

*Selection 1(2000)1–3, 57–66*  
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## On the Volunteer Dilemma I: Continuous-time Decision

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(Received: 9 February 2000,

Accepted in revised form: 4 July 2000)

It is assumed that there is a group of unrelated individuals taken at random from a large population which is exposed to the same time-continuous threat of dying. Accumulated loss of each player increases as the game goes on until at least one participant volunteers to take some extra risk on its own. The risk is taken by a volunteer in order to stop the threat may or may not depend on the time of volunteering.

This situation can be modeled as an  $n$ -player War of Attrition, which ends when one of the players volunteers. We called this sort of generalization, “The ( $n$ -player) volunteer dilemma”. Indeed, a two-player volunteer dilemma is equivalent to the original War of Attrition. It was further assumed that both the risk for the volunteer and the intensity of the risk of waiting are time dependent according to some integrable function, this instead of being constants as assumed in the original War of Attrition model of Maynard Smith.

Necessary and sufficient conditions for a strategy to be a Nash strategy are given. This strategy is characterized by a time-intensity of volunteering. In the stationary case the Nash strategy is proven to be ESS.

**Keywords:** Volunteering, ESS, War of Attrition, altruism, population games

### 1. Introduction

The War of Attrition was first introduced by Maynard Smith (1974). In the original model, two members of a population both wish to obtain the same object. Each one of them makes a sealed bid. The higher bidder wins and both must pay the low bid.

Maynard Smith conjectured that the exponential mixed strategy of bidding between  $x$  and  $x + \varepsilon$  with probability  $(1/V) \cdot e^{-x/V} \cdot \varepsilon + o(\varepsilon)$ , where  $V$  is the object value, will satisfy his requirements for evolutionary stability (see Maynard Smith and Price, 1973). This was formally demonstrated by Bishop and Cannings (1978a, b). Bishop and Cannings also pointed out that the War of Attrition model can be applied in a wide range of contexts.

Haigh and Michael (1980) introduced the model for Evolutionary Game Auctions, which is a possible generalization of the War of Attrition model. In this model, if  $x$  and  $y$  ( $x > y \geq 0$ ) represent the values the contestants are prepared to bid, then the potential

$x$ -player gains  $V$ , and pays the amount  $y + f(x - y)$ , where  $f(u)$  is the overshoot cost function. The extreme cases  $f(u) = 0$  and  $f(u) = u$  give us the War of Attrition and the Scotch Auction models, respectively. General results are found for different classes of overshoot cost functions.

The asymmetric model for the War of Attrition is discussed by Hammerstein and Parker (1982). A mixed evolutionarily stable strategy (a pair of strategies for each role) is found.

Milgrom and Weber (1985) investigated a symmetric model with  $n$  players, where each player's privately known valuations of the prize, measured in terms of the maximum stopping time that makes the contest worth the prize, are independent.

If the valuations are  $t_1 \geq t_2 \geq \dots \geq t_n$ , then the first player (i.e. the one with the highest valuation) obtains the payoff  $t_1 - t_2$  whereas all other players obtain  $-t_k$ ;  $k = 2, 3 \dots n$ .

This extension of the War of Attrition model, though natural, is only one of many possible. For example, the game may be stopped when one of the contestants drops out. These two extreme cases are, of course, identical in the case of  $n = 2$ . The latter extension can be also interpreted as a Volunteer Di-

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lemma. The following example from Dawkins (1976) will make this point clear: "Perhaps we can sympathize more directly with the reported cowardly behaviour of emperor penguins in the Antarctic. They have been seen standing on the brink of the water hesitating before diving in, because of the danger of being eaten by seals. If only one of them would dive in, the rest would know whether there was a seal there or not. Naturally nobody wants to be the guinea-pig, so they wait, and sometimes even try to push each other in."

This situation, described by Dawkins, can be modeled as an  $n$ -player War of Attrition, which ends when one of the players volunteers. We call this sort of a generalized war of attrition *The ( $n$ -player) volunteer dilemma*. Indeed, a two-player volunteer dilemma is equivalent to the original war of attrition. In this work we assume, for simplicity, that both the cost of volunteering and that of waiting are measured in terms of survival probability.

For the general case of the volunteer dilemma we further assume that both the risk for the volunteer and the intensity of the risk of waiting may change in time according to some integrable function, known to the players. This instead of being necessarily constants as assumed in the original model of Maynard Smith (indeed, the penguins' risk of dying of starvation at any given unit of time is most unlikely to remain constant). It is assumed, though, that the integral over time of the intensity of the risk of waiting is finite over any finite interval of time but it tends to infinity as time tends to infinity. This means that if no one ever volunteers, all will die in probability one. Assuming further that the risk of volunteering is always positive but never one, it follows that if at least one player ever volunteers, all participants of the game have a positive probability to survive. However, the survival probability of the volunteer is always lower than that of his non-volunteering companions.

Because of this last observation it may look, as mentioned by Dawkins for the case of the penguins, as if waiting for others to volunteer is always a rational strategy to follow; but is it really so? Clearly, if all follow this suggested pattern of behavior, all will starve to death sooner or later. In such a case it will be advantageous for any single player to take the risk and volunteer immediately, hence non-volunteering cannot possibly be an ESS or, for this matter,

even just a weak response against itself, say a Nash population strategy. But, indeed, so is immediate volunteering because, once adopted by the entire population, it becomes disadvantage against any sort of defection. In this work we characterize the Nash population strategies of the general volunteer dilemma and characterize special conditions under which these Nash strategies are shown to be ESSs.

We first analyse a special case, namely that of *the pending catastrophe*, in which death events of non-volunteering individuals, if occur, occur simultaneously. This case is relatively easier to analyse because the assumption of the simultaneous death event guarantees that the number of players  $n$  will remain fixed to the end of the game. We then take over the general case in which no specific assumption is made about how the death events of the various non-volunteering players depend on each other. This seems a much more complicated situation to cope with not only because of its generality but also because of the fact that when death events are not synchronized, the number of participants is ever decreasing as players die out during the game. This inevitable fact appears to require that each of the players, comparing his chance of survival if he volunteers versus his chance of surviving in the future if he waits, must take into consideration the possibility that in the latter case he will lose part of his companions.

Quite surprisingly, we shall see that this is generally not the case. Under a plausible assumption it is shown that with all the players following the Nash population strategy, the survival probability of each player turns out to be independent of the number of other players, participating in the game. This is so because, as the number of players increases, the Nash tendency to volunteer decreases accordingly. As a result, we shall see that even when the number of participants in the game is not constant, the individual players should be impartial to the death of their companions.

This result, obtained for non-kin players is contrary to a previous result obtained for a game among relatives where everyone seeks to increase his own inclusive fitness (Motro and Eshel, 1988; Eshel and Motro, 1988). In this case it has been shown that the survival probability of the participants is a decreasing function of their number. The loss in inclusive

fitness, due to the death of a relative, may then be partially compensated by the resulting increase in one's survival probability in the future.

For a different model with some similarity to the volunteer's dilemma, the reader is referred to the "vigilance game" (Pulliam et al., 1982).

## 2. The model

Let us consider a symmetric population-game in which  $n$  non-related individuals (say players), taken at random from a large population, are exposed to a permanent time-continuous threat. This threat comes to an end (in which case we say that the game is over) only when one of the following two events occurs: Either all players die out, or at least one of the players volunteers to take over some extra risk  $D$ .

$1 \geq D \geq 0$  and to avoid triviality we assume strict inequalities.  $D$  may or may not depend on the time  $t$  ( $t \geq 0$ ) of volunteering.

As long as the game continues it is assumed that at any short time interval  $[t, t+h)$  ( $t \geq 0, h > 0$ ), each individual has a probability  $h \cdot s_k(t) + o(h)$  of dying where  $k$  ( $k > 0$ ) is the number of players still surviving at time  $t$ . Death events of different players in the group may or may not be independent. For mathematical definiteness we assume, generally, that  $D = D(t)$  and  $s = s_n(t)$  are smooth functions of  $t$ . We further assume

$$\int_0^{\infty} s_n(t) dt = \infty \quad (2.1)$$

for all  $n = 1, 2, \dots$ . This precludes the possibility that the game will continue forever: If nobody ever volunteers, all players should eventually die for sure.

We assume that the payment function which each individual seeks to maximize is its survival probability to the end of the game.

A pure strategy in the population game is a set of volunteering times  $\{T_n\}_{n=1}^{\infty}$ , such that the player volunteers at  $T_n$  if at that time the group size is  $n$  and if the game is not yet over. We define  $T_n = \infty$  a decision never to volunteer if the group size is  $n$ . A mixed strategy is a set  $\{F_n(t)\}_{n=1}^{\infty}$  of semi-distributions such that  $F_n(t) = P(T_n \leq t)$ ,  $n = 1, 2, \dots; t \geq 0$ .

This defines a  $n$ -player non-cooperative game, i.e. a game in which there are no possibilities for communication, correlation or precommitment.

If, for any  $n$ ,  $F_n(t)$  is continuous with density  $f_n(t) = F'_n(t)$ , we say that  $\{F_n(t)\}_{n=1}^{\infty}$  is a fully mixed strategy.

We are interested in strategies which are evolutionary stable or at least Nash solution of this population game. Under the assumption of the model, it is easy to see that no strategy with the Nash property can have an atom at any time  $t_0$ . Indeed, one can easily see that if the entire population has a  $p$ -atom at the time  $t_0$ , then any single player can strictly gain by slightly shifting his  $p$ -atom to the right, thus giving the other a positive chance to volunteer before him.

Without loss of generality we, therefore, restrict our attention to strategies  $F_n(t)$  with a density  $f_n(t) = F'_n(t)$ .

For any strategy  $F_n(t)$  with a density-distribution  $f_n(t) = F'_n(t)$  and for any  $t > 0$  such that  $F_n(t) < 1$  it is convenient to define

$$\lambda_n(t) = \frac{f_n(t)}{1 - F_n(t)} = \frac{f_n(t)}{\int_t^{\infty} f_n(x) dx}. \quad (2.2)$$

The set  $\lambda = \{\lambda_n(t)\}_{n=1}^{\infty}$  is the volunteering intensities of the individual player for any moment  $t \geq 0$  and group size  $n = 1, 2, 3, \dots$

Thus, a fully-mixed strategy can more easily be given by the set of functions  $\lambda = \{\lambda_n(t)\}_{n=1}^{\infty}$  where  $\lambda_n(t) \cdot h + o(h)$  is the probability of volunteering during the short-time interval  $[t, t+h)$ , when the number of players in the encounter equals  $n$  and given that the game is not yet over.

Here, in ESS we mean a strategy  $\lambda = \{\lambda_n(t)\}_{n=1}^{\infty}$  such that for any alternative strategy  $\mu = \{\mu_n(t)\}_{n=1}^{\infty}$  there exists a value  $\delta > 0$  so that if a minority  $\varepsilon$  of the entire population, where  $0 < \varepsilon < \delta$ , is playing  $\mu$  and the rest  $1 - \varepsilon$  of the population in playing  $\lambda$ , there is an average advantage, over all encounters, to  $\lambda$ -players over  $\mu$ -players (Hamilton, 1967; Maynard Smith and Price, 1973).

We denote by  $V_n(\lambda | \lambda, \mu, \varepsilon)$  the average payoff to the  $\lambda$ -player in a random  $n$ -player encounter, given that a frequency  $\varepsilon > 0$  of all individuals in this population choose the strategy  $\mu$  and  $1 - \varepsilon$  choose the strategy  $\lambda$ . A strategy  $\{\lambda\}$  is an ESS iff for all  $n = 2, 3, 4, \dots$

$$V_n(\lambda | \lambda, \mu, \varepsilon) > V_n(\mu | \lambda, \mu, \varepsilon) \quad (2.3)$$

holds for an  $\varepsilon$  small enough.



- (iii) There is a probability  $s_n(t)h + o(h)$  for the simultaneous death event during the time-interval  $[t, t + h]$ .
- (iv) There is a probability  $1 - [(n - k)\lambda(t) + k\mu(t) + s_n(t)]h + o(h)$  that the game will continue to the time  $t + h$ , at which time each participant will have a conditioned probability  $\phi_{t+h}(\lambda | k \bullet \mu, (n - k) \bullet \lambda)$  to survive to the end of the game.

We therefore get:

$$\begin{aligned} \phi_t(\lambda | k \bullet \mu, (n - k - 1) \bullet \lambda) &= \lambda_n(t)(1 - D(t))h + \\ &+ (n - k - 1)\lambda_n(t)h + k\mu_n(t)h \\ &+ \{1 - [(n - k)\lambda_n(t) + k\mu_n(t) + \\ &+ s_n(t)]h\}\phi_{t+h}(\lambda | k \bullet \mu, (n - k - 1) \bullet \lambda) + \\ &+ o(h). \end{aligned} \quad (3.1)$$

As  $h$  tends to 0, (3.1) yields the differential equation

$$\begin{aligned} \phi'(\lambda | k \bullet \mu, (n - k - 1) \bullet \lambda) &= \\ &= [(n - k)\lambda_n(t) + k\mu_n(t) + s_n(t)]\phi_t \\ &\quad (\lambda | k \bullet \mu, (n - k - 1) \bullet \lambda) \\ &- (n - k - D(t))\lambda_n(t) - k\mu_n(t). \end{aligned} \quad (3.2)$$

Where the prime stands for a derivation in respect to  $t$ .

**Theorem 3.1.**  $\lambda_n(t)$  is a Nash strategy if and only if

$$\lambda_n(t) = \begin{cases} \frac{D'(t) + s_n(t)[1 - D(t)]}{(n - 1)D(t)} & s_n(t) > [\ln(1 - D(t))]' \\ 0 & s_n(t) > [\ln(1 - D(t))]' \end{cases}. \quad (3.3)$$

**Proof.**

- (i) Suppose  $\lambda$  is a Nash strategy, then for an strategy  $\mu$ ,  $\lambda$  must satisfy the inequality

$$\phi_t(\lambda | (n - 1) \bullet \lambda) \geq \phi_t(\mu | (n - 1) \bullet \lambda). \quad (3.4)$$

As a special case this is true for any strategy  $\mu$  such that

$$\mu_n(t) = \begin{cases} \lambda_n(\tau) & \tau \in [0, t) \cup [t + h, \infty) \\ \lambda_n(\tau) + c & \tau \in [t, t + h) \end{cases}, \quad (3.5)$$

where  $h > 0$  and  $c$  is any constant different from zero.

But for such a strategy  $\mu$  we know that, by definition,  $\mu = \lambda$  on  $[t + h, \infty)$ , hence

$$\phi_{t+h}(\lambda | (n - 1) \bullet \lambda) = \phi_{t+h}(\mu | (n - 1) \bullet \lambda). \quad (3.6)$$

From (3.1), (3.5) and (3.6) it, therefore, follows by straightforward calculation that

$$\begin{aligned} \phi_t(\lambda | (n - 1) \bullet \lambda) - \phi_t(\mu | (n - 1) \bullet \lambda) &= \\ &= \{\phi_t(\lambda | (n - 1) \bullet \lambda) - (1 - D(t))\} \cdot c \cdot h + o(h) \end{aligned}$$

and with the Nash condition (3.4) this implies

$$\{\phi_t(\lambda | (n - 1) \bullet \lambda) - (1 - D(t))\} \cdot c \geq 0 \quad (3.7)$$

for any constant  $c$ .

We now distinguish between two different cases:

1. If  $\lambda_n(t) \equiv 0$  on the interval  $[t, t + h)$ , then the constant  $c$  must be positive and therefore the Nash condition becomes:

$$\phi_t(\lambda | (n - 1) \bullet \lambda) \geq 1 - D(t). \quad (3.8)$$

2. If  $\lambda_n(t) > 0$  on  $[t, t + h)$  then the constant  $c$  may be either positive or negative. In this case (3.8) can hold, both directions, only as an equality, hence the Nash condition for  $\lambda_n(t)$  becomes:

$$\phi_t(\lambda | (n - 1) \bullet \lambda) = 1 - D(t). \quad (3.9)$$

Now, as a special case of (3.2) we get:

$$\begin{aligned} \phi'_t(\lambda | (n - 1) \bullet \lambda) &= (n\lambda(t) + s_n(t))\phi_t(\lambda | (n - 1) \bullet \lambda) - \\ &- (n - D(t))\lambda_n(t). \end{aligned} \quad (3.10)$$

If  $\lambda_n(t) > 0$  we now insert (3.9) in (3.10) to get:

$$-D'(t) = [n\lambda_n(t) + s_n(t)](1 - D(t)) - [n - D(t)]\lambda_n(t). \quad (3.11)$$

Hence:

$$\lambda_n(t) = \frac{D'(t) + s_n(t)[1 - D(t)]}{(n - 1)D(t)}. \quad (3.12)$$

But the right-hand side of (3.12) is positive if and only if  $s_n(t) > \frac{-D'(t)}{1 - D(t)} = [\ln(1 - D(t))]'$ . If, on the

other hand  $s_n(t) \leq \frac{-D'(t)}{1 - D(t)}$ , then no Nash strategy

with  $\lambda_n(t) > 0$  can possibly exist and the only possi-



ble Nash strategy, if exists, must be  $\lambda_n(t) = 0$ . We have, thus, proved that if  $\lambda_n(t)$  is a Nash strategy, than it must be given by (3.3).

(ii) We now suppose that  $\lambda$  is given by (3.3) and show it is a Nash strategy. Equivalently, we have to show that inequality (3.4) holds for all  $t$  and for any alternative strategy  $\mu_n(t)$ . From (3.10), after straightforward calculations, we get:

$$\begin{aligned} & \phi'_t(\mu|(n-1) \bullet \lambda) - \phi'_t(\lambda|(n-1) \bullet \lambda) = \\ & = [(n-1)\lambda_n(t) + s_n(t) + \mu_n(t)] \cdot \\ & \cdot [\phi_t(\mu|(n-1) \bullet \lambda) - \phi_t(\lambda|(n-1) \bullet \lambda)]. \end{aligned} \quad (3.13)$$

Denote:

$$\phi_t(\mu|(n-1) \bullet \lambda) - \phi_t(\lambda|(n-1) \bullet \lambda) = y(t) \quad (3.14)$$

(3.13) becomes:

$$\begin{aligned} y'(t) & = \\ & = [(n-1)\lambda_n(t) + s_n(t) + \mu_n(t)] \cdot y(t). \end{aligned} \quad (3.15)$$

This is a simple linear differential equation in  $y$  and as  $(n-1)\lambda_n(t) + s_n(t) + \mu_n(t) \geq 0$ , the sign of  $y'(t)$  is always equal to that of  $y(t)$  hence  $y(t)$ , if different from zero at one time should always be different from zero and, moreover, it cannot possibly change its sign. Equivalently, if (3.4) does not always hold and for some  $\mu$ , at some moment

$$\phi_t(\mu|(n-1) \bullet \lambda) > \phi_t(\lambda|(n-1) \bullet \lambda),$$

then this inequality should hold for all  $t > 0$ . But if  $\lambda$  is a Nash strategy, we already know that  $\phi_t(\lambda|(n-1) \bullet \lambda) = 1 - D(t)$ , hence, confronting  $n-1$   $\lambda$ -players, one cannot possibly increase his survival probability by increasing his probability of volunteering, in which case he will have just a probability  $1 - D(t)$  to survive. On the other hand, if one can increase his survival probability by decreasing his probability of volunteering, than he clearly can do better by decreasing it to zero as one's long-term survival probability is a linear function of his intensity of volunteering at any short-time interval. It is, therefore sufficient to show that (3.4) holds for  $t = 0$  and  $\mu_n(t) \equiv 0$ . Denote by

$$r(t) = (n-1)\lambda_n(t) + S_n(t) \quad (3.16)$$

the intensity of the event that either one of the players volunteers or all die.

Let  $\lambda$  satisfy (3.3) and let us assume first that  $\lambda_n(t) > 0$  for all  $t \geq 0$ . Thus:

$$\lambda_n(t) = \frac{D'(t) + S_n(t)[1 - D(t)]}{(n-1)D(t)}$$

and therefore:

$$r(t) = \frac{D'(t) + S_n(t)}{D(t)}. \quad (3.17)$$

We get:

$$\begin{aligned} \int_0^t r(u) du & = \int_0^t \frac{D'(u) + S_n(u)}{D(u)} du = \\ & = I_n D(t) - I_n D(0) + \int_0^t \frac{S_n(u)}{D(u)} du \end{aligned} \quad (3.18)$$

The distribution function of the length of the game is:

$$F(t) = 1 - e^{-\int_0^t r(u) du} = 1 - \frac{D(0)}{D(t)} e^{-\int_0^t \frac{S_n(u)}{D(u)} du} \quad (3.19)$$

Hence, the total death probability of a non-volunteering individual is

$$\begin{aligned} \int_0^\infty [1 - F(t)] S_n(t) dt & = D(0) \int_0^\infty \frac{S(t)}{D(t)} e^{-\int_0^t \frac{S_n(u)}{D(u)} du} dt = \\ & = D(0) \int_0^\infty \frac{d}{dt} \left\{ e^{-\int_0^t \frac{S_n(u)}{D(u)} du} \right\} dt = \\ & = D(0) \left[ 1 - e^{-\int_0^\infty \frac{S_n(t)}{D(t)} dt} \right]. \end{aligned} \quad (3.20)$$

But  $\frac{S_n(t)}{D(t)} \geq S_n(t)$ , hence it follows from (2.1) that

$$\int_0^\infty \frac{S_n(t)}{D(t)} dt = \infty \quad (3.21)$$

and the right-hand side of (3.20) becomes  $D(0)$ . Hence, if  $\lambda_n(t) > 0$  is a NASH strategy and  $\mu \equiv 0$ , we get

$$\phi_0(\mu|(n-1) \bullet \lambda) = 1 - D(0) = \phi_0(\lambda|(n-1) \bullet \lambda) \quad (3.22)$$

and from (3.15) we infer that

$$\phi_t(\mu|(n-1) \bullet \lambda) = \phi_t(\lambda|(n-1) \bullet \lambda).$$

This remains true if at some interval  $\lambda_n(t) = 0 = \mu(t)$ , hence a strategy  $\lambda_n(t)$  that satisfies (3.3) must be a Nash strategy. This completes the proof of Theorem 3.1.  $\square$

**Remark.** Note, though, that if  $\lambda_n(t) > 0$  is a Nash strategy, then it is a weak Nash strategy in the sense that if it is followed by all players except for one, then the survival probability of this one is independent of the choice of his own strategy. This may not be true, at least locally for the case where  $\lambda_n(t) = 0$ . It is not surprising that if the risk  $D(t)$  of volunteering is decreasing fast enough at the vicinity of  $t$  and if the local risk of waiting  $s_n(t)$  is relatively small, then it is strictly disadvantageous to volunteer at the time-vicinity of  $t$ , regardless of what other players do. Indeed, in such a case it would have been advantageous to postpone at least a little bit the moment of volunteering, thereby decreasing the risk of volunteering without adding too much to the risk of dying while waiting. On the other hand, we see that if  $D(t)$  is not decreasing fast enough and if  $s_n(t)$  is not small enough as to satisfy  $s_n(t) > \frac{-D'(t)}{1-D(t)}$ , then the only

Nash strategy of the game requires a positive rate of volunteering at  $t$ .

**Corollary.** *The survival probability of a group of players, fixed on the Nash population strategy is independent of the group size.*

### Theorem 3.2.

When the risk  $D(t)$  of volunteering is not decreasing in time (or, more generally, when  $s_n(t) > \frac{-D'(t)}{1-D(t)} = [\ln(1-D(t))]'$ ), then the Nash intensity of volunteering is always positive; Moreover, it then adjusts itself so that the players' survival probability at any moment remains  $1-D(t)$ , regardless of the intensity  $s_n(t)$  of the risk of waiting.

**Proof.** Immediate from (3.9) and Theorem 3.1.

### The stationary case

If  $D(t) = D$  and  $s_n(t) = s_n$ , (3.3) becomes

$$\lambda = \frac{s_n(1-D)}{(n-1)D}. \quad (3.23)$$

**Theorem 3.3.** *In the stationary case the strategy  $\lambda_n$ , given by (3.23), is an ESS.*

**Proof.** In the stationary case  $\phi_t = \phi$ . Substituting this in (3.1) one gets:

$$\begin{aligned} \phi(\lambda|k \bullet \mu, (n-k-1) \bullet \lambda) &= \\ &= \frac{\lambda(1-D) + (n-k-1)\lambda + k\mu}{(n-k)\lambda + k\mu + s_n} \end{aligned} \quad (3.24)$$

and similarly,

$$\begin{aligned} \phi(\mu|k \bullet \mu, (n-k-1) \bullet \lambda) &= \\ &= \frac{\mu(1-D) + (n-k-1)\lambda + k\mu}{(n-k-1)\lambda + (k+1)\mu + s_n}. \end{aligned} \quad (3.25)$$

In order to prove that  $\lambda = \frac{s_n(1-D)}{(n-1)D}$  is an ESS, we

must show that it obeys the (2.6) series of conditions. As we have seen, the condition (2.6.1) is always satisfied. Thus, it will be sufficient to show that (2.6.2) is fulfilled as strong inequality, i.e. for all  $t \geq 0$

$$\phi(\lambda|\mu, (n-2) \bullet \lambda) > \phi(\mu|\mu, (n-2) \bullet \lambda). \quad (3.26)$$

From (3.24) and (3.25) we get that (3.26) is equivalent to the following:

$$\begin{aligned} \frac{\lambda(1-D) + (n-2)\lambda + \mu}{(n-1)\lambda + \mu + s_n} > \\ > \frac{\mu(1-D) + (n-2)\lambda + \mu}{(n-2)\lambda + 2\mu + s_n}. \end{aligned} \quad (3.27)$$

This is equivalent to

$$(\mu - \lambda) \{D(n-2)\lambda + D\mu - (1-D)s_n\} > 0. \quad (3.28)$$

However, from (3.23) it follows that  $(1-D)s_n = (n-1)D\lambda$ . Inserting this in (3.28), the left side of it becomes  $D(\mu - \lambda)^2$  which is strictly positive for all  $\mu \neq \lambda$  as long as  $D_n \neq 0$ .

## 4. The general case

We now assume, most generally, that if at the moment  $t$ , there are  $n$  participants present at the game and if, within the short-time interval  $[t, t+h)$  none of them volunteers, then there is a probability

$$hr_{n,n-k}(t) + o(h) \quad (4.1)$$

that  $k$  out of them would die within this time interval,

$$k = 1, 2, 3, \dots, n.$$

If  $k < n$ , the game would then continue with  $n - k$  participants.

The function  $r_{n,k-k}$  will, therefore, be called the *transition-intensities* of the game. It is further assumed that once a death-event of  $k$  participants occur, the conditional death probability of each participant is  $\frac{k}{n}$ . For each participant, the intensity of the

death event is therefore:

$$S_n(t) = \sum_{k=1}^n r_{n,n-k}(t). \quad (4.2)$$

Examples:

(i) The case of the pending catastrophe

$$r_{n,n-k}(t) = \begin{cases} S_n(t) & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}.$$

(ii) The case of independent death events

$$r_{n,n-k}(t) = \begin{cases} nS_n(t) & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}.$$

A general strategy  $\underline{\lambda} = \{\lambda_k(t)\}_{k=1}^n$  of a player, in such a case, should determine his behaviour at any moment, given that nobody has volunteered before and that exactly  $k$  out of the original  $n$  participants have still remained.

Apparently one may expect that in such a case a decision to volunteer should be affected not only by the direct individual risk functions  $S_n(t)$  and  $D(t)$ , but also by increase (or decrease) of the individual risk due to the loss of one's partners (see for comparison, Eshel et al., 1994; Eshel and Shaked, 2000).

For example, if for any  $t \geq 0$ , the individual intensity of death  $S_k(t)$  increases as  $k$  decreases (i.e. if one's survival depends on the presence of other players), and if for some  $1 \leq k \leq n$   $r_{n,n-k}(t) > 0$ , then the incentive to volunteer should increase. We see, though, that under plausible conditions, this is not true. Quite surprisingly, then, the intensity  $\lambda_n(t)$  of volunteering at Nash equilibrium with the presence of  $n$  participants at moment  $t$ , depends only on the risk functions  $S_n(t)$  and  $D(t)$ , regardless of either  $\{r_{n,n-k}(t)\}_{k=1}^n$  or  $\{S_k(t)\}_{k=1}^{n-1}$ . In such a case,  $\lambda_n(t)$  is given by the equality (3.3), obtained for the special case of the pending catastrophe.

**Theorem 4.1.** Assume any set of transition intensities  $\{r_{n,n-k}(t)\}$  such that for all  $t \geq 0$  and for all  $n = 1, 2, 3, \dots$

$$S_n(t) > \frac{-D'(t)}{1-D(t)} \quad (4.3)$$

then  $\underline{\lambda} = \{\lambda_n(t)\}$  is a global Nash strategy iff for all  $n = 2, 3, 4, \dots$   $\lambda_n(t)$  is given by (3.3) and  $\lambda_1(t) = \infty$  is the strategy of immediate volunteering. Moreover, if all participants follow this strategy, then the survival probability of each participant is  $1 - D(t)$ , regardless of the number of players at a given moment.

**Proof.** Induction. Indeed, the theorem holds for  $n = 1$ , in which case condition (4.3) readily indicates that an immediate volunteering should yield a single participant with his best survival probability  $1 - D(t)$ . Now assume that for a given  $n$  ( $n = 2, 3, 4, \dots$ ) the assertion of the theorem holds for any game that starts with  $k$  participants ( $k = 1, 2, \dots, n - 1$ ) and let us look at a game with  $n$  participants.

Assume that all participants except, maybe, for the first one, follow the global strategy  $\underline{\lambda}$ . We already know that in case of the pending catastrophe with death intensity  $S_n(t)$  the survival probability of the first player, if nothing happened till the moment  $t$ , will be  $1 - D(t)$ , regardless of his own strategy. The only difference between the special case of the pending death and that of a general transition intensity is that in the latter case, the first participant, even if surviving, can find himself with  $K - 1$  rather than with  $n - 1$  other participants ( $K = 1, 2, \dots, n - 1$ ) all following the appropriate Nash strategy. But then it follows from the induction assumption that his survival probability still remains  $1 - D(t)$ , hence this is his survival probability regardless of which other strategy he chooses and  $\underline{\lambda}$  is a NASH strategy.  $\square$

Condition (4.3) precludes a situation at which, at some time interval, the risk  $D(t)$  of the volunteers decreases so drastically and the death intensity is so small, that it is worthwhile for a player to postpone volunteering even if he knows that nobody else volunteers. Theorem (4.1) asserts that whenever this condition is satisfied (for all  $n$ ), the Nash volunteering intensity  $\lambda_n(t)$  of a player should not take into consideration the question whether the death of other players increases or decreases his own risk, while waiting for a volunteering act. This somewhat counter intuitive result may be explained by the fact that a player should know that as the death intensity



$S_k(t)$  increases, so should the tendency of his partners to volunteer. Moreover, this tendency would increase exactly to the level which balances the effect of the increase in death-intensity, keeping the survival probability of the players  $1 - D(t)$ , regardless of the number of players and of the risk of waiting.

Note, moreover, that if at a given time-interval, one mutant player or more, out of the  $n$ , tend to volunteer more than the prescription of the Nash strategy, it is advantageous for the other to reduce his tendency to volunteer and vice versa if the mutant player or players is below the Nash prescription. In both cases the Nash strategy is better than the mutant one. Employing the criterion (2.6) for ESS, this argument readily indicates that the Nash strategy  $\{\lambda_n(t)\}$  has the local property of an agent ESS, namely of stability against those mutant strategies that deviate from it only at given time intervals and only in specific direction (see Eshel and Shaked, 2000). For the special case in which  $D(t) = D$  is independent of time, and only time-homogeneous strategies  $\lambda_n(t) = \lambda_n$  are allowed, the ESS property of the Nash strategy follows immediately.

## 5. Summary

The motivation for this work stems from attempts to explain a seemingly altruistic behaviour in some natural situations in which neither the conditions for kin selection, nor those of group selection or reciprocal altruism are met.

It was assumed that there is a group of unrelated individuals taken at random from a large population which is exposed to the same time-continuous threat of dying. Accumulated loss of each player increases as the game goes on until at least one participant volunteers to take some extra risk on its own. The risk taken by a volunteer in order to stop the threat may or may not depend on the time of volunteering.

This situation can be modeled as an  $n$ -player War of Attrition, which ends when one of the players volunteers. We called this sort of generalization, “The ( $n$ -player) volunteer dilemma”. Indeed, a two-player volunteer dilemma is equivalent to the original War of Attrition. It was further assumed that both the risk for the volunteer and the intensity of the risk of waiting are time-dependent according to some integrable function, this instead of being constants as assumed in the original War of Attrition model of Maynard

Smith. The integral over the time of the intensity of the danger to which all non-volunteering individuals are exposed while waiting is finite over every finite interval of time, but it tends to infinity as time does so, i.e. if no-one volunteers all will die.

In the first model under consideration, “The Pending Catastrophe” it was assumed that the number of players in the encounter group remains constant, i.e. death event, if occurs simultaneously to all participants. This is a special case of an out-most dependence among death events of non-volunteering individuals.

Under this assumptions necessary and sufficient conditions for a strategy to be a Nash strategy are given. This strategy is characterized by a time-intensity of volunteering.

It was also shown that if Nash intensity of volunteering is strictly positive, then it is a weak Nash strategy in the sense that when followed by all players except one, then the survival probability of this one is independent of the choice of his own strategy. A somewhat surprising result is that if the risk of volunteering is not dependent on the number of players, then the survival probability of individuals from a population fixed on the Nash strategy is also not dependent on the number of contestants.

In the stationary case the Nash strategy is proven to be evolutionarily stable.

In the second model a much more general case was analyzed. It was assumed that if at some time moment there are  $n$  participants present at the game and if none of them volunteers, then there is a probability that  $K$  out of them would die and the game will continue with  $n - k$  players. Apparently one may expect that in such a case a decision to volunteer should be affected not only by the direct individual’s risk of functions, but also by increase (or decrease) of the individual risk due to the loss of one’s partners. However, under plausible conditions, the intensity of volunteering at Nash equilibrium depends only on individual risk functions. In such a case it is proved that the Nash intensity obtained for the special case of “The Pending Catastrophe” is also a solution of this population game, i.e. a global Nash strategy.

The somewhat counter intuitive result that the Nash volunteering intensity of a player should not take into consideration the question whether the death of other players increases or decreases his own risk while waiting, may be elucidated by the fact that

a player should know that as the death intensity increases, so should the tendency of his partners to volunteer.

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