

Research Article

Cyclic Iterative Method for Strictly Pseudononspreading in Hilbert Space

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Let $\{T_i\}_{i=1}^N$ be N strictly pseudononspreading mappings defined on closed convex subset C of a real Hilbert space H . Consider the problem of finding a common fixed point of these mappings and introduce cyclic algorithms based on general viscosity iteration method for solving this problem. We will prove the strong convergence of these cyclic algorithm. Moreover, the common fixed point is the solution of the variational inequality $\langle (\gamma f - \mu B)x^*, v - x^* \rangle \leq 0, \forall v \in \bigcap_{i=1}^N F_{ix}(T_i)$.

1. Introduction

Throughout this paper, we always assume that C is a nonempty, closed, and convex subset of a real Hilbert space H . Let $B : C \rightarrow H$ be a nonlinear mapping. Recall the following definitions.

Definition 1.1. B is said to be

(i) monotone if

$$\langle Bx - By, x - y \rangle \geq 0, \quad \forall x, y \in C, \quad (1.1)$$

(ii) strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C, \quad (1.2)$$

for such a case, B is said to be α -strongly-monotone,

(iii) inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \alpha \|Bx - By\|^2, \quad \forall x, y \in C, \quad (1.3)$$

for such a case, B is said to be α -inverse-strongly monotone,

(iv) k -Lipschitz continuous if there exists a constant $k \geq 0$ such that

$$\|Bx - By\| \leq k \|x - y\| \quad \forall x, y \in C. \quad (1.4)$$

Remark 1.2. Let $F = \mu B - \gamma f$, where B is a k -Lipschitz and η -strongly monotone operator on H with $k > 0$ and f is a Lipschitz mapping on H with coefficient $L > 0$, $0 < \gamma \leq \mu\eta/L$. It is a simple matter to see that the operator F is $(\mu\eta - \gamma L)$ -strongly monotone over H , that is:

$$\langle Fx - Fy, x - y \rangle \geq (\mu\eta - \gamma L) \|x - y\|^2, \quad \forall (x, y) \in H \times H. \quad (1.5)$$

Following the terminology of Browder-Petryshyn [1], we say that a mapping $T : D(T) \subseteq H \rightarrow H$ is

(1) k -strict pseudocontraction if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|x - Tx - (y - Ty)\|^2, \quad \forall x, y \in D(T), \quad (1.6)$$

(2) k -strictly pseudononspreading if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty \rangle, \quad (1.7)$$

for all $x, y \in D(T)$,

(3) nonspreading in [2] if

$$\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C. \quad (1.8)$$

It is shown in [3] that (1.8) is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C. \quad (1.9)$$

Clearly every nonspreading mapping is 0-strictly pseudononspreading. Iterative methods for strictly pseudononspreading mapping have been extensively investigated; see [2, 4–6].

Let C be a closed convex subset of H , and let $\{T_i\}_{i=1}^N$ be n k_i -strictly pseudocontractive mappings on C such that $\bigcap_{i=1}^N F_{ix}(T_i) \neq \emptyset$. Let $x_1 \in C$ and $\{\alpha_n\}_{n=1}^\infty$ be a sequence in $(0, 1)$. In [7], Acedo and Xu introduced an explicit iteration scheme called the following cyclic algorithm

for iterative approximation of common fixed points of $\{T_i\}_{i=1}^N$ in Hilbert spaces. They define the sequence $\{x_n\}$ cyclically by

$$\begin{aligned}x_1 &= \alpha_0 x_0 + (1 - \alpha_0) T_0 x_0; \\x_2 &= \alpha_1 x_1 + (1 - \alpha_1) T_1 x_1; \\&\vdots \\x_N &= \alpha_{N-1} x_{N-1} + (1 - \alpha_{N-1}) T_{N-1} x_{N-1};\end{aligned}\tag{1.10}$$

$$\begin{aligned}x_{N+1} &= \alpha_N x_N + (1 - \alpha_N) T_0 x_N; \\&\vdots\end{aligned}\tag{1.11}$$

In a more compact form, they rewrite x_{n+1} as

$$x_{N+1} = \alpha_N x_N + (1 - \alpha_N) T_N x_n,\tag{1.12}$$

where $T_N = T_i$, with $i = n \pmod{N}$, $0 \leq i \leq N - 1$. Using the cyclic algorithm (1.12), Acedo and Xu [7] show that this cyclic algorithm (1.12) is weakly convergent if the sequence $\{\alpha_n\}$ of parameters is appropriately chosen.

Motivated and inspired by Acedo and Xu [7], we consider the following cyclic algorithm for finding a common element of the set of solutions of k_i -strictly pseudononspreading mappings $\{T_i\}_{i=1}^N$. The sequence $\{x_n\}_{n=1}^\infty$ generated from an arbitrary $x_1 \in H$ as follows:

$$\begin{aligned}x_1 &= \alpha_0 \gamma f(x_0) + (I - \mu \alpha_0 B) T_{\omega_0} x_0; \\x_2 &= \alpha_1 \gamma f(x_1) + (I - \mu \alpha_1 B) T_{\omega_1} x_1; \\&\vdots \\x_N &= \alpha_{N-1} \gamma f(x_{N-1}) + (I - \mu \alpha_{N-1} B) T_{\omega_{N-1}} x_{N-1};\end{aligned}\tag{1.13}$$

$$\begin{aligned}x_{N+1} &= \alpha_N \gamma f(x_N) + (I - \mu \alpha_N B) T_{\omega_0} x_N; \\&\vdots\end{aligned}\tag{1.14}$$

Indeed, the algorithm above can be rewritten as

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n B) T_{\omega_{[n]}} x_n,\tag{1.15}$$

where $T_{\omega_{[n]}} = (I - \omega_{[n]})I + \omega_{[n]}T_{[n]}$, $T_{[n]} = T_{n \pmod{N}}$; namely, $T_{[N]}$ is one of T_1, T_2, \dots, T_N circularly.

2. Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x . $x_n \rightarrow x$ implies that $\{x_n\}$ converges strongly to x . The following definitions and lemmas are useful for main results.

Definition 2.1. A mapping T is said to be demiclosed, if for any sequence $\{x_n\}$ which weakly converges to y , and if the sequence $\{Tx_n\}$ strongly converges to z , then $T(y) = z$.

Definition 2.2. $T : H \rightarrow H$ is called demicontractive on H , if there exists a constant $\alpha < 1$ such that

$$\|Tx - q\|^2 \leq \|x - q\|^2 + \alpha\|x - Tx\|^2, \quad \forall (x, q) \in H \times F_{ix}(T). \quad (2.1)$$

Remark 2.3. Every k -strictly pseudononspreading mapping with a nonempty fixed point set $F_{ix}(T)$ is demicontractive (see [8, 9]).

Remark 2.4 (See [10]). Let T be a α -demicontractive mapping on H with $F_{ix}(T) \neq \emptyset$ and $T_\omega = (1 - \omega)I + \omega T$ for $\omega \in (0, \infty)$:

(A1) $T\alpha$ -demicontractive is equivalent to

$$\langle x - Tx, x - q \rangle \geq \frac{1}{2}(1 - \alpha)\|x - Tx\|^2, \quad \forall (x, q) \in H \times F_{ix}(T), \quad (2.2)$$

(A2) $F_{ix}(T) = F_{ix}(T_\omega)$ if $\omega \neq 0$.

Remark 2.5. According to $I - T_\omega = \omega(I - T)$ with T being a k -strictly pseudononspreading mapping, we obtain

$$\langle x - T_\omega x, x - q \rangle \geq \frac{\omega(1 - k)}{2}\|x - Tx\|^2, \quad \forall (x, q) \in H \times F_{ix}(T). \quad (2.3)$$

Proposition 2.6 (see [2]). Let C be a nonempty closed convex subset of a real Hilbert space H , and let $T : C \rightarrow C$ be a k -strictly pseudononspreading mapping. If $F_{ix}(T) \neq \emptyset$, then it is closed and convex.

Proposition 2.7 (see [2]). Let C be a nonempty closed convex subset of a real Hilbert space H , and let $T : C \rightarrow C$ be a k -strictly pseudononspreading mapping. Then $(I - T)$ is demiclosed at 0.

Lemma 2.8 (see [11]). Let $\{\mathcal{T}_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\mathcal{T}_{n_j}\}_{j \geq 0}$ of $\{\mathcal{T}_n\}$ which satisfies $\mathcal{T}_{n_j} < \mathcal{T}_{n_j+1}$ for all $j \geq 0$. Also consider the sequence of integers $\{\delta(n)\}_{n \geq n_0}$ defined by

$$\delta(n) = \max\{k \leq n \mid \mathcal{T}_k < \mathcal{T}_{k+1}\}. \quad (2.4)$$

Then $\{\delta(n)\}_{n \geq n_0}$ is a nondecreasing sequence verifying $\lim_{n \rightarrow \infty} \delta(n) = \infty$, $\forall n \geq n_0$; it holds that $\mathcal{T}_{\delta(n)} < \mathcal{T}_{\delta(n)+1}$ and we have

$$\mathcal{T}_n < \mathcal{T}_{\delta(n)+1}. \quad (2.5)$$

Lemma 2.9. Let H be a real Hilbert space. The following expressions hold:

- (i) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \quad \forall x, y \in H, \forall t \in [0, 1],$
(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle, \quad \forall x, y \in H.$

Lemma 2.10 (see [6]). Let C be a closed convex subset of a Hilbert space H , and let $T : H \rightarrow H$ be a k -strictly pseudononspreading mapping with a nonempty fixed point set. Let $k \leq \omega < 1$ be fixed and define $T_\omega C \rightarrow C$ by

$$T_\omega(x) = (1-\omega)(x) + \omega T(x), \quad \forall x \in C. \quad (2.6)$$

Then $F_{ix}(T_\omega) = F_{ix}(T)$.

Lemma 2.11. Assume C is a closed convex subset of a Hilbert space H .

- (a) Given an integer $N \geq 1$, assume, $T_i : H \rightarrow H$ is a k_i -strictly pseudononspreading mapping for some $k_i \in [0, 1)$, ($i = 1, 2, \dots, N$). Let $\{\lambda_i\}_{i=1}^N$ be a positive sequence such that $\sum_{i=1}^N \lambda_i = 1$. Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point and $\bigcap_{i=1}^N F_{ix}(T_i) \neq \emptyset$. Then,

$$F_{ix}\left(\sum_{i=1}^N \lambda_i T_i\right) = \bigcap_{i=1}^N F_{ix}(T_i). \quad (2.7)$$

- (b) Assuming $T_i : H \rightarrow H$ is a k_i -strictly pseudononspreading mapping for some $k_i \in [0, 1)$, ($i = 1, 2, \dots, N$), let $T_{\omega_i} = (1-\omega_i)I + \omega_i T_i$, $1 \leq i \leq N$. If $\bigcap_{i=1}^N F_{ix}(T_i) \neq \emptyset$, then

$$F_{ix}(T_{\omega_1} T_{\omega_2} \cdots T_{\omega_N}) = \bigcap_{i=1}^N F_{ix}(T_{\omega_i}). \quad (2.8)$$

Proof. To prove (a), we can assume $N = 2$. It suffices to prove that $F_{ix}(F) \subset F_{ix}(T_1) \cap F_{ix}(T_2)$, where $F = (1-\lambda)T_1 + \lambda T_2$ with $0 < \lambda < 1$. Let $x \in F_{ix}(F)$ and write $V_1 = I - T_1$ and $V_2 = I - T_2$.

From Lemma 2.9 and taking $z \in F_{ix}(T_1) \cap F_{ix}(T_2)$ to deduce that

$$\begin{aligned} \|z-x\|^2 &= \|(1-\lambda)(z-T_1x) + \lambda(z-T_2x)\|^2 \\ &= (1-\lambda)\|z-T_1x\|^2 + \lambda\|z-T_2x\|^2 - \lambda(1-\lambda)\|T_1x-T_2x\|^2 \\ &\leq (1-\lambda)\left(\|z-x\|^2 + k\|x-T_1x\|^2\right) + \lambda\left(\|z-x\|^2 + k\|x-T_2x\|^2\right) \\ &\quad - \lambda(1-\lambda)\|T_1x-T_2x\|^2 \\ &= \|z-x\|^2 + k\left[(1-\lambda)\|V_1x\|^2 + \lambda\|V_2x\|^2\right] - \lambda(1-\lambda)\|V_1x-V_2x\|^2, \end{aligned} \quad (2.9)$$

it follows that

$$\lambda(1-\lambda)\|V_1x-V_2x\|^2 \leq k\left[(1-\lambda)\|V_1x\|^2 + \lambda\|V_2x\|^2\right]. \quad (2.10)$$

Since $(1 - \lambda)V_1x + \lambda V_2x = 0$, we obtain

$$(1 - \lambda)\|V_1x\|^2 + \lambda\|V_2x\|^2 = \lambda(1 - \lambda)\|V_1x - V_2x\|^2. \quad (2.11)$$

This together with (2.10) implies that

$$(1 - k)\lambda(1 - \lambda)\|V_1x - V_2x\|^2 \leq 0. \quad (2.12)$$

Since $0 < \lambda < 1$ and $k < 1$, we get $\|V_1x - V_2x\| = 0$ which implies $T_1x = T_2x$ which in turn implies that $T_1x = T_2x = x$ since $(1 - \lambda)T_1x + \lambda T_2x = x$. Thus, $x \in F_{ix}(T_1) \cap F_{ix}(T_2)$.

By induction, we also claim that $F_{ix}(\sum_{i=1}^N \lambda_i T_i) = \bigcap_{i=1}^N F_{ix}(T_i)$ with $\{\lambda_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \lambda_i = 1$, $(i = 1, 2, \dots, N)$.

To prove (b), we can assume $N = 2$. Set $T_{\omega_1} = (1 - \omega_1)I + \omega_1 T_1$ and $T_{\omega_2} = (1 - \omega_2)I + \omega_2 T_2$, $0 < k_i < \omega_i < 1/2$, $i = 1, 2$. Obviously

$$F_{ix}(T_{\omega_1}) \cap F_{ix}(T_{\omega_2}) \subset F_{ix}(T_{\omega_1} T_{\omega_2}). \quad (2.13)$$

Now we prove

$$F_{ix}(T_{\omega_1}) \cap F_{ix}(T_{\omega_2}) \supset F_{ix}(T_{\omega_1} T_{\omega_2}), \quad (2.14)$$

for all $q \in F_{ix}(T_{\omega_1} T_{\omega_2})$ and $T_{\omega_1} T_{\omega_2} q = q$. If $T_{\omega_2} q = q$, then $T_{\omega_1} q = q$; the conclusion holds. From Lemma 2.10, we can know that $F_{ix}(T_{\omega_1}) \cap F_{ix}(T_{\omega_2}) = F_{ix}(T_1) \cap F_{ix}(T_2) \neq \emptyset$. Taking $p \in F_{ix}(T_{\omega_1}) \cap F_{ix}(T_{\omega_2})$, then

$$\begin{aligned} \|p - q\|^2 &= \|p - T_{\omega_1} T_{\omega_2} q\|^2 = \|p - [(1 - \omega_1)(T_{\omega_2} q) + \omega_1 T_1 T_{\omega_2} q]\|^2 \\ &= \|(1 - \omega_1)(p - T_{\omega_2} q) + \omega_1(p - T_1 T_{\omega_2} q)\|^2 \\ &= (1 - \omega_1)\|p - T_{\omega_2} q\|^2 + \omega_1\|p - T_1 T_{\omega_2} q\|^2 - \omega_1(1 - \omega_1)\|T_{\omega_2} q - T_1 T_{\omega_2} q\|^2 \\ &\leq (1 - \omega_1)\|p - T_{\omega_2} q\|^2 - \omega_1(1 - \omega_1)\|T_{\omega_2} q - T_1 T_{\omega_2} q\|^2 \\ &\quad + \omega_1 \left[\|p - T_{\omega_2} q\|^2 + k_1 \|T_{\omega_2} q - T_1 T_{\omega_2} q\|^2 + 2\langle p - T_1 T_{\omega_2} p, T_{\omega_2} q - T_1 T_{\omega_2} q \rangle \right] \quad (2.15) \\ &= (1 - \omega_1)\|p - T_{\omega_2} q\|^2 - \omega_1(1 - \omega_1)\|T_{\omega_2} q - T_1 T_{\omega_2} q\|^2 \\ &\quad + \omega_1 \left[\|p - T_{\omega_2} q\|^2 + k_1 \|T_{\omega_2} q - T_1 T_{\omega_2} q\|^2 \right] \\ &\leq \|p - T_{\omega_2} q\|^2 - \omega_1(1 - \omega_1 - k_1)\|T_{\omega_2} q - T_1 T_{\omega_2} q\|^2 \\ &\leq \|p - q\|^2 - \omega_1(1 - \omega_1 - k_1)\|T_{\omega_2} q - T_1 T_{\omega_2} q\|^2. \end{aligned}$$

Since $0 < k_1 < \omega_1 < 1/2$, we obtain

$$\|T_{\omega_2} q - T_1 T_{\omega_2} q\|^2 \leq 0. \quad (2.16)$$

Namely, $T_{\omega_2}q = T_1T_{\omega_2}q$, that is:

$$T_{\omega_2}q \in F_{ix}(T_1) = F_{ix}(T_{\omega_1}), \quad T_{\omega_2}q = T_{\omega_1}T_{\omega_2}q. \quad (2.17)$$

By induction, we also claim that the Lemma 2.11(b) holds. \square

Lemma 2.12. *Let K be a closed convex subset of a real Hilbert space H , given $x \in H$ and $y \in K$. Then $y = P_Kx$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in K. \quad (2.18)$$

3. Cyclic Algorithm

In this section, we are concerned with the problem of finding a point p such that

$$p \in \bigcap_{i=1}^N F_{ix}(T_{\omega_i}) = \bigcap_{i=1}^N F_{ix}(T_i), \quad N \geq 1, \quad (3.1)$$

where $T_{\omega_i} = (1 - \omega_i)I + \omega_iT_i$, $\{\omega_i\}_{i=1}^N \in (0, 1/2]$ and $\{T_i\}_{i=1}^N$ are k_i -strictly pseudononspreading mappings with $k_i \in [0, \omega_i)$, ($i = 1, 2, \dots, N$), defined on a closed convex subset C in Hilbert space H . Here $F_{ix}(T_{\omega_i}) = \{q \in C : T_{\omega_i}q = q\}$ is the set of fixed points of T_i , $1 \leq i \leq N$.

Let H be a real Hilbert space, and let $B : H \rightarrow H$ be η -strongly monotone and ρ -Lipschitzian on H with $\rho > 0$, $\eta > 0$. Let $0 < \mu < 2\eta/\rho^2$, $0 < \gamma < \mu(\eta - (\mu\rho^2/2))/L = \tau/L$. Let N be a positive integer, and let $T_i : H \rightarrow H$ be a k_i -strictly pseudononspreading mapping for some $k_i \in [0, 1)$, ($i = 1, 2, \dots, N$), such that $\bigcap_{i=1}^N F_{ix}(T_i) \neq \emptyset$. We consider the problem of finding $p \in \bigcap_{i=1}^N F_{ix}(T_i)$ such that

$$\langle (\gamma f - \mu B)p, v - p \rangle \leq 0, \quad \forall v \in \bigcap_{i=1}^N F_{ix}(T_i). \quad (3.2)$$

Since $\bigcap_{i=1}^N F_{ix}(T_i)$ is a nonempty closed convex subset of H , VI (3.2) has a unique solution. The variational inequality has been extensively studied in literature; see, for example, [12–16].

Remark 3.1. Let H be a real Hilbert space. Let B be a ρ -Lipschitzian and η -strongly monotone operator on H with $\rho > 0$, $\eta > 0$. Letting $0 < \mu < 2\eta/\rho^2$ and letting $S = (I - t\mu B)$ and $\mu(\eta - (\mu\rho^2/2)) = \tau$, then for $\forall x, y \in H$ and $t \in (0, \min\{1, 1/\tau\})$, S is a contraction.

Proof. Consider

$$\begin{aligned} \|Sx - Sy\|^2 &= \|(I - t\mu B)x - (I - t\mu B)y\|^2 \\ &= \langle (I - t\mu B)x - (I - t\mu B)y, (I - t\mu B)x - (I - t\mu B)y \rangle \\ &= \|x - y\|^2 + t^2\mu^2\|Bx - By\|^2 - 2t\mu\langle x - y, Bx - By \rangle \end{aligned}$$

$$\begin{aligned}
&\leq \|x - y\|^2 + t^2 \mu^2 \rho^2 \|x - y\|^2 - 2t\mu\eta \|x - y\|^2 \\
&\leq \left[1 - 2t\mu \left(\eta - \frac{\mu\rho^2}{2} \right) \right] \|x - y\|^2 \\
&= (1 - 2t\tau) \|x - y\|^2 \\
&\leq (1 - t\tau)^2 \|x - y\|^2.
\end{aligned} \tag{3.3}$$

It follows that

$$\|Sx - Sy\| \leq (1 - t\tau) \|x - y\|. \tag{3.4}$$

So S is a contraction. □

Next, we consider the cyclic algorithm (1.15), respectively, for solving the variational inequality over the set of the common fixed points of finite strictly pseudononspreading mappings.

Lemma 3.2. *Assume that $\{x_n\}$ is defined by (1.15); if p is solution of (3.2) with $T : H \rightarrow H$ being strictly pseudononspreading mapping and demiclosed and $\{y_n\} \subset H$ is a bounded sequence such that $\|Ty_n - y_n\| \rightarrow 0$, then*

$$\liminf_{n \rightarrow \infty} \langle (\gamma f - \mu B)p, y_n - p \rangle \leq 0. \tag{3.5}$$

Proof. By $\|Ty_n - y_n\| \rightarrow 0$ and $T : H \rightarrow H$ demi-closed, we know that any weak cluster point of $\{y_n\}$ belongs to $\bigcap_{i=1}^N F_{ix}(T_i)$. Furthermore, we can also obtain that there exists \bar{y} and a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $y_{n_j} \rightharpoonup \bar{y}$ as $j \rightarrow \infty$ (hence $\bar{y} \in \bigcap_{i=1}^N F_{ix}(T_i)$) and

$$\liminf_{n \rightarrow \infty} \langle (\gamma f - \mu B)p, y_n - p \rangle = \lim_{j \rightarrow \infty} \langle (\gamma f - \mu B)p, y_{n_j} - p \rangle. \tag{3.6}$$

From (3.2), we can derive that

$$\liminf_{n \rightarrow \infty} \langle (\gamma f - \mu B)p, y_n - p \rangle = \langle (\gamma f - \mu B)p, \bar{y} - p \rangle \leq 0. \tag{3.7}$$

It is the desired result. In addition, the variational inequality (3.7) can be written as

$$\langle (I - \mu B + \gamma f)p - p, \bar{y} - p \rangle \leq 0, \quad \bar{y} \in \bigcap_{i=1}^N F_{ix}(T_i). \tag{3.8}$$

So, by Lemma 2.12, it is equivalent to the fixed point equation

$$P_{\bigcap_{i=1}^N F_{ix}(T_i)}(I - \mu B + \gamma f)p = p. \quad (3.9)$$

□

Theorem 3.3. *Let C be a nonempty closed convex subset of H and for $1 \leq i \leq N$. Let $T_i : H \rightarrow H$ be k_i -strictly pseudononspreading mappings for some $k_i \in [0, \omega_i)$, $\omega_i \in (0, 1/2)$, ($i = 1, 2, \dots, N$), and $k = \max\{k_i : 1 \leq i \leq N\}$. Let f be L -Lipschitz mapping on H with coefficient $L > 0$, and let $B : H \rightarrow H$ be η -strongly monotone and ρ -Lipschitzian on H with $\rho > 0$, $\eta > 0$. Let $\{\alpha_n\}$ being a sequence in $(0, \min\{1, 1/\tau\})$ satisfying the following conditions:*

$$(c1) \lim_{n \rightarrow \infty} \alpha_n = 0,$$

$$(c2) \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Given $x_0 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by the cyclic algorithm (1.15). Then $\{x_n\}$ converges strongly to the unique element p in $\bigcap_{i=1}^N F_{ix}(T_i)$ verifying

$$p = P_{\bigcap_{i=1}^N F_{ix}(T_i)}(I - \mu B + \gamma f)p, \quad (3.10)$$

which equivalently solves the variational inequality problem (3.2).

Proof. Take a $p \in \bigcap_{i=1}^N F_{ix}(T_i)$. Let $T_{\omega}x = (1 - \omega)x + \omega Tx$ and $0 < k < \omega < 1/2$. Then $\forall x, y \in C$, we have

$$\begin{aligned} \|T_{\omega}x - T_{\omega}y\|^2 &= \omega \|x - y\|^2 + (1 - \omega) \|Tx - Ty\|^2 - \omega(1 - \omega) \|x - Tx - (y - Ty)\|^2 \\ &\leq \omega \|x - y\|^2 + (1 - \omega) \left[\|x - y\|^2 + k \|x - Tx - (y - Ty)\|^2 + 2 \langle x - Tx, y - Ty \rangle \right] \\ &\quad - \omega(1 - \omega) \|x - Tx - (y - Ty)\|^2 \\ &= \|x - y\|^2 + 2(1 - \omega) \langle x - Tx, y - Ty \rangle - (1 - \omega)(\omega - k) \|x - Tx - (y - Ty)\|^2 \\ &\leq \|x - y\|^2 + 2(1 - \omega) \langle x - Tx, y - Ty \rangle \\ &= \|x - y\|^2 + \frac{2(1 - \omega)}{\omega^2} \langle x - T_{\omega}x, y - T_{\omega}y \rangle. \end{aligned} \quad (3.11)$$

From $p \in F_{ix}(T)$ and (3.11), we also have

$$\|T_{\omega}x_n - p\| \leq \|x_n - p\|. \quad (3.12)$$

Using (1.15) and (3.12), we obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n \gamma (f(x_n) - f(p)) + \alpha_n (\gamma f(p) - p) + (I - \mu \alpha_n B)(T_{\omega_{[n]}} x_n - p)\| \\ &\leq \alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - p\| + (1 - \alpha_n \tau) \|x_n - p\|, \end{aligned} \quad (3.13)$$

which combined with (3.12) and $\|f(x_n) - f(p)\| \leq L\|x_n - p\|$ amounts to

$$\|x_{n+1} - p\| \leq (1 - \alpha_n(\tau - \gamma L))\|x_n - p\| + \alpha_n\|\gamma f(p) - p\|. \quad (3.14)$$

Putting $M_1 = \max\{\|x_0 - p\|, \|\gamma f(p) - p\|\}$, we clearly obtain $\|x_n - p\| \leq M_1$. Hence $\{x_n\}$ is bounded. We can also prove that the sequences $\{f(x_n)\}$ and $\{T_{\omega_{[n]}}x_n\}$ are all bounded.

From (1.15) we obtain that

$$x_{n+1} - x_n + \alpha_n(\mu Bx_n - \gamma f(x_n)) = (I - \mu\alpha_n B)(T_{\omega_{[n]}}x_n - x_n), \quad (3.15)$$

hence

$$\begin{aligned} & \langle x_{n+1} - x_n + \alpha_n(\mu Bx_n - \gamma f(x_n)), x_n - p \rangle \\ &= (1 - \alpha_n)\langle T_{\omega_{[n]}}x_n - x_n, x_n - p \rangle + \alpha_n\langle (I - \mu B)(T_{\omega_{[n]}}x_n - x_n), x_n - p \rangle \\ &= (1 - \alpha_n)\langle T_{\omega_{[n]}}x_n - x_n, x_n - p \rangle + \alpha_n\|(I - \mu B)(T_{\omega_{[n]}}x_n - x_n)\|\|x_n - p\| \\ &= (1 - \alpha_n)\langle T_{\omega_{[n]}}x_n - x_n, x_n - p \rangle + \alpha_n(1 - \tau)\|T_{\omega_{[n]}}x_n - x_n\|\|x_n - p\| \\ &= (1 - \alpha_n)\langle T_{\omega_{[n]}}x_n - x_n, x_n - p \rangle + \omega_{[n]}\alpha_n(1 - \tau)\|T_{[n]}x_n - x_n\|\|x_n - p\|. \end{aligned} \quad (3.16)$$

Moreover, by $p \in \bigcap_{i=1}^N \text{Fix}(T_i)$ and using Remark 2.5, we obtain

$$\langle x_n - T_{\omega_{[n]}}x_n, x_n - p \rangle \geq \frac{1}{2}\omega_{[n]}(1 - k_{[n]})\|x_n - T_{[n]}x_n\|^2, \quad (3.17)$$

which combined with (3.16) entails

$$\begin{aligned} \langle x_{n+1} - x_n + \alpha_n(\mu B - \gamma f)x_n, x_n - p \rangle &\leq -\frac{1}{2}\omega_{[n]}(1 - k_{[n]})(1 - \alpha_n)\|x_n - T_{[n]}x_n\|^2 \\ &\quad + \omega_{[n]}\alpha_n(1 - \tau)\|T_{[n]}x_n - x_n\|\|x_n - p\|, \end{aligned} \quad (3.18)$$

or equivalently

$$\begin{aligned} & -\langle x_n - x_{n+1}, x_n - p \rangle \\ &\leq -\alpha_n\langle (\mu B - \gamma f)x_n, x_n - p \rangle - \frac{1}{2}\omega_{[n]}(1 - k_{[n]})(1 - \alpha_n)\|x_n - T_{[n]}x_n\|^2 \\ &\quad + \omega_{[n]}\alpha_n(1 - \tau)\|T_{[n]}x_n - x_n\|\|x_n - p\|. \end{aligned} \quad (3.19)$$

Furthermore, using the following classical equality

$$\langle u, v \rangle = \frac{1}{2}\|u\|^2 - \frac{1}{2}\|u - v\|^2 + \frac{1}{2}\|v\|^2, \quad \forall u, v \in C, \quad (3.20)$$

and setting $\tau_n = 1/2\|x_n - p\|^2$, we have

$$\langle x_n - x_{n+1}, x_n - p \rangle = \tau_n - \tau_{n+1} + \frac{1}{2}\|x_n - x_{n+1}\|^2. \quad (3.21)$$

So (3.19) can be equivalently rewritten as

$$\begin{aligned} \tau_{n+1} - \tau_n - \frac{1}{2}\|x_n - x_{n+1}\|^2 &\leq -\alpha_n \langle (\mu B - \gamma f)x_n, x_n - p \rangle \\ &\quad - \frac{1}{2}\omega_{[n]}(1 - k_{[n]})(1 - \alpha_n)\|x_n - T_{[n]}x_n\|^2 \\ &\quad + \omega_{[n]}\alpha_n(1 - \tau)\|T_{[n]}x_n - x_n\|\|x_n - p\|. \end{aligned} \quad (3.22)$$

Now using (3.15) again, we have

$$\|x_{n+1} - x_n\|^2 = \|\alpha_n(\gamma f(x_n) - \mu Bx_n) + (I - \mu\alpha_n B)(T_{\omega_{[n]}}x_n - x_n)\|^2. \quad (3.23)$$

Since $B : H \rightarrow H$ is η -strongly monotone and k -Lipschitzian on H , hence it is a classical matter to see that

$$\|x_{n+1} - x_n\|^2 \leq 2\alpha_n^2\|\gamma f(x_n) - \mu Bx_n\|^2 + 2(1 - \alpha_n\tau)^2\|T_{\omega_{[n]}}x_n - x_n\|^2, \quad (3.24)$$

which by $\|T_{\omega_{[n]}}x_n - x_n\| = \omega_{[n]}\|x_n - T_{[n]}x_n\|$ and $(1 - \alpha_n\tau)^2 \leq (1 - \alpha_n\tau)$ yields

$$\frac{1}{2}\|x_{n+1} - x_n\|^2 \leq \alpha_n^2\|\gamma f(x_n) - \mu Bx_n\|^2 + (1 - \alpha_n\tau)\omega_{[n]}^2\|x_n - T_{[n]}x_n\|^2. \quad (3.25)$$

Then from (3.22) and (3.25), we have

$$\begin{aligned} &\tau_{n+1} - \tau_n + \omega_{[n]}\left(\frac{1}{2}(1 - k_{[n]})(1 - \alpha_n) - \omega_{[n]}(1 - \alpha_n\tau)\right)\|x_n - T_{[n]}x_n\|^2 \\ &\leq \alpha_n\left(\alpha_n\|\gamma f(x_n) - \mu Bx_n\|^2 - \langle (\mu B - \gamma f)x_n, x_n - p \rangle + \omega_{[n]}(1 - \tau)\|T_{[n]}x_n - x_n\|\|x_n - p\|\right). \end{aligned} \quad (3.26)$$

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists n_0 such that $\{\tau_n\}_{n \geq n_0}$ is nonincreasing. In this situation, $\{\tau_n\}$ is then convergent because it is also nonnegative (hence it is bounded from below), so that $\lim_{n \rightarrow \infty}(\tau_{n+1} - \tau_n) = 0$; hence, in light of (3.26) together with $\lim_{n \rightarrow \infty}\alpha_n = 0$ and the boundedness of $\{x_n\}$, we obtain

$$\lim_{n \rightarrow \infty}\|x_n - T_{[n]}x_n\| = 0. \quad (3.27)$$

It also follows from (3.26) that

$$\begin{aligned} \tau_n - \tau_{n+1} \geq & \alpha_n \left(-\alpha_n \|\gamma f(x_n) - \mu B x_n\|^2 + \langle (\mu B - \gamma f)x_n, x_n - p \rangle \right. \\ & \left. + \omega_{[n]}(1 - \tau) \|T_{[n]}x_n - x_n\| \|x_n - p\| \right). \end{aligned} \quad (3.28)$$

Then, by $\sum_{n=0}^{\infty} \alpha_n = \infty$, we obviously deduce that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \alpha_n \left(-\alpha_n \|\gamma f(x_n) - \mu B x_n\|^2 + \langle (\mu B - \gamma f)x_n, x_n - p \rangle \right. \\ \left. + \omega_{[n]}(1 - \tau) \|T_{[n]}x_n - x_n\| \|x_n - p\| \right) \leq 0, \end{aligned} \quad (3.29)$$

or equivalently (as $\alpha_n \|\gamma f(x_n) - \mu B x_n\|^2 \rightarrow 0$)

$$\liminf_{n \rightarrow \infty} \langle (\mu B - \gamma f)x_n, x_n - p \rangle \leq 0. \quad (3.30)$$

Moreover, by Remark 1.2, we have

$$2(\mu\eta - \gamma L)\tau_n + \langle (\mu B - \gamma f)p, x_n - p \rangle \leq \langle (\mu B - \gamma f)x_n, x_n - p \rangle, \quad (3.31)$$

which by (3.30) entails

$$\liminf_{n \rightarrow \infty} \langle 2(\mu\eta - \gamma L)\tau_n + (\mu B - \gamma f)p, x_n - p \rangle \leq 0, \quad (3.32)$$

hence, recalling that $\lim_{n \rightarrow \infty} \tau_n$ exists, we equivalently obtain

$$2(\mu\eta - \gamma L) \lim_{n \rightarrow \infty} \tau_n + \liminf_{n \rightarrow \infty} \langle (\mu B - \gamma f)p, x_n - p \rangle \leq 0, \quad (3.33)$$

namely

$$2(\mu\eta - \gamma L) \lim_{n \rightarrow \infty} \tau_n \leq -\liminf_{n \rightarrow \infty} \langle (\mu B - \gamma f)p, x_n - p \rangle. \quad (3.34)$$

From (3.27) and Lemma 3.2, we obtain

$$\liminf_{n \rightarrow \infty} \langle (\mu B - \gamma f)p, x_n - p \rangle \geq 0, \quad (3.35)$$

which yields $\lim_{n \rightarrow \infty} \tau_n = 0$, so that $\{x_n\}$ converges strongly to p .

Case 2. Suppose there exists a subsequence $\{\tau_{n_k}\}_{k \geq 0}$ of $\{\tau_n\}_{n \geq 0}$ such that $\tau_{n_k} \leq \tau_{n_{k+1}}$ for all $k \geq 0$. In this situation, we consider the sequence of indices $\{\delta(n)\}$ as defined in Lemma 2.8. It follows that $\tau_{\delta(n+1)} - \tau_{\delta(n)} > 0$, which by (3.26) amounts to

$$\begin{aligned} & \omega_{[n]} \left(\frac{1}{2} (1 - k_{[n]}) (1 - \alpha_{\delta(n)}) - \omega_{[n]} (1 - \alpha_{\delta(n)} \tau) \right) \|x_{\delta(n)} - T_{[n]} x_{\delta(n)}\|^2 \\ & < \alpha_{\delta(n)} \left(\alpha_{\delta(n)} \|\gamma f(x_{\delta(n)}) - \mu B x_{\delta(n)}\|^2 - \langle (\mu B - \gamma f)x_{\delta(n)}, x_{\delta(n)} - p \rangle \right. \\ & \quad \left. + \omega_{[n]} (1 - \tau) \|T_{[n]} x_{\delta(n)} - x_{\delta(n)}\| \|x_{\delta(n)} - p\| \right). \end{aligned} \quad (3.36)$$

By the boundedness of $\{x_n\}$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$, we immediately obtain

$$\lim_{n \rightarrow \infty} \|x_{\delta(n)} - T_{[n]} x_{\delta(n)}\| = 0. \quad (3.37)$$

Using (1.15), we have

$$\begin{aligned} \|x_{\delta(n+1)} - x_{\delta(n)}\| & \leq \alpha_{\delta(n)} \|\gamma f(x_{\delta(n)}) - \mu B x_{\delta(n)}\| + (1 - \alpha_{\delta(n)} \tau) \|T_{\omega_{[n]}} x_{\delta(n)} - x_{\delta(n)}\| \\ & \leq \alpha_{\delta(n)} \|\gamma f(x_{\delta(n)}) - \mu B x_{\delta(n)}\| + \omega_{[n]} (1 - \alpha_{\delta(n)} \tau) \|T_{[n]} x_{\delta(n)} - x_{\delta(n)}\|, \end{aligned} \quad (3.38)$$

which together with (3.37) and $\lim_{n \rightarrow \infty} \alpha_n = 0$ yields

$$\lim_{n \rightarrow \infty} \|x_{\delta(n+1)} - x_{\delta(n)}\| = 0. \quad (3.39)$$

Now by (3.36) we clearly have

$$\begin{aligned} & \alpha_{\delta(n)} \|\gamma f(x_{\delta(n)}) - \mu B x_{\delta(n)}\|^2 + \omega_{[n]} (1 - \tau) \|T_{[n]} x_{\delta(n)} - x_{\delta(n)}\| \|x_{\delta(n)} - p\| \\ & \geq \langle (\mu B - \gamma f)x_{\delta(n)}, x_{\delta(n)} - p \rangle, \end{aligned} \quad (3.40)$$

which in the light of (3.31) yields

$$\begin{aligned} & 2(\mu\eta - \gamma L)\tau_{\delta(n)} + \langle (\mu B - \gamma f)p, x_{\delta(n)} - p \rangle \\ & \leq \alpha_{\delta(n)} \|\gamma f(x_{\delta(n)}) - \mu B x_{\delta(n)}\|^2 + \omega_{[n]} (1 - \tau) \|T_{[n]} x_{\delta(n)} - x_{[n]}\| \|x_{\delta(n)} - p\|, \end{aligned} \quad (3.41)$$

hence (as $\lim_{n \rightarrow \infty} \alpha_{\delta(n)} \|\gamma f(x_{\delta(n)}) - \mu B x_{\delta(n)}\|^2 = 0$ and (3.37)) it follows that

$$2(\mu\eta - \gamma L) \limsup_{n \rightarrow \infty} \tau_{\delta(n)} \leq -\liminf_{n \rightarrow \infty} \langle (\mu B - \gamma f)p, x_{\delta(n)} - p \rangle. \quad (3.42)$$

From (3.37) and Lemma 3.2, we obtain

$$\lim_{n \rightarrow \infty} \langle (\mu B - \gamma f)p, x_{\delta(n)} - p \rangle \geq 0, \quad (3.43)$$

which by (3.42) yields $\limsup_{n \rightarrow \infty} \mathcal{T}_{\delta(n)} = 0$, so that $\lim_{n \rightarrow \infty} \mathcal{T}_{\delta(n)} = 0$. Combining (3.39), we have $\lim_{n \rightarrow \infty} \mathcal{T}_{\delta(n)+1} = 0$. Then, recalling that $\mathcal{T}_n < \mathcal{T}_{\delta(n)+1}$ (by Lemma 2.8), we get $\lim_{n \rightarrow \infty} \mathcal{T}_n = 0$, so that $x_n \rightarrow p$ strongly. \square

Taking $k_i = 0$, we know that k_i -strictly pseudononspreading mapping is nonspreading mapping and $i = n \pmod{N}$, $0 \leq i \leq N - 1$. According to the proof Theorem 3.3, we obtain the following corollary.

Corollary 3.4. *Let C be a nonempty closed convex subset of H . Let $T_i : C \rightarrow C$ be nonspreading mappings and $\omega_i \in (0, 1/2)$, ($i = 1, 2, \dots, N$). Let f be L -Lipschitz mapping on H with coefficient $L > 0$ and let $B : H \rightarrow H$ be η -strongly monotone and ρ -Lipschitzian on H with $\rho > 0$, $\eta > 0$. Let $\{\alpha_n\}$ be a sequence in $(0, \min\{1, 1/\tau\})$ satisfying the following conditions:*

- (c1) $\lim_{n \rightarrow \infty} \alpha_n = 0$,
- (c2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Given $x_0 \in C$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence generated by the cyclic algorithm (1.15). Then $\{x_n\}$ converges strongly to the unique element p in $\bigcap_{i=1}^N \text{Fix}(T_i)$ verifying

$$p = P_{\bigcap_{i=1}^N \text{Fix}(T_i)}(I - \mu B + \gamma f)p, \quad (3.44)$$

which equivalently solves the variational inequality problem (3.2).

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