2), \ldots, (m-1)X_m/\sum_{j=1}^{m-1} X_j)$ as the β_j's change for any given $2 \leqslant m \leqslant k$. With this result and the assumption of effect sparsity, we construct a step-up simultaneous testing procedure that strongly controls experimentwise error rate for a sequence of null hypotheses regarding the number of negligible effects (zero β_j's) in orthogonal saturated designs.

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Keywords: Stochastic ordering; Effect sparsity; Experimentwise error rate; Non-central chi-squared distribution

1. Introduction

Assume a linear model

$$Y_i = \mu + \beta_1 x_{i1} + \cdots + \beta_k x_{ik} + \varepsilon_i \quad \text{for } i = 1, \ldots, M, \tag{1.1}$$

where $\varepsilon_i \sim$ iid $\mathrm{N}(0, \sigma^2)$. The goal is to identify how many and which of the β_i's are non-zero using least-squares estimators $\{\hat{\beta}_i\}_{i=1}^k$, where $\hat{\beta}_i$ follows $\mathrm{N}(\beta_i, a_i\sigma^2)$ for some known positive constant a_i. The design is called *orthogonal* if the statistics $\{\hat{\beta}_i\}_{i=1}^k$ are uncorrelated, and is said to be *saturated* if $M = k+1$, which leaves no degrees of freedom to estimate the error variance σ^2. Orthogonal saturated designs may occur in two-level fractional factorial designs with single replicate. Such problem may also arise in the context of outlier detection and signal diagnosing such as identification of non-zero coefficients in a wavelet transformed image. Without loss of generality, we assume $a_i = 1$ for $1 \leqslant i \leqslant k$ because otherwise statistics $\{\hat{\beta}_i/\sqrt{a_i}\}_{i=1}^k$ would be considered.

In order to find active (non-zero) β_i's, typically one has to use the assumption of effect sparsity, i.e., only a small number of the β_i's are active. We assume at least v of the β_i's equal zero for a predetermined positive integer v, which is either half or 60% of k, see Wang and Voss (2003) or Box and Meyer (1986). Therefore, those small $\hat{\beta}_i^2$'s, including

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the smallest v of $\hat{\beta}_i^2$'s, should be used to estimate σ^2. Any other $\hat{\beta}_i$, whose square is substantially larger, is likely to have a non-zero mean, and corresponds to an active effect.

There exist many statistical procedures to identify active effects for orthogonal saturated designs. Hamada and Balakrishnan (1998) provided a thorough review on the analysis methods for saturated designs. For example, Daniel (1959) considered the half-normal probability plot, which is still being used in the preliminary analysis. Lenth (1989) and Wang and Voss (2001, 2003) proposed adaptive confidence intervals that utilize the data to determine which and how many of $\hat{\beta}_i$'s should be used to estimate σ^2. Voss and Wang (2006) derived adaptive step-down tests with the experimentwise error rate controlled at a given level in the strong sense. As pointed out in Venter and Steel (1998) and Finner and Roters (1999), the step-up tests are generally more powerful than the step-down ones. Langsrud and Naes (1998) and Venter and Steel (1998) (LNVS) provided step-up tests to detect active effects without showing experimentwise error rates control in the strong sense, which was accomplished in Wu and Wang (2007) by using the same form of rejection regions but with a different set of cutoff points. However, the LNVS test is intuitively more attractive and easier to implement.

In this article, we show that the LNVS test also strongly controls experimentwise error rate for a sequence of null hypotheses regarding the number of negligible effects (zero effects). In Section 2, we describe the LNVS testing procedure and the corresponding hypotheses in terms of the number of negligible effects. Section 3 contains the main results: first establish a monotone property of the test statistics, which identifies the least favorable distributions (given in (2.8)) in the null hypotheses; then prove that the LNVS test strongly controls experimentwise error rate for the set of null hypotheses proposed in Section 2. The technical details are deferred to the Appendix.

2. Testing procedures and hypotheses

Let X_i be the ith order statistics of $\hat{\beta}_j^2$ for $1 \leqslant j \leqslant k$. Under the assumption of effect sparsity, the step-up procedure starts a comparison between the $(v+1)$th smallest order statistic X_{v+1} and the average of X_1 through X_v using a ratio. If the ratio is large, then conclude that there are $k-v$ active effects and stop; otherwise, compare X_{v+2} (go up) with the average of X_1 through X_{v+1} and so on until a significant ratio is found. To be more precise, for any two integers $v \leqslant n < m \leqslant k$, let $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = S_n/n$ and define a statistic

$$T_{n,m} = \frac{nX_m}{\sum_{i=1}^n X_i} = \frac{nX_m}{S_n} = \frac{X_m}{\bar{X}_n}. \tag{2.1}$$

And for a sequence of constants $\{c_m\}_{m=v+1}^k$, let

$$R_m = \bigcup_{i=v+1}^{m} \{T_{i-1,i} > c_i\} \tag{2.2}$$

for any $m \in [v+1, k]$. Then a step-up testing procedure can be conducted as follows:

Step 1: If $R_{v+1} (= \{T_{v,v+1} > c_v\})$ occurs, then conclude there are $k-v$ active effects and stop; otherwise go to step 2.

Step 2: If $R_{v+2} (= \{T_{v,v+1} > c_v\} \cup \{T_{v+1,v+2} > c_{v+1}\})$ occurs, then conclude there are $k-v-1$ active effects and stop; otherwise go to step 3.

⋮

Step $k-v$: If R_k occurs, then conclude there is one active effect and stop; otherwise conclude no active effect and stop.

Two sets of thresholds $\{c_m\}_{m=v+1}^k$ have been proposed, denoted by $\{c_m^{\text{LNVS}}\}_{m=v+1}^k$ and $\{c_m^{\text{WW}}\}_{m=v+1}^k$, corresponding to the LNVS and the Wu and Wang procedures, respectively. For a given $\alpha \in (0,1)$, constants c_m^{LNVS} are determined iteratively by solving equation

$$P(R_m^{\text{LNVS}}) = P\left(\bigcup_{i=v+1}^{m} \{T_{i-1,i} > c_i^{\text{LNVS}}\}\right) = P\left(\bigcup_{i=v+1}^{m} \left\{\frac{X_i}{\bar{X}_{i-1}} > c_i^{\text{LNVS}}\right\}\right) = \alpha \tag{2.3}$$

one by one, starting from $m = v+1$ and ending at k, where R_m^{LNVS} is the region in (2.2) using constants c_m^{LNVS}, and $\{X_i\}_{i=1}^m$ are the ordered statistics of m independent χ_1^2 random variables. For example, first when $m = v+1$, (2.3) reduces to $P(\frac{X_{v+1}}{\bar{X}_v} > c_{v+1}^{\mathrm{LNVS}}) = \alpha$, which involves only one unknown constant c_{v+1}^{LNVS}, so c_{v+1}^{LNVS} is solved with $v+1$ independent χ_1^2 random variables. Next, when $m = v+2$, (2.3) only involves one unknown c_{v+2}^{LNVS} (note c_{v+1}^{LNVS} is known now), and then it is solved. Repeat this until all $\{c_m^{\mathrm{LNVS}}\}_{m=v+1}^k$ are solved. Due to the complicated form of R_m^{LNVS}, c_m^{LNVS} is obtained by simulations (not numerical integration) based on m independent χ_1^2 random variables. On the contrary, $\{c_i^{\mathrm{WW}}\}_{i=v+1}^k$ are obtained by solving

$$\sum_{i=v+1}^{m} P\left(\max\left\{S_{i-1}, \left\{\frac{(j-1)X_j}{c_j^{\mathrm{WW}}} + S_{i-1} - S_{j-1}\right\}_{j=v}^{i-1}\right\} < \frac{(i-1)X_i}{c_i^{\mathrm{WW}}}\right) = \alpha, \tag{2.4}$$

iteratively ($c_v^{\mathrm{WW}} =: \infty$) in a similar way. Intuitively, constants c_m^{WW} would be larger than their correspondent c_m^{LNVS}, and then result in smaller rejection regions. However, the difference between the two sets of cutoff points is small as shown in the simulation of Wu and Wang (2007). Nevertheless, c_i^{LNVS} is also simpler to evaluate. The main contribution of this paper is to prove that the LNVS test with $\{c_m^{\mathrm{LNVS}}\}_{m=v+1}^k$ strongly control the experimentwise error rate.

Now we formulate a sequence of null hypotheses for the LNVS tests. For a fixed parameter vector $\underline{\boldsymbol{\beta}} = (\beta_1, \ldots, \beta_k)$, let

$$N = \text{the number of } \beta_i\text{' s being zero}, \tag{2.5}$$

thus the number of non-zero β_i's is equal to $k - N$. When effect sparsity is assumed, the entire parameter space is $H = \{\underline{\boldsymbol{\beta}} = (\beta_1, \ldots, \beta_k) : N \geqslant v\}$. For each integer $m \in [v+1, k]$, consider a testing problem:

$$\mathrm{H}_{0,m} : N \geqslant m \quad \text{vs.} \quad \mathrm{H}_{A,m} : N \leqslant m-1 \tag{2.6}$$

and define a parameter configuration in each $\mathrm{H}_{0,m}$,

$$\underline{\boldsymbol{\beta}}_m =: (0, \ldots, 0, +\infty, \ldots, +\infty), \tag{2.7}$$

where the first m components are zero. Then (2.3) can be rewritten as

$$P_{\underline{\boldsymbol{\beta}}_m}(R_m^{\mathrm{LNVS}}) = \alpha. \tag{2.8}$$

Let

$$\mathscr{B} = \{\mathrm{H}_{0,m} : v+1 \leqslant m \leqslant k\}, \tag{2.9}$$

which contains all null hypotheses of interest in this paper.

Because $\mathrm{H}_{0,i}$ is a subset of $\mathrm{H}_{0,j}$ for any $i > j$, if $\mathrm{H}_{0,j}$ is incorrect, so is $\mathrm{H}_{0,i}$. This implies that a reasonable testing process should be terminated as soon as a rejection occurs for some null hypothesis. Starting from $m = v+1$, we test these hypotheses one at a time as m goes up to k. If $\mathrm{H}_{0,v+1}$ is rejected, we then conclude that there are $k - v$ active effects (i.e., $H \cap \mathrm{H}_{A,v+1}$) and no longer test any other hypotheses; otherwise, test the next hypothesis $\mathrm{H}_{0,v+2}$. In general, if H_{0,m_0} is the first hypothesis being rejected for some $m_0 \leqslant k$, we stop and conclude that there are $k - m_0 + 1$ non-zero effects (i.e., $\mathrm{H}_{0,m_0-1} \cap \mathrm{H}_{A,m_0}$); otherwise, all hypotheses in $\mathscr{B}$ are accepted and we conclude no active effect. Clearly, this is a step-up testing procedure and provides the same result as the step-up tests $\{R_m\}_{m=v+1}^k$ described earlier.

Notice that $\mathrm{H}_{0,m}$ is decreasing and R_m^{LNVS} is increasing in m. To establish that the procedure $\{R_m^{\mathrm{LNVS}}\}_{m=v+1}^k$ controls the experimentwise error rate in the strong sense for all hypotheses in $\mathscr{B}$, following Marcus et al. (1976), we only need to show that R_m^{LNVS} is a level-α rejection region for the individual hypothesis $\mathrm{H}_{0,m}$. This will be done in the next section, and we will also show that its least favorable distribution for each $\mathrm{H}_{0,m}$ is at $\underline{\boldsymbol{\beta}}_m$.

Remark. Hypotheses in $\mathscr{B}$ deal with N, the number of zero β_i's. It would be a more interesting problem to show strong control of experimentwise error rate for the step-up tests to identify all active β_i's. This is very challenging and deserves further study. Nevertheless, if one makes a type I error regarding N, then an error is made for identifying the set of active effects. Therefore, a strong control of experimentwise error rate on N is a necessary condition for that on the set of active effects.

3. Main results

3.1. A monotone property

In this section, we establish a stochastic ordering of test statistics $T_{i,i+1}$ as the β's change. The proofs of our results are mainly based on probability inequalities on coordinatewise ordered spaces.

Definition 1. We define a partial ordering of two vectors $\boldsymbol{x}(=(x_1, \ldots, x_d))$ and $\boldsymbol{y}$ in R^d, denoted by $\boldsymbol{x} \leqslant \boldsymbol{y}$ if $x_i \leqslant y_i$, $i = 1, \ldots, d$. A function $h : R^d \to R^1$ is non-decreasing to this coordinatewise ordering if $\boldsymbol{x} \leqslant \boldsymbol{y}$ implies $h(\boldsymbol{x}) \leqslant h(\boldsymbol{y})$.

Definition 2. We say that random vector $\boldsymbol{X}$ with a pdf $p(\boldsymbol{x})$ is stochastically smaller than $\boldsymbol{Y}$ with a pdf $q(\boldsymbol{y})$ in the coordinatewise order, denoted by $\boldsymbol{X} \prec \boldsymbol{Y}$ or $p(\boldsymbol{x}) \prec q(\boldsymbol{y})$, if $E[h(\boldsymbol{X})] \leqslant E[h(\boldsymbol{Y})]$ for any non-negative bounded non-decreasing function h.

Our first main result is given below.

Theorem 1. *For any $m \leqslant k$, random vector $\boldsymbol{T} = (T_{1,2}, T_{2,3}, \ldots, T_{m-1,m})$ is stochastically largest at $\underline{\boldsymbol{\beta}}_m$ for all $\underline{\boldsymbol{\beta}} \in \mathrm{H}_{0,m}$.*

Note that the above result is a stronger result than Theorem 1 in Wu and Wang (2007), which only establishes the stochastic ordering of each coordinate of $\boldsymbol{T}$. Also Theorem 1 implies:

Theorem 2. *For all $\underline{\boldsymbol{\beta}} \in \mathrm{H}_{0,m}$ and for any constants $\{c_i\}_{i=v+1}^{m}$, we have*

$$P_{\underline{\boldsymbol{\beta}}}(R_m) \leqslant P_{\underline{\boldsymbol{\beta}}_m}(R_m) \quad \text{with equality only if } \underline{\boldsymbol{\beta}} = \underline{\boldsymbol{\beta}}_m. \tag{3.1}$$

Theorem 2 warrants the existence of finite solutions $\{c_i^{\mathrm{LNVS}}\}_{i=v+1}^{k}$ for equations in (2.3). Specifically, the sequence of critical values can be determined iteratively (c_i depends on c_j for $j < i$) as follows:

(1) For testing $\mathrm{H}_{0,v+1}$, since $P_{\underline{\boldsymbol{\beta}}_{v+1}}\{T_{v,v+1} > c\}$ decreases from 1 to 0 as c goes from zero to infinity, there exist a unique c_{v+1} such that $P_{\underline{\boldsymbol{\beta}}_{v+1}}(R_{v+1}^{\mathrm{LNVS}}) = \alpha$, which guarantees the type I error to be α due to Theorem 2.
(2) For any $v+2 \leqslant i \leqslant k$, suppose finite $\{c_j\}_{j=v+1}^{i-1}$ are available such that $P_{\underline{\boldsymbol{\beta}}_j}(R_j^{\mathrm{LNVS}}) = \alpha$, $v+1 \leqslant j \leqslant i-1$. Since $\underline{\boldsymbol{\beta}}_i \in \mathrm{H}_{0,i-1}$, Theorem 2 implies $P_{\underline{\boldsymbol{\beta}}_i}(R_{i-1}^{\mathrm{LNVS}}) < P_{\underline{\boldsymbol{\beta}}_{i-1}}(R_{i-1}^{\mathrm{LNVS}}) = \alpha$. Hence $P_{\underline{\boldsymbol{\beta}}_i}(T_{i-1,i} > c_i^{\mathrm{LNVS}}) > 0$, which warrants c_i^{LNVS} to be finite and implies that rejection region R_i^{LNVS} is larger than R_{i-1}^{LNVS}.

3.2. Strong control of experimentwise error rate

To conduct the simultaneous tests for $\mathscr{B} = \{\mathrm{H}_{0,m} : v+1 \leqslant m \leqslant k\}$, assert not $\mathrm{H}_{0,m}$ (i.e., assert $\mathrm{H}_{A,m}$),

$$\text{if } R_m^{\mathrm{LNVS}} \text{ is true.} \tag{3.2}$$

Notice two facts: (i) $\mathscr{B}$ is closed under the operation of intersection and (ii) for each $v+1 \leqslant m \leqslant k$, $R_m^{\mathrm{LNVS}} = \bigcap_{i=m}^{k} R_i^{\mathrm{LNVS}}$ is a level-α test for $\mathrm{H}_{0,m}$. Therefore, the experimentwise error rate is no greater than α by the closed test procedure proposed by Marcus et al. (1976). In summary, we have the second main result below.

Theorem 3. *The rejection region R_m^{LNVS} increases when m gets larger, and each defines a level-α test for $\mathrm{H}_{0,m}$. If one conducts simultaneous tests for $\mathscr{B}$ using* (3.2), *then the experimentwise error rate is controlled at α in the strong sense.*

4. Discussion

In this paper, a stochastic ordering is established for the ratios of order statistics of non-central chi-squared distributed random variables. With this result and the assumption of effect sparsity, we construct a step-up simultaneous testing procedure such that its least favorable distribution is located at $\underline{\boldsymbol{\beta}}_m$ and it strongly controls experimentwise error rate for testing a sequence of null hypotheses regarding the number of zero effects in orthogonal saturated designs.

All theorems proved in this paper hold for any value of v. Therefore, there is no need for "assumption of effect sparsity" regarding the result on the experimentwise error rate. However, the choice of v does affect the cutoff points.

The step-up testing procedure considered in this paper is called step-up test with sequential scaling by Venter and Steel (1998). However, we are unable to prove similar results for the step-up procedure with fixed scaling, which is based on $R_m^{\dagger} = \bigcup_{i=v+1}^{m}\{T_{v,i} > c_i^{\dagger}\}$. Even though the test seems to be simpler because it always uses $\bar{X}_v$ to estimate σ^2, but we cannot establish the stochastic ordering for the random vector $(T_{v,v+1}, T_{v,v+2}, \ldots, T_{v,m})$. Fortunately, the step-up tests with sequential scaling are usually more powerful than the ones with fixed scaling. Other variants of the step-up tests, including "Q vector", were proposed in Langsrud and Naes (1998). Again, whether these procedures control the experimentwise error rate is open.

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Appendix A. Proof of Theorem 1

The proof is based on probability inequalities on coordinatewise ordered spaces. In particular, we rely on the following property, which is a special case of Proposition 1 in Kamae et al. (1977).

Theorem 4. *Let $p(\boldsymbol{x}, \boldsymbol{y})$ and $q(\boldsymbol{x}, \boldsymbol{y})$ be the two densities with marginals $p_1(\boldsymbol{x})$ and $q_1(\boldsymbol{x})$, and conditional probability distributions $p(\boldsymbol{y}|\boldsymbol{x})$ and $q(\boldsymbol{y}|\boldsymbol{x})$, respectively. Suppose $p_1(\boldsymbol{x}) \prec q_1(\boldsymbol{x})$ and $p(\cdot|\boldsymbol{x}) \prec p(\cdot|\boldsymbol{x}') \prec q(\cdot|\boldsymbol{x}')$ for $\boldsymbol{x} \leqslant \boldsymbol{x}'$, then $p(\boldsymbol{x}, \boldsymbol{y}) \prec q(\boldsymbol{x}, \boldsymbol{y})$.*

W.l.o.g., assume $\sigma = 1$ because the distribution of $\boldsymbol{T}$ depends on β_i/σ. Let $\phi(x)$ and $\Phi(x)$ be the pdf and cdf of standard normal distribution, then $\hat{\beta}_i^2$ has cdf $G_{\beta_i}(x) = \Phi(\sqrt{x}+\beta_i) + \Phi(\sqrt{x}-\beta_i) - 1$ and pdf $g_{\beta_i}(x) = \frac{1}{2}x^{-1/2}[\phi(\sqrt{x}+\beta_i) + \phi(\sqrt{x}-\beta_i)]$. Also to simply the notation, let $f(x) = g_0(x) = x^{-1/2}\mathrm{e}^{-x/2}/\sqrt{2\pi}$ and $F(x) = G_0(x)$. Recall that X_i is the ith order statistics of $\hat{\beta}_j^2$, $1 \leqslant j \leqslant k$. If we define $W_i = \frac{\sum_{j=1}^{i} X_j}{X_{i+1}} = i/T_{i,i+1}$, then Theorem 1 is equivalent to show that $\boldsymbol{W} = (W_1, \ldots, W_{m-1})$ is stochastically smallest at $\boldsymbol{\beta}_m$ among $\mathrm{H}_{0,m}$.

First, we derive the absolute Jacobian for transformation from $(X_1, \ldots, X_n, X_{n+1}, \ldots, X_m)$ to $(W_1, \ldots, W_n, Z_{n+1}, \ldots, Z_m)$, where $Z_i = 1/X_i$.

Lemma 1. $J_n(\boldsymbol{w}, \boldsymbol{z}) = |\det \frac{\partial(x_1,\ldots,x_n,x_{n+1},\ldots,x_m)}{\partial(w_1,\ldots,w_n,z_{n+1},\ldots,z_m)}| = \prod_{i=1}^{n-1}(\frac{1}{w_i+1}\prod_{j=i+1}^{n}\frac{w_j}{w_{j-1}+1})\frac{1}{z_{n+1}^{n+2}}\prod_{i=n+2}^{m}\frac{1}{z_i^2}$.

Proof. Let $u_i = \frac{x_i}{x_{i+1}}$, in other words $x_i = (\prod_{j=i}^{n} u_j)/z_{n+1}$. Hence $\frac{\partial(x_1,\ldots,x_n,x_{n+1},\ldots,x_m)}{\partial(u_1,\ldots,u_n,z_{n+1},\ldots,z_m)}$ has diagonal elements $\frac{\mathrm{d}x_i}{\mathrm{d}u_i} = (\prod_{j=i+1}^{n} u_j)/z_{n+1},\ i \leqslant n-1$; $\frac{\mathrm{d}x_n}{\mathrm{d}u_n} = 1/z_{n+1}$; $\frac{\mathrm{d}x_i}{\mathrm{d}z_i} = -1/z_i^2,\ i \geqslant n+1$; and all of its lower triangular elements are equal to zero. Therefore,

$$\left|\det \frac{\partial(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m)}{\partial(u_1, \ldots, u_n, z_{n+1}, \ldots, z_m)}\right| = \prod_{i=1}^{n-1}\left(\prod_{j=i+1}^{n} u_j\right)\frac{1}{z_{n+1}^{n+2}}\prod_{i=n+2}^{m}\frac{1}{z_i^2}.$$

Similarly, since $u_i = \frac{w_i}{w_{i-1}+1}$, all upper triangular elements of $\frac{\partial(u_1,\ldots,u_n,z_{n+1},\ldots,z_m)}{\partial(w_1,\ldots,w_n,z_{n+1},\ldots,z_m)}$ are equal to zero. Consequently,

$$\left|\det \frac{\partial(u_1, \ldots, u_n, z_{n+1}, \ldots, z_m)}{\partial(w_1, \ldots, w_n, z_{n+1}, \ldots, z_m)}\right| = \prod_{i=1}^{n-1}\frac{1}{w_i+1}.$$

The proof completes because $J_n(\boldsymbol{w}, \boldsymbol{z}) = |\frac{\partial(x_1,\ldots,x_n,x_{n+1},\ldots,x_m)}{\partial(u_1,\ldots,u_n,z_{n+1},\ldots,z_m)}||\frac{\partial(u_1,\ldots,u_n,z_{n+1},\ldots,z_m)}{\partial(w_1,\ldots,w_n,z_{n+1},\ldots,z_m)}|$. $\square$

Now, we are ready to prove Theorem 1, which is a special case of the following:

Lemma 2.

$$(W_1, \dots, W_n, Z_{n+1}, \dots, Z_m)|\underline{\boldsymbol{\beta}}_m \prec (W_1, \dots, W_n, Z_{n+1}, \dots, Z_m)|\underline{\boldsymbol{\beta}}, \tag{A.1}$$

for all $1 \leqslant n \leqslant m-1$ *and* $\underline{\boldsymbol{\beta}} \in \mathrm{H}_{0,m}$.

Proof. We prove Lemma 2 by an induction on n.

First we derive the densities of the two distributions at $\underline{\boldsymbol{\beta}}_m$ and $\underline{\boldsymbol{\beta}}$. Under $\underline{\boldsymbol{\beta}}_m$, random vector $(X_1, \dots, X_m)$ has pdf $m!\prod_{i=1}^m f(x_i)$ for $0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_m < \infty$. Apply transformation

$$w_i = \frac{\sum_{j=1}^i x_j}{x_{i+1}}, \quad i \leqslant n, \quad z_i = 1/x_i, \quad i \geqslant n+1,$$

we have

$$x_i = \left[\prod_{j=i}^n \frac{w_j}{w_{j-1}+1}\right] \frac{1}{z_{n+1}}, \quad i \leqslant n, \quad x_i = 1/z_i, \quad i \geqslant n+1.$$

Thus $(W_1, \dots, W_n, Z_{n+1}, \dots, Z_m)$ has a pdf

$$p(\mathbf{w}, \mathbf{z}) = m! \prod_{i=1}^n f\left(\frac{1}{z_{n+1}} \prod_{j=i}^n \frac{w_j}{w_{j-1}+1}\right) \prod_{i=n+1}^m f(1/z_i) J_n(\mathbf{w}, \mathbf{z}), \tag{A.2}$$

over the domain $D = \{(\mathbf{w}, \mathbf{z}) : 0 < w_i \leqslant w_{i-1}+1, i \leqslant n; \infty > z_{n+1} \geqslant \cdots \geqslant z_m > 0\}$ with $w_0 = 0$. On the other hand, under parameter configuration $\underline{\boldsymbol{\beta}}$, random vector $(X_1, \dots, X_m)$ has pdf

$$\sum_\tau \left[\prod_{i=1}^m g_{t_i}(x_i)\right] \left[\prod_{i=m+1}^k (1 - G_{t_i}(x_m))\right], \quad 0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_m < \infty, \tag{A.3}$$

where summation is over all permutations of $\{\beta_1, \dots, \beta_k\}$ $(\tau = (t_1, t_2, \dots, t_k))$. Hence $(W_1, \dots, W_n, Z_{n+1}, \dots, Z_m)|\underline{\boldsymbol{\beta}}$ has pdf

$$q(\mathbf{w}, \mathbf{z}) = \sum_\tau \left[\prod_{i=1}^n g_{t_i}\left(\frac{1}{z_{n+1}} \prod_{j=i}^n \frac{w_j}{w_{j-1}+1}\right) \prod_{i=n+1}^m g_{t_i}(1/z_i) \prod_{i=m+1}^k (1 - G_{t_i}(1/z_m))\right] J_n(\mathbf{w}, \mathbf{z}), \tag{A.4}$$

over region D.

Next we prove the stochastic ordering (A.1) for $n = 1$.

Since $(z_1, \dots, z_m)|\underline{\boldsymbol{\beta}}_m \prec (z_1, \dots, z_m)|\underline{\boldsymbol{\beta}}$ due to the Stochastic Ordering Lemma, see, for example, Lemma 2 in Wu and Wang (2007), it is obvious that the marginal distributions satisfy $(z_2, \dots, z_m)|\underline{\boldsymbol{\beta}}_m \prec (z_2, \dots, z_m)|\underline{\boldsymbol{\beta}}$. And in this case, we have

$$p(\mathbf{w}, \mathbf{z}) = m! f\left(\frac{w_1}{z_2}\right) \prod_{i=2}^m f(1/z_i) J_1(\mathbf{w}, \mathbf{z}),$$

$$q(\mathbf{w}, \mathbf{z}) = \sum_\tau \left[g_{t_1}\left(\frac{w_1}{z_2}\right) \prod_{i=2}^m g_{t_i}(1/z_i) \prod_{i=m+1}^k (1 - G_{t_i}(1/z_m))\right] J_1(\mathbf{w}, \mathbf{z}). \tag{A.5}$$

This, along with the facts that (1) $J_1(\mathbf{w}, \mathbf{z}')/J_1(\mathbf{w}, \mathbf{z})$ does not depend on $\mathbf{w}$; (2) $f(\frac{w}{z'})/f(\frac{w}{z})$ is non-decreasing in w when $z \leqslant z'$; and (3) $g_{t_1}(\frac{w}{z'})/f(\frac{w}{z'})$ is non-decreasing in w, we conclude that both $\frac{p(w_1|\mathbf{z}')}{p(w_1|\mathbf{z})}$ and $\frac{q(w_1|\mathbf{z}')}{p(w_1|\mathbf{z}')}$ are non-decreasing functions of w_1 when $z \leqslant z'$. Therefore, we have $p(w_1|z) \prec p(w_1|z') \prec q(w_1|z')$, and the inequality (A.1) follows Theorem 4 for $n = 1$.

Lastly, suppose the stochastic ordering (A.1) holds for n. Then the marginal distributions satisfy

$$p(w_1, \ldots, w_n, z_{n+2}, \ldots, z_m) \prec q(w_1, \ldots, w_n, z_{n+2}, \ldots, z_m). \tag{A.6}$$

In addition, $(W_1, \ldots, W_n, W_{n+1}, Z_{n+2}, \ldots, Z_m)$ has densities

$$p(\boldsymbol{w}, \boldsymbol{z}) = m! \prod_{i=1}^{n+1} f\left(\frac{1}{z_{n+2}} \prod_{j=i}^{n+1} \frac{w_j}{w_{j-1}+1}\right) \prod_{i=n+2}^{m} f(1/z_i) J_{n+1}(\boldsymbol{w}, \boldsymbol{z}),$$

$$q(\boldsymbol{w}, \boldsymbol{z}) = \sum_{\tau} \left[\prod_{i=1}^{n+1} g_{t_i}\left(\frac{1}{z_{n+2}} \prod_{j=i}^{n+1} \frac{w_j}{w_{j-1}+1}\right) \prod_{i=n+2}^{m} g_{t_i}(1/z_i) \prod_{i=m+1}^{k} (1 - G_{t_i}(1/z_m))\right] J_{n+1}(\boldsymbol{w}, \boldsymbol{z}), \tag{A.7}$$

under parameter configurations $\underline{\boldsymbol{\beta}}_m$ and $\underline{\boldsymbol{\beta}}$, respectively. Therefore,

$$\frac{p(w_{n+1}|w'_1, \ldots, w'_n, z'_{n+2}, \ldots, z'_m)}{p(w_{n+1}|w_1, \ldots, w_n, z_{n+2}, \ldots, z_m)} = A(w'_1, \ldots, w'_n, w_1, \ldots, w_n, \boldsymbol{z}', \boldsymbol{z}) \prod_{i=1}^{n+1} \left[\frac{f\left(\frac{w_{n+1}}{z'_{n+2}(w'_{n-1}+1)} \prod_{j=i}^{n} \frac{w'_j}{w'_j+1}\right)}{f\left(\frac{w_{n+1}}{z_{n+2}(w_{n-1}+1)} \prod_{j=i}^{n} \frac{w_j}{w_j+1}\right)}\right].$$

Note that

$$\prod_{i=1}^{n+1} f\left(\frac{1}{z_{n+2}} \prod_{j=i}^{n+1} \frac{w_j}{w_{j-1}+1}\right) = B(w_1, \ldots, w_n, z_{n+2}) w_{n+1}^{-1/2} \mathrm{e}^{-w_{n+1}/(2z_{n+2})},$$

then

$$\frac{p(w_{n+1}|w'_1, \ldots, w'_n, z'_{n+2}, \ldots, z'_m)}{p(w_{n+1}|w_1, \ldots, w_n, z_{n+2}, \ldots, z_m)}$$

is a non-decreasing function of w_{n+1} when $z_{n+2} \leqslant z'_{n+2}$. Consequently, we have

$$p(w_{n+1}|w_1, \ldots, w_n, z_{n+2}, \ldots, z_m) \prec p(w_{n+1}|w'_1, \ldots, w'_n, z'_{n+2}, \ldots, z'_m), \quad \forall z_{n+2} \leqslant z'_{n+2}. \tag{A.8}$$

Consider the ratio

$$\frac{q(w_{n+1}|w'_1, \ldots, w'_n, z'_{n+2}, \ldots, z'_m)}{p(w_{n+1}|w'_1, \ldots, w'_n, z'_{n+2}, \ldots, z'_m)} = C(w'_1, \ldots, w'_n, \boldsymbol{z}') \sum_{\tau} \prod_{i=1}^{n+1} \left[\frac{g_{t_i}\left(\frac{w_{n+1}}{z'_{n+2}(w'_{n-1}+1)} \prod_{j=i}^{n} \frac{w'_j}{w'_j+1}\right)}{f\left(\frac{w_{n+1}}{z'_{n+2}(w'_{n-1}+1)} \prod_{j=i}^{n} \frac{w'_j}{w'_j+1}\right)}\right].$$

As a function of w_{n+1}, this ratio is non-decreasing since each $g_{t_i}(x)/f(x)$ is non-decreasing. Therefore,

$$p(w_{n+1}|w'_1, \ldots, w'_n, z'_{n+2}, \ldots, z'_m) \prec q(w_{n+1}|w'_1, \ldots, w'_n, z'_{n+2}, \ldots, z'_m). \tag{A.9}$$

Combining (A.6), (A.8) and (A.9), the inequality (A.1) follows Theorem 4. □

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