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POSITIVE SOLUTION OF CRITICAL HARDY–SOBOLEV ELLIPTIC SYSTEMS WITH THE BOUNDARY SINGULARITY

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Abstract. In this paper, we are concerned with the existence of positive solutions to the system

$$\begin{cases} -\Delta u = \frac{2p}{p+q} u^{p-1} v^q + \frac{2\lambda\alpha}{\alpha+\beta} \frac{u^{\alpha-1} v^\beta}{|x|^s}, & \text{in} \quad \Omega, \\ -\Delta v = \frac{2q}{p+q} u^p v^{q-1} + \frac{2\lambda\beta}{\alpha+\beta} \frac{u^\alpha v^{\beta-1}}{|x|^s}, & \text{in} \quad \Omega, \\ u > 0, v > 0, & \text{in} \quad \Omega, \\ u = v = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a C^2 domain in $\mathbb{R}^N$ with $0 \in \partial\Omega$, $0 < s < 2$, $\lambda > 0$, $p+q = 2^* = \frac{2N}{N-2}$, $\alpha+\beta = 2^*(s) = \frac{2(N-s)}{N-2}$, $N \geqslant 3$. We show that if $\Omega = \mathbb{R}^N_+$, problem (0.1) possesses a least energy solution and if Ω is bounded, $0 \in \partial\Omega$, there exists $\lambda^* > 0$ such that problem (0.1) has at least a positive solution provided $0 < \lambda < \lambda^*$.

1. Introduction

In this paper, we are concerned with the existence of positive solutions to the system

$$\begin{cases} -\Delta u = \frac{2p}{p+q} u^{p-1} v^q + \frac{2\lambda\alpha}{\alpha+\beta} \frac{u^{\alpha-1} v^\beta}{|x|^s}, & \text{in} \quad \Omega, \\ -\Delta v = \frac{2q}{p+q} u^p v^{q-1} + \frac{2\lambda\beta}{\alpha+\beta} \frac{u^\alpha v^{\beta-1}}{|x|^s}, & \text{in} \quad \Omega, \\ u > 0, v > 0, & \text{in} \quad \Omega, \\ u = v = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a C^2 bounded domain in $\mathbb{R}^N$ with $0 \in \partial\Omega$, $0 < s < 2$, $\lambda > 0$,

$$p+q = 2^* = \frac{2N}{N-2}, \ \alpha+\beta = 2^*(s) = \frac{2(N-s)}{N-2}, \ N \geqslant 3.$$

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The exponent $2^*(s)$ is the critical exponent for the Hardy-Sobolev inequality

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^*(s)}} \leqslant C\int_{\mathbb{R}^N} |\nabla u|^2\,dx \tag{1.2}$$

for $u\in D^{1,2}(\mathbb{R}^N)$, where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\Big)^{\frac{1}{2}}.$$

Let Ω be a domain in $\mathbb{R}^N$ and denote by $H_0^1(\Omega)$ the usual Sobolev space, the best constant $\mu_s(\Omega)$ of the Hardy-Sobolev inequality is defined by

$$\mu_s(\Omega) = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^*(s)}}}. \tag{1.3}$$

If $s=0$, (1.2) is reduced to the Sobolev inequality. The best constant $\mu_s(\Omega)$ becomes the Sobolev constant

$$S = S(\Omega) = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*}\,dx\right)^{\frac{2}{2^*}}}. \tag{1.4}$$

Due to the scaling invariance, $S(\Omega)=S(\mathbb{R}^N)$, that is, $S(\Omega)$ is independent of the domain Ω. It is well known that S is achieved if and only if $\Omega=\mathbb{R}^N$, and by the function

$$U_\varepsilon(x) = \left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)^{\frac{N-2}{2}}. \tag{1.5}$$

Similarly, if $s\neq 0$ and $0\in\Omega$, we also have $\mu_s(\Omega)=\mu_s(\mathbb{R}^N)$. Thus $\mu_s(\Omega)$ is never attained unless $\Omega=\mathbb{R}^N$. However, if $s\neq 0$ and $0\in\partial\Omega$, the quantity $\mu_s(\Omega)$ may depend on the domain Ω. In fact, Ghoussoub and Robert [6, 7] proved that $\mu_s(\Omega)$ is attained if, among other things, the mean curvature $H(0)$ of $\partial\Omega$ at 0 is negative. This fact was used in [5] to study the existence of positive solutions of the critical problem

$$-\Delta u = \frac{u^{2^*(s)-1}}{|x|^s} + \lambda u^p, \quad u>0 \quad \text{in} \quad \Omega, \quad u=0, \quad \text{on} \quad \partial\Omega, \tag{1.6}$$

where $\lambda>0, 1<p<\frac{N+2}{N-2}$, and Ω is a bounded domain in $\mathbb{R}^N, 0\in\partial\Omega$. In the spirit of [2], it was shown in [5] that the associated functional of (1.6) satisfies the $(PS)_c$ condition for $c\in\left(0,\frac{2-s}{2(N-s)}\mu_s(\Omega)^{\frac{N-s}{2-s}}\right)$. Then the existence result can be obtained as [2] by the mountain pass theorem. It was discussed in [11] the existence of a similar problem with Neumann boundary condition and $0\in\partial\Omega$. In [9], the existence of positive solutions for the problem with double critical nonlinearities

$$-\Delta u = \frac{u^{2^*(s)-1}}{|x|^s} + \lambda u^{\frac{N+2}{N-2}}, \quad u>0 \quad \text{in} \quad \Omega, \quad u=0, \quad \text{on} \quad \partial\Omega, \tag{1.7}$$

was considered. As a replacement of the energy level related to the best constant, the least energy c_0 of solutions of the problem

$$-\Delta u = \frac{u^{2^*(s)-1}}{|x|^s} + \lambda u^{\frac{N+2}{N-2}}, \quad u>0 \quad \text{in} \quad \mathbb{R}^N_+, \quad u=0, \quad \text{on} \quad \partial\mathbb{R}^N_+, \tag{1.8}$$

was taken into account. It was proved that the $(PS)_c$ condition holds for the functional related to (1.7) and $c \in (0, c_0)$.

In this paper, we consider the existence of positive solutions of system (1.1) with double critical exponents and $0 \in \partial\Omega$. In [3], the existence of solutions for a critical singular system was considered in Ω with $0 \in \Omega$. In our case, the geometry at the singularity should be considered.

Suppose throughout this paper that $\partial\Omega$ is C^2 at 0, the mean curvature of $\partial\Omega$ at 0 is negative, and $0<s<1$ if $N=3$, $0<s<2$ if $N \geqslant 4$.

Let

$$S_{p,q}(\Omega) = \inf_{(u,v)\in(H_0^1(\Omega))^2\setminus\{0\}} \frac{\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx}{\left(\int_\Omega |u|^p|v|^q dx\right)^{\frac{2}{2^*}}}, \tag{1.9}$$

where $2 \leqslant p+q \leqslant 2^*$. We know from [1] that

$$S_{p,q}(\Omega) = \left[\left(\frac{p}{q}\right)^{\frac{q}{p+q}} + \left(\frac{p}{q}\right)^{\frac{-p}{p+q}}\right] S(\Omega). \tag{1.10}$$

Since S is independent of the domain Ω, so is $S_{p,q}(\Omega)$. Moreover, if w_0 is a minimizer of $S(\mathbb{R}^N)$, then $(p_1 w_0, q_1 w_0)$ is a minimizer of $S_{p,q}(\mathbb{R}^N)$ with $p_1, q_1 \in \mathbb{R}$ satisfying $\frac{p_1}{q_1} = \sqrt{\frac{p}{q}}$.

We first consider the existence of the least energy solution of the problem

$$\begin{cases} -\Delta u = \frac{2p}{p+q} u^{p-1} v^q + \frac{2\lambda\alpha}{\alpha+\beta} \frac{u^{\alpha-1}v^\beta}{|x|^s}, & \text{in} \quad \mathbb{R}^N_+, \\ -\Delta v = \frac{2q}{p+q} u^p v^{q-1} + \frac{2\lambda\beta}{\alpha+\beta} \frac{u^\alpha v^{\beta-1}}{|x|^s}, & \text{in} \quad \mathbb{R}^N_+, \\ u>0, v>0, & \text{in} \quad \mathbb{R}^N_+, \\ u=v=0, & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases} \tag{1.11}$$

where $\mathbb{R}^N_+ = \{x=(x_1,\cdots,x_{N-1},x_N) \in \mathbb{R}^N, x_N>0\}$ is the half space. The functional associated to system (1.11)

$$J(u,v) = \int_{\mathbb{R}^N_+} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2 - \frac{2}{p+q} u^p v^q - \frac{2\lambda}{\alpha+\beta} \frac{u^\alpha v^\beta}{|x|^s}\right) dx \tag{1.12}$$

is C^1 on $H_0^1(\mathbb{R}^N_+) \times H_0^1(\mathbb{R}^N_+)$. We firstly construct an approximating sequence of solutions by solving a subcritical system, then using blow up argument, we analyse the limiting behavior of the sequence. In contrast with one equation case, there are two components of the approximating sequence of solutions for the system, so we need to carefully study the limiting behavior of both components. We find that both components of approximating solutions have the same blow up rate. Eventually, we show that there exists a least energy solution of problem (1.11). By a least energy solution of problem (1.11), we mean a solution with the energy level

$$c_0 = \inf\{J(u,v) \mid (u,v) \text{ is a positive solution of } (1.11) \text{ and } J(u,v) > 0\}, \tag{1.13}$$

that is, it is a solution with the least energy among all solutions. We have the following result.

THEOREM 1.1. *For $N \geqslant 3$, $\lambda > 0$, there exists a least energy solution (u,v) of system* (1.11)*. Furthermore, the energy c_0 of the least energy solution satisfies*

$$c_0 = J(u,v) < \frac{1}{N} 2^{\frac{2-N}{2}} S_{p,q}^{\frac{N}{2}}. \tag{1.14}$$

Next, we turn to the existence of positive solutions for problem (1.1). We will show that the functional associated to problem (1.1)

$$I(u,v) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2 - \frac{2}{p+q} u^p v^q - \frac{2\lambda}{\alpha+\beta} \frac{u^\alpha v^\beta}{|x|^s} \Big) dx \tag{1.15}$$

defined on $H_0^1(\Omega) \times H_0^1(\Omega)$, satisfies the $(PS)_c$ condition for $c \in (0, c_0)$. Using the blow up argument again, we obtain

THEOREM 1.2. *Suppose that the mean curvature of $\partial\Omega$ at 0 is negative. There exists $\lambda^* > 0$ such that system (1.1) has at least a positive solution provided $0 < \lambda < \lambda^*$.*

In section 2, we find an upper bound of the mountain pass level of I and give a nonexistence result for problem (1.1). Then, we prove Theorem 1.1 in section 3 and Theorem 1.2 in section 4 respectively.

2. Some estimates and nonexistence results

In this section, we find an upper bound of the mountain pass level of I, which will be used in the proof of Theorem 1.2. We also establish a nonexistence result for problem (1.1).

LEMMA 2.1. *For $\lambda > 0$, there exist nonnegative functions u_0 and v_0 in $H_0^1(\Omega) \setminus \{0\}$, such that $I(u_0, v_0) < 0$, and*

$$\max_{t \geqslant 0} I(tu_0, tv_0) < \frac{1}{N} 2^{\frac{2-N}{2}} S_{p,q}^{\frac{N}{2}}. \tag{2.1}$$

Proof. Let U be given in (1.5) and $\frac{p_1}{q_1} = \sqrt{\frac{p}{q}}$. Then $(p_1 U, q_1 U)$ is the minimizer of $S_{p,q}(\mathbb{R}^N)$. Let x_0 be an interior point of Ω such that $B_{2r}(x_0) \subset \Omega$. Take $\varphi \in C_0^\infty(B_{3r}(x_0))$ be a cutoff function such that $\varphi|_{B_r(x_0)} \equiv 1$ and $0 \leqslant \varphi \leqslant 1$. Let

$$u_\varepsilon(x) = p_1 \varepsilon^{-\frac{N-2}{2}} \varphi(x) U\Big(\frac{x-x_0}{\varepsilon}\Big),\ v_\varepsilon(x) = q_1 \varepsilon^{-\frac{N-2}{2}} \varphi(x) U\Big(\frac{x-x_0}{\varepsilon}\Big).$$

We have $u_\varepsilon, v_\varepsilon \in H_0^1(\Omega)$.

Now, we estimate each term in $I(u_\varepsilon, v_\varepsilon)$. Note that

$$\int_{B_{2r}(x_0)} \varphi(x)U(\frac{x-x_0}{\varepsilon})\nabla\varphi(x)\nabla U(\frac{x-x_0}{\varepsilon})\,dx$$
$$= -\frac{1}{2}\int_{B_{2r}(x_0)} |\nabla\varphi(x)|^2|U(\frac{x-x_0}{\varepsilon})|^2\,dx - \frac{1}{2}\int_{B_{2r}(x_0)} \varphi(x)\Delta\varphi(x)U^2(\frac{x-x_0}{\varepsilon})\,dx,$$

we deduce that

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^2 dx &= p_1^2\int_\Omega |\nabla(\varepsilon^{-\frac{N-2}{2}}\varphi(x)U(\frac{x-x_0}{\varepsilon}))|^2 dx\\ &= p_1^2\varepsilon^{-N}\int_{B_{2r}(x_0)} \left|\varphi(x)\nabla U(\frac{x-x_0}{\varepsilon})\right|^2 dx\\ &\quad - p_1^2\varepsilon^{2-N}\int_{B_{2r}(x_0)\setminus B_r(x_0)} U^2(\frac{x-x_0}{\varepsilon})\varphi(x)\Delta\varphi(x)\,dx\\ &= p_1^2\int_{B_{\frac{2r}{\varepsilon}}(0)} |\nabla U(y)|^2\varphi^2(x_0+\varepsilon y)\,dy\\ &\quad - \varepsilon^2 p_1^2\int_{B_{\frac{2r}{\varepsilon}}(0)\setminus B_{\frac{r}{\varepsilon}}(0)} U^2(y)\varphi(x_0+\varepsilon y)\Delta\varphi(x_0+\varepsilon y)\,dy.\end{aligned}$$

Direct calculation gives

$$p_1^2\int_{B_{\frac{2r}{\varepsilon}}(0)} |\nabla U(y)|^2\,dy = p_1^2\int_{\mathbb{R}^N} |\nabla U(y)|^2\,dy + O(\varepsilon^{N-2})$$

and

$$\varepsilon^2 p_1^2\int_{B_{\frac{2r}{\varepsilon}}(0)\setminus B_{\frac{r}{\varepsilon}}(0)} U^2(y)\varphi(x_0+\varepsilon y)\Delta\varphi(x_0+\varepsilon y)\,dy = O(\varepsilon^{N-2}).$$

Hence,

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = p_1^2\int_{\mathbb{R}^N} |\nabla U(y)|^2 dy + O(\varepsilon^{N-2}). \tag{2.2}$$

Similarly,

$$\int_\Omega |\nabla v_\varepsilon|^2 dx = q_1^2\int_{\mathbb{R}^N} |\nabla U(y)|^2 dy + O(\varepsilon^{N-2}). \tag{2.3}$$

In the same way, we have

$$\int_\Omega u_\varepsilon^p v_\varepsilon^q\,dx = p_1^p q_1^q\int_{B_{\frac{r}{\varepsilon}}(0)} U^{\frac{2N}{N-2}}(y)\,dy + p_1^p q_1^q\int_{B_{\frac{2r}{\varepsilon}}(0)\setminus B_{\frac{r}{\varepsilon}}(0)} (U(y)\varphi(x_0+\varepsilon y))^{\frac{2N}{N-2}}\,dy,$$

$$p_1^p q_1^q\int_{B_{\frac{2r}{\varepsilon}}(0)\setminus B_{\frac{r}{\varepsilon}}(0)} (U(y)\varphi(x_0+\varepsilon y))^{\frac{2N}{N-2}}\,dy = O(\varepsilon^N),$$

and

$$p_1^p q_1^q \int_{\mathbb{R}^N \setminus B_{\frac{2r}{\varepsilon}}(0)} U^{\frac{2N}{N-2}}(y)\,dy = O(\varepsilon^N).$$

Therefore,

$$\begin{aligned}\int_\Omega u_\varepsilon^p v_\varepsilon^q dx &= p_1^p q_1^q \left(\int_{\mathbb{R}^N} U^{\frac{2N}{N-2}}(y)\,dy - \int_{\mathbb{R}^N \setminus B_{\frac{r}{\varepsilon}(0)}} U^{\frac{2N}{N-2}}(y)\,dy \right) + O(\varepsilon^N) \\ &= p_1^p q_1^q \int_{\mathbb{R}^N} U^{\frac{2N}{N-2}}(y)\,dy + O(\varepsilon^N). \end{aligned} \tag{2.4}$$

We can also infer that

$$\int_\Omega \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^s}\,dx = C\varepsilon^s \int_{B_{\frac{2r}{\varepsilon}}(0)} \frac{U^{2^*(s)}(y)}{|x_0+\varepsilon y|^s} \varphi^{2^*(s)}(x_0+\varepsilon y)\,dy. \tag{2.5}$$

We remark that the integral on the righthand side is positive and independent of ε. Now, we prove (2.1). Obviously, there exists $T > 0$ large such that $I(tu_\varepsilon, tv_\varepsilon) < 0$ if $t \geqslant T$. By (2.2)-(2.5), for $t > 0$ and $0 < s < N-2$, i.e. $0 < s < 1$ if $N = 3$ and $0 < s < 2$ if $N \geqslant 4$ we have

$$\begin{aligned} I(tu_\varepsilon, tv_\varepsilon) = {} & \frac{t^2 p_1^2}{2} \int_{\mathbb{R}^N} |\nabla U(y)|^2\,dy + O(\varepsilon^{N-2}) + \frac{t^2 q_1^2}{2} \int_{\mathbb{R}^N} |\nabla U(y)|^2\,dy + O(\varepsilon^{N-2}) \\ & - \frac{2t^{2^*}}{2^*} p_1^p q_1^q \int_{\mathbb{R}^N} U^{\frac{2N}{N-2}}(y)\,dy + O(\varepsilon^N) \\ & - C\varepsilon^s \int_{B_{\frac{2r}{\varepsilon}}(0)} \frac{U^{2^*(s)}(y)}{|x_0+\varepsilon y|^s} \varphi^{2^*(s)}(x_0+\varepsilon y)\,dy \\ < {} & \frac{t^2}{2}(p_1^2+q_1^2) \int_{\mathbb{R}^N} |\nabla U(y)|^2\,dy - \frac{2t^{2^*}}{2^*} p_1^p q_1^q \int_{\mathbb{R}^N} U^{\frac{2N}{N-2}}(y)\,dy := j(t). \end{aligned}$$

The function $j(t)$ attains its maximum value at

$$t = \left(\frac{p_1^2+q_1^2}{2p_1^p q_1^q} \right)^{\frac{N-2}{4}} \left(\frac{\int_{\mathbb{R}^N} |\nabla U(y)|^2\,dy}{\int_{\mathbb{R}^N} U^{\frac{2N}{N-2}}(y)\,dy} \right)^{\frac{N-2}{4}}$$

with the maximum value

$$\begin{aligned} \max_{t \geqslant 0} j(t) &= \frac{1}{N}(p_1^2+q_1^2) \left(\frac{p_1^2+q_1^2}{2p_1^p q_1^q} \right)^{\frac{N-2}{2}} S^{\frac{N}{2}} \\ &= \frac{1}{N}(p_1^2+q_1^2) \left(\frac{p_1^2+q_1^2}{2p_1^p q_1^q} \right)^{\frac{N-2}{2}} \left[\left(\frac{p}{q}\right)^{\frac{q}{p+q}} + \left(\frac{p}{q}\right)^{\frac{-p}{p+q}} \right]^{-\frac{N}{2}} S_{p,q}^{\frac{N}{2}}. \end{aligned}$$

Since $\frac{p_1^2}{q_1^2} = \frac{p}{q}$, we find

$$\max_{t \geqslant 0} j(t) = \frac{1}{N} 2^{\frac{2-N}{2}} S_{p,q}^{\frac{N}{2}}.$$

As a result, for $0 < \varepsilon < \varepsilon_0$,

$$\max_{t \geqslant 0} I(tu_\varepsilon, tv_\varepsilon) < \max_{t \geqslant 0} j(t) = \frac{1}{N} 2^{\frac{2-N}{2}} S_{p,q}^{\frac{N}{2}}.$$

Choosing $u_0 = Tu_{\varepsilon_0}$, $v_0 = Tv_{\varepsilon_0}$, we obtain (2.1). The proof is complete.

LEMMA 2.2. *If the domain $\Omega \subset \mathbb{R}^N$ is bounded and star-shaped around the origin, then system (1.1) has no positive solution.*

Proof. Multiplying the first equation in (1.1) by $x \cdot \nabla u$ and the second one by $x \cdot \nabla v$ respectively, integrating by part and adding both of them, we obtain

$$\frac{N-2}{2} \int_\Omega (|\nabla u|^2 + |\nabla v|^2)\, dx + \frac{1}{2} \int_{\partial\Omega} \left[\left(\frac{\partial u}{\partial \nu} \right)^2 + \left(\frac{\partial v}{\partial \nu} \right)^2 \right] (x, \nu)\, dS$$
$$= (N-2) \int_\Omega \left(u^p v^q + \lambda \frac{u^\alpha v^\beta}{|x|^s} \right) dx, \quad (2.6)$$

which and (1.1) lead to

$$\int_{\partial\Omega} \left[\left(\frac{\partial u}{\partial \nu} \right)^2 + \left(\frac{\partial v}{\partial \nu} \right)^2 \right] (x, \nu)\, dS = 0. \quad (2.7)$$

Since Ω is a star-shaped around the origin, then $(x \cdot \nu) > 0$. We deduce that

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu} = 0 \text{ a.e on } \partial\Omega$$

and by (1.1)

$$\int_\Omega -\Delta u\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, dS = \int_\Omega \left(\frac{2p}{p+q} u^{p-1} v^q + \frac{2\lambda\alpha}{\alpha+\beta} \frac{u^{\alpha-1} v^\beta}{|x|^s} \right) dx = 0.$$

Hence, the result follows.

3. Problem in the half space

In this section, we prove Theorem 1.1 by the blow up argument and the mountain pass theorem. We start with the following lemma, which is a counterpart of Lemma 2.6 in [10], for reader's convenience, we sketch the proof.

LEMMA 3.1. *Let (u, v) be a positive solution of system (1.11). Then the following conclusions hold:*
(1) $u, v \in C^{1,\beta}(\overline{\mathbb{R}}_+^N)$;
(2) There is a constant C, such that

$$|u(y)|, |v(y)| \leqslant C(1+|y|)^{1-N}, \quad |\nabla u(y)|, |\nabla v(y)| \leqslant C(1+|y|)^{-N}.$$

Proof. We consider the regularity result first. It is enough to consider the regularity at $0 \in \partial\mathbb{R}^N_+$. By the Nash-Moser iteration method, u and v are locally bounded. Then we have $u \in C^\alpha(\bar{B}_1^+)$ for $0 < \alpha < \min\{2-s, 1\}$, where $B_1^+ := B_1(0) \cap \mathbb{R}^N_+$. Set

$$\alpha_0 := \sup\{\alpha; \sup_{B_1^+} \frac{|u(x)|}{|x|^\alpha} < \infty, 0 < \alpha < 1\}.$$

Then for any $0 < \alpha < \alpha_0$, we have $|u(x)| \leqslant C|x|^\alpha$ for $x \in B_1^+$, and

$$\frac{|u(x)|^{2^*(s)-1}}{|x|^s} \leqslant C|x|^{(2^*(s)-1)\alpha - 1} \quad \text{for} \quad x \in B_1^+. \tag{3.1}$$

We may prove $\alpha_0 = 1$. So (3.1) holds for any $0 < \alpha < 1$.

Furthermore, if $2^*(s) - 1 - s \geqslant 0$, i.e., $s \leqslant \frac{(N+2)}{N}$, by taking α close to 1, we see that

$$\frac{|u|^{2^*(s)-1}}{|x|^s} \in L^q(B_1^+) \quad \text{for} \quad 1 < q < \infty,$$

Similarly,

$$\frac{|v|^{2^*(s)-1}}{|x|^s} \in L^q(B_1^+) \quad \text{for} \quad 1 < q < \infty.$$

By Hölder's inequality,

$$\int_{B_1^+} \left(\frac{|u|^{\alpha-1}|v|^\beta}{|x|^s}\right)^q \leqslant \left(\int_{B_1^+} (\frac{|u|^{2^*(s)-1}}{|x|^s})^q\right)^{\frac{(\alpha-1)q}{2^*(s)-1}} \left(\int_{B_1^+} (\frac{|v|^{2^*(s)-1}}{|x|^s})^q\right)^{\frac{q\beta}{2^*(s)-1}},$$

that is

$$\frac{|u|^{\alpha-1}|v|^\beta}{|x|^s} \in L^q(B_1^+) \quad \text{for} \quad 1 < q < \infty.$$

Therefore, $u \in C^{1;\beta}(B_{\frac{1}{2}}^+)$ for $0 < \beta < 1$. The same conclusion also holds for v.

To show (2), by the Kelvin transformation, we see that

$$\tilde{u} = \frac{1}{|x|^{N-2}} u(\frac{x}{|x|^2}) \quad \text{and} \quad \tilde{v} = \frac{1}{|x|^{N-2}} v(\frac{x}{|x|^2})$$

satisfy (1.11) and $\tilde{u}, \tilde{v} \in H_0^1(\mathbb{R}^N_+)$. By (1) of the lemma, $|\tilde{u}(y)| \leqslant C|y|$ for $y \in B_1^+$, it yields

$$|u(y)| \leqslant C(1+|y|)^{1-N}, \quad \forall y \in \mathbb{R}^N_+.$$

The gradient estimate enables us to find $|\nabla u(y)| \leqslant C|y|^{-N}$ for $y \in \mathbb{R}^N_+$. The proof is complete.

Proof of Theorem 1.1 We use the blowing up argument to show the result. Let Ω be a star-shaped domain with respect to 0 and $0 \in \partial\Omega$. For any $\varepsilon > 0$, by applying

Lemma 2.1 and the mountain pass theorem, we can find a positive solution $(u_\varepsilon, v_\varepsilon)$ of the following subcritical system

$$\begin{cases} -\Delta u_\varepsilon = \frac{2p_\varepsilon}{p_\varepsilon+q} u_\varepsilon^{p_\varepsilon-1} v_\varepsilon^q + \frac{2\lambda\alpha}{\alpha+\beta-\varepsilon} \frac{u_\varepsilon^{\alpha-1} v_\varepsilon^{\beta-\varepsilon}}{|x|^s}, & x \in \Omega, \\ -\Delta v_\varepsilon = \frac{2q}{p_\varepsilon+q} u_\varepsilon^{p_\varepsilon} v_\varepsilon^{q-1} + \frac{2\lambda(\beta-\varepsilon)}{\alpha+\beta-\varepsilon} \frac{u_\varepsilon^{\alpha} v_\varepsilon^{\beta-1-\varepsilon}}{|x|^s}, & x \in \Omega, \\ u_\varepsilon > 0, v_\varepsilon > 0, & x \in \Omega, \\ u_\varepsilon = v_\varepsilon = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

The mountain pass level c_ε satisfies

$$0 < \delta \leqslant c_\varepsilon = I_\varepsilon(u_\varepsilon, v_\varepsilon) < \frac{1}{N} 2^{\frac{2-N}{2}} S_{p,q}^{\frac{N}{2}} \tag{3.3}$$

for some $\delta > 0$ independent of $\varepsilon > 0$ small, where $p_\varepsilon + q = \frac{2N}{N-2} - \frac{2\varepsilon}{2-s}$ and

$$I_\varepsilon(u_\varepsilon, v_\varepsilon) = \int_\Omega \Big(\frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{2}|\nabla v_\varepsilon|^2 - \frac{2}{p_\varepsilon+q} u_\varepsilon^{p_\varepsilon} v_\varepsilon^q - \frac{2\lambda}{2^*(s)-\varepsilon} \frac{u_\varepsilon^\alpha v_\varepsilon^{\beta-\varepsilon}}{|x|^s} \Big) dx.$$

By (3.2) and (3.3), we may verify that both $\|u_\varepsilon\|_{H_0^1(\Omega)}$ and $\|v_\varepsilon\|_{H_0^1(\Omega)}$ are uniformly bounded in ε for $\varepsilon > 0$ small. Thus, there is a subsequence $\{(u_j, v_j)\}$ of $\{(u_\varepsilon, v_\varepsilon)\}$ such that

$$\begin{aligned} &u_j \rightharpoonup u, \quad v_j \rightharpoonup v, \quad \text{in} \quad H_0^1(\Omega), \\ &u_j \rightharpoonup u, \quad v_j \rightharpoonup v, \quad \text{in} \quad L^{\frac{2N}{N-2}}(\Omega), \\ &u_j \rightharpoonup u, \quad v_j \rightharpoonup v, \quad \text{in} \quad L^{2^*(s)}(\Omega, |x|^{-s}dx) \end{aligned} \tag{3.4}$$

with (u,v) satisfies (1.1). By Lemma 2.2, $u \equiv v \equiv 0$ since Ω is a star-shaped. Let

$$m_j := u_j(x_j) = \max_{\bar{\Omega}} u_j(x), \quad n_j := v_j(y_j) = \max_{\bar{\Omega}} v_j(x).$$

Then, we have either $m_j \to \infty$ or $n_j \to \infty$ as $j \to \infty$. Indeed, on the contrary we would have $m_j \leqslant C$ and $n_j \leqslant C$ for a positive constant C. By the Sobolev embedding,

$$\int_\Omega u_j^{p_{\varepsilon_j}} v_j^q dx \leqslant C \int_\Omega v_j^q dx \to 0, \quad \int_\Omega \frac{u_j^\alpha v_j^{\beta-\varepsilon_j}}{|x|^s} dx \leqslant C \int_\Omega \frac{v_j^\alpha}{|x|^s} dx \to 0$$

as $j \to \infty$. This implies

$$\int_\Omega (|\nabla u_j|^2 + |\nabla v_j|^2)\, dx = 2 \int_\Omega u_j^{p_\varepsilon} v_j^q dx + 2\lambda \int_\Omega \frac{u_j^\alpha v_j^{\beta-\varepsilon}}{|x|^s} dx \to 0,$$

that is, $u_j \to 0$, $v_j \to 0$ strongly in $H_0^1(\Omega)$. It yields

$$0 = \lim_{j\to\infty} \frac{1}{2} \int_\Omega (|\nabla u_j|^2 + |\nabla v_j|^2) dx \geqslant \delta$$

a contradiction.

We will show that $m_j = O(1)n_j$, and $x_j \to 0$, $y_j \to 0$ at the same time, which implies that the origin is the only blow up point. Suppose $n_j \leqslant m_j \to \infty$ and denote

$$\tilde{u}_j(y) = m_j^{-1}u_j(k_jy + x_j), \quad \tilde{v}_j(y) = m_j^{-1}v_j(k_jy + x_j),$$

where

$$k_j = m_j^{-\frac{p_{\varepsilon_j}+q-2}{2}} \quad \text{and} \quad p_{\varepsilon_j} + q = \frac{2N}{N-2} - \frac{2\varepsilon_j}{2-s}.$$

Then $(\tilde{u}_j, \tilde{v}_j)$ satisfies

$$\begin{cases} -\Delta \tilde{u}_j = \frac{2p_{\varepsilon_j}}{p_{\varepsilon_j}+q}\tilde{u}_j^{p_{\varepsilon_j}-1}\tilde{v}_j^q + \frac{2\lambda\alpha}{\alpha+\beta-\varepsilon_j}\frac{\tilde{u}_j^{\alpha-1}\tilde{v}_j^{\beta-\varepsilon_j}}{|\frac{x_j}{k_j}+x|^s}, & \text{in} \quad \Omega_j, \\ -\Delta \tilde{v}_j = \frac{2q}{p_{\varepsilon_j}+q}\tilde{u}_j^{p_{\varepsilon_j}}\tilde{v}_j^{q-1} + \frac{2\lambda(\beta-\varepsilon_j)}{\alpha+\beta-\varepsilon_j}\frac{\tilde{u}_j^{\alpha}\tilde{v}_j^{\beta-1-\varepsilon_j}}{|\frac{x_j}{k_j}+x|^s}, & \text{in} \quad \Omega_j, \\ 0 \leqslant \tilde{u}_j, \tilde{v}_j \leqslant 1, & \text{in} \quad \Omega_j, \\ \tilde{u}_j = \tilde{v}_j = 0, & \text{on} \quad \partial\Omega_j, \end{cases} \tag{3.5}$$

where $\Omega_j = \{x \in \mathbb{R}^N \mid x_j + k_jx \in \Omega\}$.

We claim that $|x_j| = O(k_j)$ and $x_j \to 0$ as $j \to \infty$. Suppose on the contrary that

$$\limsup_{j\to\infty} \frac{|x_j|}{k_j} = \infty.$$

Since $m_j \to \infty$, $k_j \to 0$ as $j \to \infty$. Because $(\tilde{u}_j, \tilde{v}_j)$ is uniformly bounded in $C^{2,\alpha}_{loc}$, we may assume that $\tilde{u}_j \to u, \tilde{v}_j \to v$ in C^2_{loc}.

Suppose $x_j \to x_0 \in \bar{\Omega}$. There are two cases: (1) $x_0 \in \Omega$ or $x_0 \in \partial\Omega$ and

$$\frac{\text{dist}(x_j, \partial\Omega)}{k_j} \to \infty;$$

and (2) $x_0 \in \partial\Omega$ and $\frac{\text{dist}(x_j,\partial\Omega)}{k_j} \to \sigma \geqslant 0$.

In the case (1), we have $\Omega_j \to \mathbb{R}^N$ as $j \to \infty$ and (u,v) with $u(0) = 1$ satisfies

$$-\Delta u = \frac{2p}{p+q}u^{p-1}v^q, \quad -\Delta v = \frac{2q}{p+q}u^pv^{q-1}, \quad 0 \leqslant u, v \leqslant 1 \quad \text{in} \quad \mathbb{R}^N. \tag{3.6}$$

Furthermore, we have

$$\lim_{j\to\infty}\int_\Omega |\nabla u_j|^2\,dx = \lim_{j\to\infty}(m_j^{\frac{\varepsilon_j(N-2)}{2-s}}\int_{\Omega_j}|\nabla\tilde{u}_j|^2\,dy) \geqslant \int_{\mathbb{R}^N}|\nabla u|^2\,dy,$$

$$\lim_{j\to\infty}\int_\Omega |\nabla v_j|^2\,dx = \lim_{j\to\infty}(m_j^{\frac{\varepsilon_j(N-2)}{2-s}}\int_{\Omega_j}|\nabla\tilde{v}_j|^2\,dy) \geqslant \int_{\mathbb{R}^N}|\nabla v|^2\,dy,$$

$$\lim_{j\to\infty}\int_\Omega u_j^{p_\varepsilon} v_j^q\,dx = \lim_{j\to\infty}(m_j^{\frac{\varepsilon_j(N-2)}{2-s}}\int_{\Omega_j}\tilde{u}_j^{p_\varepsilon}\tilde{v}_j^q dy) \geqslant \int_{\mathbb{R}^N} u^p v^q\,dy,$$

$$\lim_{j\to\infty}\int_\Omega \frac{u_j^\alpha v_j^{\beta-\varepsilon_j}}{|x|^s}\,dx = \lim_{j\to\infty}\left(m_j^{\frac{\varepsilon_j(N-2)}{2-s}}\int_{\Omega_j}\frac{\tilde{u}_j^\alpha \tilde{v}_j^{\beta-\varepsilon_j}}{|\frac{x_j}{k_j}+y|^s}\,dy\right).$$

Using these facts and (3.5), we deduce

$$\begin{aligned} c &= \lim_{j\to\infty} c_{\varepsilon_j} = \lim_{j\to\infty} I_{\varepsilon_j}(u_j,v_j) \\ &= \left(\frac{1}{2}-\frac{1}{2^*(s)}\right)\lim_{j\to\infty}\int_\Omega \left(|\nabla u_j|^2+|\nabla v_j|^2\right)dx \\ &\quad + \left(\frac{2}{2^*(s)}-\frac{2}{2^*}\right)\lim_{j\to\infty}\int_\Omega u_j^{p\varepsilon_j} v_j^q\,dx \\ &\geqslant \left(\frac{1}{2}-\frac{1}{2^*(s)}\right)\int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)dx + \left(\frac{2}{2^*(s)}-\frac{2}{2^*}\right)\int_{\mathbb{R}^N} u^p v^q\,dx. \end{aligned} \tag{3.7}$$

On the other hand, by the definition of $S_{p,q}$, we see that

$$S_{p,q}(\mathbb{R}^N)\left(\int_{\mathbb{R}^N} u^p v^q\,dx\right)^{\frac{2}{2^*}} \leqslant \int_{\mathbb{R}^N}\left(|\nabla u|^2+|\nabla v|^2\right)dx = 2\int_{\mathbb{R}^N} u^p v^q\,dx, \tag{3.8}$$

that is

$$2^{-\frac{N}{2}} S_{p,q}^{\frac{N}{2}}(\mathbb{R}^N) \leqslant \int_{\mathbb{R}^N} u^p v^q\,dx. \tag{3.9}$$

Therefore,

$$c \geqslant \frac{2}{N}\int_{\mathbb{R}^N} u^p v^q\,dx \geqslant \frac{1}{2} 2^{\frac{2-N}{2}} S_{p,q}^{\frac{N}{2}}(\mathbb{R}^N),$$

which contradicts to the fact that

$$c \leqslant \max_{0\leqslant t\leqslant 1} I(tu_0,tv_0) < \frac{1}{2} 2^{\frac{2-N}{2}} S_{p,q}^{\frac{N}{2}}(\mathbb{R}^N).$$

In the case (2), after an orthogonal transformation, we have $\Omega_j \to \mathbb{R}_+^N = \{x = (x_1,\cdots,x_N) \mid x_1 > 0\}$ as $j\to\infty$ and $\tilde{u}_j$, $\tilde{v}_j$ converge to some u, v uniformly in every compact subset of $\mathbb{R}_+^N$. Apparently, $u(0)=1$ and $0\leqslant v(0)\leqslant 1$. Hence, (u,v) satisfies

$$\begin{cases} -\Delta u = \frac{2p}{p+q} u^{p-1} v^q & \text{in } \mathbb{R}_+^N, \\ -\Delta v = \frac{2q}{p+q} u^p v^{q-1} & \text{in } \mathbb{R}_+^N, \\ 0\leqslant u,v\leqslant 1 & \text{in } \mathbb{R}_+^N, \\ u=v=0 & \text{on } \partial\mathbb{R}_+^N. \end{cases} \tag{3.10}$$

The boundary condition violates to $u(0)=1$. Consequently, $\limsup_{j\to\infty}\frac{|x_j|}{k_j}<\infty$. Since $k_j\to 0$, we have $x_j\to 0$ as $j\to\infty$.

Next, we show that $\liminf_{j\to\infty}\frac{|x_j|}{k_j}>0$. Were it not the case, we would have, up to a subsequence, that $\lim_{j\to\infty}\frac{|x_j|}{k_j}=0$. Up to a rotation, we have $\Omega_j\to\mathbb{R}^N_+$ and $\tilde{u}_j$, $\tilde{v}_j$ converge to some u, v uniformly in compact subsets of $\mathbb{R}^N_+$ respectively, where (u,v) is a solution of (1.11) with $0\leqslant u,v\leqslant 1$. Again $u(0)=0$ contradicts to the fact $u(0)=1$. Hence, $\liminf_{j\to\infty}\frac{|x_j|}{k_j}>0$.

Now, we show that problem (1.11) has a nontrivial solution. We may assume

$$\frac{\operatorname{dist}(x_j,\partial\Omega)}{k_j}\to\sigma\geqslant 0.$$

By an affine transformation, we find $(\tilde{u}_j,\tilde{v}_j)$ converges to (u,v) uniformly in any compact subset of $\mathbb{R}^N_+$ and (u,v) satisfies (1.11) with $u(\sigma,\cdots,0)=1$. Since u is nontrivial, so is v. Indeed, otherwise if $v\equiv 0$, we would have

$$\begin{cases}\Delta u=0 \quad \text{in} \quad \mathbb{R}^N_+,\\ 0\leqslant u\leqslant 1, u(\sigma,\cdots,0)=1 \quad \text{in} \quad \mathbb{R}^N_+,\\ u=0 \quad \text{on} \quad \partial\mathbb{R}^N_+.\end{cases}$$

By the strong maximum principle, u would be a constant because it attains its maximum value inside $\mathbb{R}^N_+$. This yields a contradiction between $u(\sigma,\cdots,0)=1$ and the boundary condition. Therefore, there exists $y_0\in\mathbb{R}^N_+$ such that $v(y_0)\neq 0$. So we have proved that problem (1.11) has a nontrivial solution. As a by product, this also implies

$$\tilde{v}_j(y_0)=m_j^{-1}v_j(x_j+k_jy_0)\to v(y_0)>0,$$

and then

$$1\geqslant\frac{n_j}{m_j}\geqslant\frac{v_j(x_j+k_jy_0)}{m_j}\geqslant v(y_0)-\varepsilon>0$$

for $\varepsilon>0$ small and j large. As a result, $n_j=O(1)m_j$ as $j\to\infty$. Replacing m_j by n_j in above blow up process, we may deduce that $|y_j|=O(\tilde{k}_j)$, where

$$\tilde{k}_j=n_j^{-\frac{p\varepsilon_j+q-2}{2}}.$$

So we also have $y_j\to 0$. Consequently, the origin is the only blow up point and problem (1.11) has a solution (u,v). Observe that such a solution verifies

$$\begin{aligned}J(u,v)&=(1-\frac{2}{2^*})\int_{\mathbb{R}^N_+}u^pv^q\,dy+\lambda(1-\frac{2}{2^*(s)})\int_{\mathbb{R}^N_+}\frac{u^\alpha v^\beta}{|y|^s}\,dy\\&\leqslant\lim_{j\to\infty}c_\varepsilon<\frac{1}{N}2^{\frac{2-N}{2}}S_{p,q}^{\frac{N}{2}}\end{aligned}\tag{3.11}$$

since

$$\int_{\mathbb{R}^N}u^pv^q dy\leqslant\lim_{j\to\infty}\int_\Omega u_j^pv_j^q\,dy,\quad\int_{\mathbb{R}^N}\frac{u^\alpha v^\beta}{|y|^s}dy\leqslant\lim_{j\to\infty}\int_\Omega\frac{u_j^\alpha v_j^{\beta-\varepsilon_j}}{|y|^s}\,dy.$$

Finally, we show that there exists a least energy solution of problem (1.11). Let

$$c_0 = \inf\{J(u,v) \mid (u,v) \text{ is a positive solution of (1.11) and } J(u,v) > 0\}, \tag{3.12}$$

which is finite. For any positive solution (u,v) of (1.11), by Hölder's inequality, Sobolev and Hardy-Sobolev inequalities we deduce from

$$\frac{1}{2}\int_{\mathbb{R}^N_+}(|\nabla u|^2+|\nabla v|^2)\,dy = \int_{\mathbb{R}^N_+} u^p v^q\,dy + \lambda\int_{\mathbb{R}^N_+}\frac{u^\alpha v^\beta}{|y|^s}\,dy$$

that

$$\|u\|_{H^1_0(\mathbb{R}^N_+)} + \|v\|_{H^1_0(\mathbb{R}^N_+)} \geqslant \gamma > 0 \tag{3.13}$$

for some constant γ. This implies $c_0 > 0$.

Let (u_j,v_j) be a minimizing sequence of c_0. Denote $m_j = \max u_j(x)$, $n_j = \max v_j(x)$. By Lemma 3.1, we may assume that the maximum points of u_j or v_j are uniformly bounded. If m_j or n_j tends to infinity, we may show as before that $m_j = O(1)n_j$. Hence, $m_j \to \infty$ if and only if $n_j \to \infty$. So we need to treat two cases:(i) both u_j and v_j are uniformly bounded; (ii) both m_j and n_j tend to infinity.

In the case (i), we have $u_j \to u$ and $v_j \to v$ and (u,v) is a positive solution of problem (1.11) with $J(u,v) = c_0$. The assertion follows.

In the case (ii), since there is a solution of (1.11) such that (3.11) holds, we have

$$J(u_j,v_j) < \frac{1}{N}2^{\frac{2-N}{2}}S_{p,q}^{\frac{N}{2}}.$$

Applying the blow up argument as before, we have that $m_j = O(1)n_j$ and $x_j \to 0$. Moreover, the functions

$$u_j(y) = m_j^{-1}u_j(x_j + k_j y), \quad v_j(y) = m_j^{-1}v_j(x_j + k_j y),$$

where $k_j = m_j^{-\frac{2}{N-2}}$, converge to a positive solution (u,v) of (1.11) with $J(u,v) \leqslant \lim_{j\to\infty} J(u_j,v_j) = c_0$. This means that (u,v) is the least energy solution of problem (1.11), which satisfies (1.14). The proof is completed. □

4. Existence of solutions in bounded domains

In this section, we shall prove the existence of positive solution of system (1.1). To this end, we need the following lemma.

LEMMA 4.1. *For $\lambda > 0$ small, there exist nonnegative functions u_0, $v_0 \in H^1_0(\Omega)\setminus\{0\}$ such that $I(u_0,v_0) < 0$ and*

$$\max_{t\geqslant 0} I(tu_0,tv_0) < c_0, \tag{4.1}$$

where c_0 is defined in (1.14).

Proof. Without loss of generality, we may assume that in a neighborhood of 0, the boundary $\partial\Omega$ can be represented by $x_n = \varphi(x')$ with $\varphi(0) = 0$, $\nabla'\varphi(0) = 0$ and the outer normal of $\partial\Omega$ at 0 is $-e_N = (0,0,\cdots -1)$, where $x' = (x_1,\cdots x_{N-1})$, $\nabla' = (\partial_1,\cdots \partial_{N-1})$. Define

$$\psi(x) = (x', x_n - \varphi(x'))$$

We choose a positive number r_0 small so that there exist neighborhoods U and $\tilde{U}$ of 0, such that

$$\psi(U) = B_{r_0}(0),\ \psi(U\cap\Omega) = B^+_{r_0}(0) = B_{r_0}(0)\cap\mathbb{R}^N_+,$$
$$\psi(\tilde{U}) = B_{\frac{r_0}{2}}(0),\ \psi(\tilde{U}\cap\Omega) = B^+_{\frac{r_0}{2}}(0).$$

Suppose that (u,v) is the least energy solution of (1.11). For $\varepsilon > 0$, we define

$$u_\varepsilon(x) = \varepsilon^{-\frac{N-2}{2}}\eta(x)u\left(\frac{\psi(x)}{\varepsilon}\right), \quad v_\varepsilon(x) = \varepsilon^{-\frac{N-2}{2}}\eta(x)v\left(\frac{\psi(x)}{\varepsilon}\right),$$

where $\eta \in C_0^\infty(U)$ is a positive cut-off function with $\eta \equiv 1$ in $\tilde{U}$.

Now we estimate each term in $I(u_\varepsilon, v_\varepsilon)$.

First, by the change of the variable $y = \frac{\psi(x)}{\varepsilon} \in B^+_{\frac{r_0}{\varepsilon}}(0)$, we obtain

$$\begin{aligned}\int_\Omega u_\varepsilon^p v_\varepsilon^q\,dx &= \varepsilon^{-\frac{(N-2)(p+q)}{2}+N}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta^{p+q}(\psi^{-1}(\varepsilon y))u^p(y)v^q(y)\,dy\\ &= \int_{\mathbb{R}^N_+} u^p v^q\,dy - \int_{\mathbb{R}^N_+\setminus B^+_{\frac{r_0}{\varepsilon}}} u^p v^q\,dy\\ &= \int_{\mathbb{R}^N_+} u^p v^q\,dy + O(\varepsilon^{\frac{N(p+q)}{2}}).\end{aligned}$$

Next, we estimate

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = \int_\Omega (|\nabla\eta|^2 u_\varepsilon^2 + \eta^2|\nabla u_\varepsilon|^2 + 2\nabla\eta\nabla u_\varepsilon \eta u_\varepsilon)\,dx.$$

Since

$$\int_\Omega \eta u_\varepsilon \nabla\eta\nabla u_\varepsilon\,dx = -\int_\Omega |\nabla\eta|^2 u_\varepsilon^2\,dx - \int_\Omega \nabla\eta\eta\nabla u_\varepsilon u_\varepsilon\,dx - \int_\Omega \eta(\Delta\eta)u_\varepsilon^2\,dx,$$

we have

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx = \int_{\Omega\cap U} \eta^2|\nabla u_\varepsilon|^2\,dx - \int_{\Omega\cap U} \eta(\Delta\eta)u_\varepsilon^2\,dx.$$

By the change of the variable $y = \frac{\psi(x)}{\varepsilon}$ and Lemma 3.1,

$$\begin{aligned}|\int_{\Omega\cap U} \eta(\Delta\eta)u_\varepsilon^2\,dx| &\leqslant C\varepsilon^2\int_{B^+_{\frac{r_0}{\varepsilon}}(0)\setminus B^+_{\frac{r_0}{2\varepsilon}}(0)} \eta(\psi^{-1}(\varepsilon y))|\Delta\eta(\psi^{-1}(\varepsilon y))|u^2(y)\,dy\\ &= o(\varepsilon^2).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{\Omega\cap U}&\eta^2|\nabla u_\varepsilon(x)|^2\,dx\\ &=\varepsilon^2\int_{B^+_{\frac{r_0}{\varepsilon}}}\eta^2(\psi^{-1}(\varepsilon y))|\nabla_x u(y)|^2\,dy\\ &=\int_{B^+_{\frac{r_0}{\varepsilon}}(0)}\eta^2(\psi^{-1}(\varepsilon y))(|\nabla_y u(y)|^2-2\partial_n u(y)\nabla' u(y)(\nabla'\varphi)(\varepsilon y')\\ &\qquad\qquad+[\partial_n u(y)]^2|(\nabla'\varphi)(\varepsilon y')|^2)\,dy\\ &=I_1+I_2+I_3.\end{aligned}$$

Obviously,

$$|I_1|\leqslant\int_{\mathbb{R}^N_+}|\nabla u|^2\,dy.$$

Since $\partial\Omega$ is C^2 at 0, it holds that

$$\varphi(y')=\sum_{i=1}^{N-1}\alpha_i y_i^2+o(1)(|y'|^2).$$

By Lemma 3.1, we have

$$|I_3|\leqslant C\varepsilon^2\int_{\mathbb{R}^N}\frac{|y|^2}{(1+|y|)^{2N}}\,dy=O(\varepsilon^2).$$

Integrating by part, we obtain that

$$\begin{aligned}I_2=&\frac{4}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)}\eta(\psi^{-1}(\varepsilon y))\nabla'[\eta(\phi^{-1}(\varepsilon y))]\partial_N u(y)\nabla' u(y)\varphi(\varepsilon y')\,dy\\ &+\frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)}\eta^2(\psi^{-1}(\varepsilon y))\nabla'\partial_N u(y)\nabla' u(y)\varphi(\varepsilon y')\,dy\\ &+\frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)}\eta^2(\psi^{-1}(\varepsilon y))\partial_N u(y)\sum_{i=1}^{n-1}\partial_{ii}u(y)u(y)\varphi(\varepsilon y')\,dy\\ =&\,I_{21}+I_{22}+I_{23}.\end{aligned}$$

By Lemma 3.1, we deduce

$$|I_{21}|\leqslant c\varepsilon^2\int_{B^+_{\frac{r_0}{\varepsilon}}(0)\setminus B_{\frac{r_0}{2\varepsilon}}(0)}(1+|y|)^{-2N}|y|^2\,dy\leqslant c_2\varepsilon^N.$$

In the same way, $I_{22}=O(\varepsilon^N)$. Since (u,v) satisfies the system (1.11), we have

$$\sum_{i=1}^{n-1}\partial_{ii}u=\Delta u-\partial_{NN}u=-\frac{2p}{p+q}u^{p-1}v^q-\frac{2\lambda\alpha}{\alpha+\beta}\frac{u^{\alpha-1}v^\beta}{|x|^s}-\partial_{NN}u,$$

and then

$$\begin{aligned} I_{23} = &-\frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)} \eta^2(\psi^{-1}(\varepsilon y))\frac{2p}{p+q}u^{p-1}v^q\partial_N u(y)\varphi(\varepsilon y')\,dy \\ &-\frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)} \eta^2(\psi^{-1}(\varepsilon y))\frac{2\lambda\alpha}{\alpha+\beta}\frac{u^{\alpha-1}v^\beta}{|y|^s}\partial_N u(y)\varphi(\varepsilon y')\,dy \\ &-\frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)} \eta^2(\psi^{-1}(\varepsilon y))\partial_{NN}u(y)\partial_N u(y)\varphi(\varepsilon y')\,dy \\ =:&\, I_a+I_b+I_c. \end{aligned}$$

Using Lemma 3.1, we can show that $I_a = O(\varepsilon^{\frac{N^2-N+2}{N-2}})$. Integrating by parts, we obtain

$$\begin{aligned} I_b &= -\frac{4\lambda}{(\alpha+\beta)\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)} \partial_N\Big(\eta^2(\psi^{-1}(\varepsilon y))\frac{\varphi(\varepsilon y')v^\beta}{|y|^s}\Big)u^\alpha\,dy \\ &= \frac{4\lambda}{(\alpha+\beta)\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)} 2\eta(\psi^{-1}(\varepsilon y))\partial_N[\eta(\psi^{-1}(\varepsilon y))]\varphi(\varepsilon y')\frac{u^\alpha v^\beta}{|y|^s}\,dy \\ &\quad+\frac{4\lambda}{(\alpha+\beta)\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)} \eta^2(\psi^{-1}(\varepsilon y))\partial_N[\varphi(\varepsilon y')]\frac{u^\alpha v^\beta}{|y|^s}dy \\ &\quad+\frac{4\lambda}{(\alpha+\beta)\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)} \eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')\beta\partial_N v\frac{u^\alpha v^{\beta-1}}{|y|^s}\,dy \\ &\quad-\frac{4\lambda s}{(\alpha+\beta)\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)} \eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')y_N\frac{u^\alpha v^\beta}{|y|^{s+2}}\,dy \\ &=: I_{b1}+I_{b2}+I_{b3}+I_{b4}. \end{aligned}$$

In the same way, we have

$$I_{b1}, I_{b2} = O(\varepsilon^{\frac{N^2-N-Ns+2}{N-2}}),\ I_{b3} = O(\varepsilon^{\frac{N(N-s)}{N-2}}).$$

Hence,

$$I_b = -\frac{4\lambda s}{(\alpha+\beta)\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)} \eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')y_N\frac{u^\alpha v^\beta}{|y|^{s+2}}\,dy + O(\varepsilon^{\frac{N^2-N-Ns+2}{N-2}}).$$

Similarly,

$$I_c = \frac{1}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)\cap\partial\mathbb{R}^N_+} \eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')(\partial_N u(y))^2\,dS_y + O(\varepsilon^{N-1})$$

Therefore,

$$
\begin{aligned}
I_2 = & -\frac{4\lambda s}{(\alpha+\beta)\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)} \eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')y_N\frac{u^\alpha v^\beta}{|y|^{s+2}}dy \\
& \qquad + \frac{1}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)\cap\partial\mathbb{R}^N_+} \eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')(\partial_n u(y))^2 dS_y + O(\varepsilon^{N-1}) \\
=: & \, J_1 + J_2 + O(\varepsilon^{N-1}).
\end{aligned}
$$

We may write

$$
\begin{aligned}
J_1 = & -\frac{4\lambda s}{(\alpha+\beta)\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}(0)\setminus B^+_{\frac{r_0}{2\varepsilon}}(0)} \eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')y_N\frac{u^\alpha v^\beta}{|y|^{s+2}}\,dy \\
& \qquad - \frac{4\lambda s}{(\alpha+\beta)\varepsilon}\int_{B^+_{\frac{r_0}{2\varepsilon}}(0)} \varphi(\varepsilon y')y_N\frac{u^\alpha v^\beta}{|y|^{s+2}}\,dy =: J_{11}+J_{12}.
\end{aligned}
$$

We estimate

$$
|J_{11}| \leqslant c\varepsilon\int_{B^+_{\frac{r_0}{\varepsilon}}(0)\setminus B^+_{\frac{r_0}{2\varepsilon}}(0)} \frac{|y|^3}{|y|^{s+2}(1+|y|)^{(N-1)2^*(s)}}\,dy \leqslant c_2\varepsilon^{\frac{N(N-s)}{N-2}},
$$

and

$$
\frac{4\lambda s}{(\alpha+\beta)\varepsilon}\int_{\mathbb{R}^N_+\setminus B^+_{\frac{r_0}{\varepsilon}}(0)} \varphi(\varepsilon y')y_N\frac{u^\alpha v^\beta}{|y|^{s+2}}dy = O(\varepsilon^{\frac{N(N-s)}{N-2}}).
$$

Hence,

$$
\begin{aligned}
J_{12} = & -\frac{4\lambda s}{(\alpha+\beta)\varepsilon}\int_{\mathbb{R}^N_+} \varphi(\varepsilon y')y_N\frac{u^\alpha v^\beta}{|y|^{s+2}}\,dy + \frac{4\lambda s}{(\alpha+\beta)\varepsilon}\int_{\mathbb{R}^N_+\setminus B^+_{\frac{r_0}{\varepsilon}}(0)} \varphi(\varepsilon y')y_N\frac{u^\alpha v^\beta}{|y|^{s+2}}\,dy \\
= & -\frac{4\lambda s}{(\alpha+\beta)\varepsilon}\int_{\mathbb{R}^N_+} \varphi(\varepsilon y')y_N\frac{u^\alpha v^\beta}{|y|^{s+2}}\,dy + O(\varepsilon^{\frac{N(N-s)}{N-2}}) \\
= & -\frac{4\lambda s\varepsilon}{(\alpha+\beta)}\sum_{i=1}^{N-1}\alpha_i\int_{\mathbb{R}^N_+}\frac{y_i^2 y_N u^\alpha v^\beta}{|y|^{s+2}}\,dy(1+o(1)) + O(\varepsilon^{\frac{N(N-s)}{N-2}}) \\
= & -\frac{4\lambda s\varepsilon}{(\alpha+\beta)(N-1)}\int_{\mathbb{R}^N_+}\frac{|y'|^2 y_N u^\alpha v^\beta}{|y|^{s+2}}dy\sum_{i=1}^{N-1}\alpha_i(1+o(1)) + O(\varepsilon^{\frac{N(N-s)}{N-2}}) \\
= & -\lambda K_1 H(0)(1+o(1))\varepsilon + O(\varepsilon^{\frac{N(N-s)}{N-2}}),
\end{aligned}
$$

where

$$
H(0) = \frac{1}{N-1}\sum_{i=1}^{N-1}\alpha_i \quad \text{and} \quad K_1 = \frac{4s}{(\alpha+\beta)}\int_{\mathbb{R}^N_+}\frac{|y'|^2 y_N u^\alpha v^\beta}{|y|^{s+2}}\,dy.
$$

On the other hand, we write

$$J_2 = \frac{1}{\varepsilon}\int_{(B^+_{\frac{r_0}{\varepsilon}}(0)\setminus B^+_{\frac{r_0}{2\varepsilon}}(0))\cap\partial\mathbb{R}^N_+} \eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')(\partial_N u(y))^2\,dS_y$$
$$+\frac{1}{\varepsilon}\int_{B^+_{\frac{r_0}{2\varepsilon}}\cap\partial\mathbb{R}^N_+} \varphi(\varepsilon y')(\partial_n u(y))^2\,dS_y =: J_{21}+J_{22},$$

and estimate

$$\begin{aligned}|J_{21}| &\leqslant \frac{C}{\varepsilon}\int_{\{\frac{r_0}{2}<|\varepsilon y'|\leqslant r_0\}} |(\partial_n u)(y',0)|^2|\varphi(\varepsilon y')|\,dy'\\ &\leqslant C\varepsilon\int_{\{\frac{r_0}{2}<|\varepsilon y'|\leqslant r_0\}} |y'|^{-2N+2}dy' = O(\varepsilon^N).\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^{N-1}\setminus(B^+_{\frac{r_0}{2\varepsilon}}\cap\partial\mathbb{R}^N_+)} \varphi(\varepsilon y')(\partial_N u(y))^2\,dS_y = O(\varepsilon^N.)$$

Therefore,

$$\begin{aligned}J_{22} &= \frac{1}{\varepsilon}\int_{\mathbb{R}^{N-1}} \varphi(\varepsilon y')(\partial_N u(y))^2 dSy - \frac{1}{\varepsilon}\int_{\mathbb{R}^{N-1}\setminus(B^+_{\frac{r_0}{2\varepsilon}}\cap\partial\mathbb{R}^N_+)} \varphi(\varepsilon y')(\partial_N u(y))^2 dSy\\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}^{N-1}} \varphi(\varepsilon y')(\partial_N u(y))^2\,dS_y + O(\varepsilon^{N-1})\\ &= \varepsilon\sum_{i=1}^{N-1}\alpha_i\int_{\mathbb{R}^{N-1}}[(\partial_N u)(y',0)]^2 y_i^2\,dy'(1+o(1)) + O(\varepsilon^{N-1})\\ &= \frac{\varepsilon}{N-1}\int_{\mathbb{R}^{N-1}}|(\partial_N u)(y',0)|^2|y'|^2\,dy'\sum_{i=1}^{N-1}\alpha_i(1+o(1)) + O(\varepsilon^{N-1})\\ &= K_2H(0)(1+o(1))\varepsilon + O(\varepsilon^{N-1}),\end{aligned}$$

where $K_2 = \int_{\mathbb{R}^{N-1}}|(\partial_N u)(y',0)|^2|y'|^2\,dy'$. Consequently, we have

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx \leqslant \int_{\mathbb{R}^N_+}|\nabla u|^2\,dx - \lambda K_1H(0)(1+o(1))\varepsilon + K_2H(0)(1+o(1))\varepsilon + O(\varepsilon^2)$$

and in the same way,

$$\int_\Omega |\nabla v_\varepsilon|^2\,dx \leqslant \int_{\mathbb{R}^N_+}|\nabla v|^2\,dx - \lambda K_1H(0)(1+o(1))\varepsilon + K_2'H(0)(1+o(1))\varepsilon + O(\varepsilon^2)$$

where $K_2' = \int_{\mathbb{R}^{N-1}}|(\partial_N v)(y',0)|^2|y'|^2\,dy'$.

Finally, we estimate

$$\int_\Omega \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^s}\,dx \geqslant \int_{\Omega\cap\tilde{U}}\frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^s}\,dx = \int_{B^+_{\frac{r_0}{2\varepsilon}}(0)}\frac{u^\alpha(y)v^\beta(y)}{|\frac{\psi^{-1}(\varepsilon y)}{\varepsilon}|^s}\,dy.$$

Since $|\psi^{-1}(y)|^2 = |y|^2 + 2y_N\varphi(y') + \varphi^2(y')$, we have

$$\frac{1}{|\frac{\psi^{-1}(\varepsilon y)}{\varepsilon}|^s} = \frac{1}{|y|^s}\left(1 - \frac{sy_N\varphi(\varepsilon y')}{\varepsilon|y|^2} - \frac{s\varphi^2(\varepsilon y')}{2\varepsilon^2|y|^2}\right) + \frac{1}{|y|^s}O\left(\left(\frac{2y_N\varphi(\varepsilon y')}{\varepsilon|y|^2} + \frac{\varphi^2(\varepsilon y')}{\varepsilon^2|y|^2}\right)^2\right).$$

Therefore,

$$\int_{\Omega\cap\tilde{U}} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^s}\,dx = \int_{B^+_{\frac{r_0}{2\varepsilon}}(0)} \frac{u^\alpha v^\beta}{|y|^s}\,dy - \frac{s}{\varepsilon}\int_{B^+_{\frac{r_0}{2\varepsilon}}} \frac{y_N\varphi(\varepsilon y')u^\alpha(y)v^\beta(y)}{|y|^{s+2}}dy + O(\varepsilon^2).$$

The fact

$$\int_{\mathbb{R}^N_+\setminus B^+_{\frac{r_0}{2\varepsilon}}(0)} \frac{u^\alpha v^\beta}{|y|^s}\,dy = O(\varepsilon^{\frac{N(N-s)}{N-2}})$$

allows us to show that

$$\int_{\Omega\cap\tilde{U}} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^s}\,dx = \int_{\mathbb{R}^N_+} \frac{u^\alpha v^\beta}{|y|^s}\,dy - \frac{s}{\varepsilon}\int_{B^+_{\frac{r_0}{2\varepsilon}}(0)} \frac{y_N\varphi(\varepsilon y')u^\alpha(y)v^\beta(y)}{|y|^{s+2}}\,dy + O(\varepsilon^2).$$

While

$$\frac{s}{\varepsilon}\int_{\mathbb{R}^N_+\setminus B^+_{\frac{r_0}{2\varepsilon}}(0)} \frac{y_N\varphi(\varepsilon y')u^\alpha(y)v^\beta(y)}{|y|^{s+2}}\,dy = O(\varepsilon^{\frac{N^2-Ns+4N}{N-2}})$$

implies that

$$\begin{aligned}
&-\frac{s}{\varepsilon}\int_{B^+_{\frac{r_0}{2\varepsilon}}(0)} \frac{y_N\varphi(\varepsilon y')u^\alpha(y)v^\beta(y)}{|y|^{s+2}}\,dy \\
&\quad= -\frac{s}{\varepsilon}\int_{\mathbb{R}^N_+} \frac{y_N\varphi(\varepsilon y')u^\alpha(y)v^\beta(y)}{|y|^{s+2}}\,dy + \frac{s}{\varepsilon}\int_{\mathbb{R}^N_+\setminus B^+_{\frac{r_0}{2\varepsilon}}(0)} \frac{y_N\varphi(\varepsilon y')u^\alpha(y)v^\beta(y)}{|y|^{s+2}}\,dy \\
&\quad= -s\varepsilon\sum_{i=1}^{N-1}\alpha_i\int_{\mathbb{R}^N_+}\frac{y_Ny_i^2u^\alpha v^\beta}{|y|^{s+2}}\,dy(1+o(1)) + O(\varepsilon^{\frac{N^2-Ns+4N}{N-2}}) \\
&\quad= -\frac{s\varepsilon}{N-1}\int_{\mathbb{R}^N_+}\frac{y_N|y'|^2u^\alpha v^\beta}{|y|^{s+2}}\,dy\sum_{i=1}^{N-1}\alpha_i(1+o(1)) + O(\varepsilon^{\frac{N^2-Ns+4N}{N-2}}).
\end{aligned}$$

So we obtain

$$\int_{\Omega\cap\tilde{U}} \frac{u_\varepsilon^\alpha v_\varepsilon^\beta}{|x|^s}\,dx = \int_{\mathbb{R}^N_+} \frac{u^\alpha v^\beta}{|y|^s}\,dy - K_3H(0)(1+o(1))\varepsilon + O(\varepsilon^2),$$

where $K_3 = s\int_{\mathbb{R}^N_+}\frac{y_N|y'|^2u^\alpha v^\beta}{|y|^{s+2}}\,dy = \frac{(\alpha+\beta)}{4}K_1$.

We may verify that there exists $T>0$ such that $I(tu_\varepsilon,tv_\varepsilon)<0$ if $t\geqslant T$. For $0<t\leqslant T$,

$$I(tu_\varepsilon,tv_\varepsilon)=J(tu,tv)+\frac{H(0)}{2}((K_2+K_2'-2\lambda K_1+o(1))t^2 + \frac{4\lambda}{2^*(s)}(K_3+o(1))t^{2^*(s)})\varepsilon+O(\varepsilon^2).$$

It readily verifies that

$$\max_{t\geqslant 0} J(tu,tv)=J(u,v)=c_0$$

and

$$K_2+K_2'-2\lambda K_1+\frac{4\lambda}{2^*(s)}K_3=K_2+K_2'-\lambda K_1>0$$

for $\lambda>0$ small. Hence, for $\varepsilon>0$ small and $H(0)<0$, we conclude that

$$\max_{t\geqslant 0} I(tu_\varepsilon,tv_\varepsilon)<c_0.$$

Taking $u_0=t_0u_\varepsilon$, $v_0=t_0v_\varepsilon$, where t_0 is large enough so that $I(u_0,v_0)<0$, we obtain $\max_{t\geqslant 0} I(tu_0,tv_0)<c_0$. The lemma is proved.

Proof of Theorem 1.2 Let $\lambda^*=\sup\{\lambda>0|(4.1) \text{ holds}\}$. By the mountain pass theorem and Lemma 4.1, we can find a positive solution $(u_\varepsilon,v_\varepsilon)$ of (3.2) such that

$$c_\varepsilon=I_\varepsilon(u_\varepsilon,v_\varepsilon)<c_0 \tag{4.2}$$

for $\varepsilon>0$ small. We may show that $\|u_\varepsilon\|_{H_0^1(\Omega)}$, $\|v_\varepsilon\|_{H_0^1(\Omega)}\leqslant C$, where C is independent of $\varepsilon>0$. Thus, there is a subsequence (u_j,v_j) of $(u_\varepsilon,v_\varepsilon)$ such that

$$u_j\rightharpoonup u,\quad v_j\rightharpoonup v,\quad \text{in}\quad H_0^1(\Omega),\quad L^{\frac{2N}{N-2}}(\Omega),\quad L^{2^*(s)}(\Omega,|x|^{-s}\,dx),$$

and (u,v) with $u,v\geqslant 0$ is a solution of (1.1). If (u,v) is nontrivial, by the strong maximum principle, $u,v>0$, the theorem is proved.

In what follows, we shall prove that (u,v) is a nontrivial solution. We will use the blow up argument as the proof of Theorem 1.1. We sketch the proof, the details may be worked out as the proof of Theorem 1.1.

Suppose, on the contrary, that $u=v=0$. Let

$$m_j=u_j(x_j)=\max_{\bar\Omega} u_j(x),\quad n_j=v_j(y_j)=\max_{\bar\Omega} v_j(x),$$

we have either m_j or n_j tends to infinity, we might assume $n_j\leqslant m_j\to\infty$. Set

$$\tilde u_j(y)=m_j^{-1}u_j(k_jy+x_j),\tilde v_j(y)=m_j^{-1}v_j(k_jy+x_j),$$

where

$$k_j=m_j^{-\frac{p_\varepsilon+q-2}{2}}\quad\text{and}\quad p_\varepsilon+q=\frac{2N}{N-2}-\frac{2\varepsilon_j}{2-s}.$$

Then, $(\tilde{u}_j, \tilde{v}_j)$ satisfies (3.5). Using the fact that

$$c_\varepsilon < c_0 < \frac{1}{N} 2^{\frac{2-N}{2}} S_{p,q}^{\frac{N}{2}},$$

we may show as the proof of Theorem 1.1 that $0 < \lim_{j\to\infty} \frac{|x_j|}{k_j} < \infty$, $m_j = O(1)n_j$ and $x_j \to 0$, $y_j \to 0$. Suppose $\frac{x_j}{k_j} \to y_0 \neq 0$, and up to an affine transformation, we see that and $\tilde{u}_j$ and $\tilde{v}_j$ uniformly converge to u and v respectively in compact subsets of $\mathbb{R}^N_+$ with $(u,v) \not\equiv (0,0)$, which satisfies (1.11). Inferring as (3.7), we obtain $c = \lim_{j\to\infty} c_{\varepsilon_j} \geqslant c_0$, which contradicts to the fact $c < c_0$. So (u,v) is a nontrivial solution of (1.1), the proof is complete. □

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