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THREE CONDITIONS LYAPUNOV EXPONENTS SHOULD SATISFY

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ABSTRACT

In contrast to the unilateral claim in some papers that a positive Lyapunov exponent means chaos, it was claimed in this paper that this is just one of the three conditions that Lyapunov exponent should satisfy in a dissipative dynamical system when the chaotic motion appears. The other two conditions, any continuous dynamical system without a fixed point has at least one zero exponent, and any dissipative dynamical system has at least one negative exponent and the sum of all of the 1-dimensional Lyapunov exponents id negative, are also discussed. In order to verify the conclusion, a MATLAB scheme was developed for the computation of the 1-dimensional and 3-dimensional Lyapunov exponents of the Duffing system with square and cubic nonlinearity.

KEYWORDS: Lyapunov exponent, chaos, three conditions, Duffing system

INTRODUCTION

Recently, chaotic motions that arise from the nonlinearity of dissipative dynamical systems have received a great concern in both physical and non-physical fields. The most striking feature of chaos is the unpredictability of the future despite a deterministic time evolution. This unpredictability is a consequence of the inherent instability of the solutions, reflected by what is called sensitive dependence on initial conditions. The tiny deviations between the initial conditions of all the trajectories are blown up after a short time.

A more careful investigation of this instability leads to two different, although related, concepts. One is the loss of information related to unpredictability, quantified by the Kolmogorov-Sinai entropy. The other is a simple geometric one, namely, that nearby trajectories separate very fast, or more precisely, separate exponentially over time. The properly averaged exponent of this increase is the characteristics for the dynamical system and quantifiers the strength of chaos. It is called the Lyapunov exponent.

In many papers there was a unilateral claim that a positive Lyapunov exponent means an exponential divergence of nearby trajectories, i.e. chaos. In many situations the exist of the maximal positive Lyapunov exponent only is completely justified. However, when a dynamical system is defined as a mathematical object in a given state space, there exist as many different Lyapunov exponents as there are phase dimensions and only the complete set of Lyapunov exponents can characterize the asymptotic behavior of the dissipative dynamical systems.

The purpose of this paper is to present a comprehensive analysis of the Lyapunov exponent spectrum. As a result, three conditions, which Lyapunov exponents should satisfy in a dissipative dynamical system when the chaotic motion appears, were derived. The conclusion was verified through the computation of Lyapunov exponents in a Duffing system with square and cubic nonlinearity.

NOMENCLATURE

C	damping coefficient
F	excitation amplitude
f	n-dimensional nonlinear vector
J	the $n \times n$ Jacobian matrix
M	mass
w	the deviation between x and y
x	n-dimensional state variable
y	n-dimensional state variable
ω	excitation frequency
e_i	eigenvector
K_i	stiffness
Λ_i	eigenvalue

λ_i	Lyapunov exponent
$\lambda(e^n)$	n-dimensional Lyapunov exponent
$\|\bullet\|$	norm

1. LYAPUNOV EXPONENT SPECTRUM

Chaotic systems have the property of sensitive dependence on initial conditions, which means that infinitesimally close vectors in phase space give rise to two trajectories that separate exponentially. The time evolution of infinitesimal difference, however, is completely described by the time evolution in tangent space, that is, by the linearized dynamics [1].

Let us consider the dynamical system of which the time evolution is described by a set of differential equations in n-dimensional Euclidian space

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \cdots x_n) \quad i=1, 2,\ldots, n \tag{1}$$

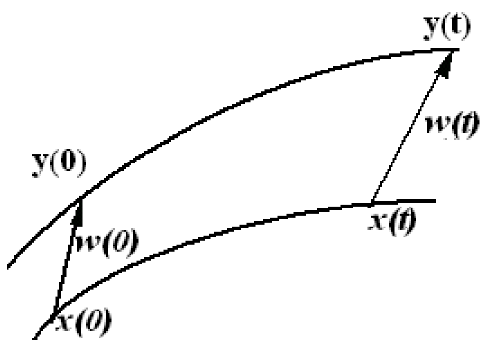

Fig. 1 Sketch for the divergence of two trajectories

where t is time, $\{x_1, x_2, \ldots, x_n\}$ is a n-dimensional state variable which constitutes a n-dimensional phase space, and f_i is a n-dimensional nonlinear vector.

The solution of system (1) under the initial condition $x(0)=x_0$ is a trajectory $x(t)$. Supposing a deviation $w(0)$ of the initial condition $x(0)$, another trajectory $y(t)$ originating from $y(0)$ is formed, as illustrated in Fig. 1, where $y(0)=x(0)+w(0)$ and $y(t)= x(t)+w(t)$. $w(t)=\{w_1(t), w_2(t), \ldots, w_i(t)\}$ constitutes a phase space called the tangent space.

As far as $\{w_i(t)\}$ is small enough and dynamical system (1) is dissipative, the time evolution of the deviation $w(t)$ obeys the following linear differential equations

$$\frac{dw_i}{dt} = Jw_i \tag{2}$$

where J is the $n \times n$ Jacobian matrix of dynamical system (1) with

$$J_{ij} = \frac{\partial f_i}{\partial x_j} \quad i, j=1, 2,\ldots, n \tag{3}$$

Let e_i be an eigenvector of J and Λ_i be its eigenvalue. If $w_i(t)$ is parallel to one of the eigenvectors e_i, it is either stretched or compressed by a factor of Λ_i. Thus n different local stretching factors are found and the phase space is decomposed into n linear subspaces. Up to now the analysis is local in space and for a fixed time. However, the Jacobian matrix is in general position dependent. In order to characterize the dynamical system as a whole one, the Lyapunov exponents λ_i is defined as an average stretching factor of the infinitely long trajectory $w_i(t)$ [1,2], that is

$$\lambda_i = \lim_{N\to\infty} \frac{1}{N} \ln\left|\Lambda_i^{(N)}\right| \quad i=1, 2,\ldots, n \tag{4}$$

or

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|w_i(t)\|}{\|w_i(0)\|} \quad i=1, 2,\ldots, n \tag{5}$$

where $\Lambda_i^{(N)}$ means the ith eigenvalue of the product of all Jacobian matrices along the trajectory, and $\|w_i(0)\|$ and $\|w_i(t)\|$ are the lengths or norms of $w_i(0)$ and $w_i(t)$, respectively.

Thus n different exponents of the n-dimensional dynamical system (1) are defined and the set of exponents is often called the spectrum of the Lyapunov exponent. It is often notated that $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$, where λ_1 is called the maximal Lyapunov exponent.

2. THREE CONDITIONS

2.1 The Maximal Lyapunov Exponent

The Lyapunov exponents give a typical time scale for the divergence or convergence of nearby trajectories. If λ_i is zero, the nearby trajectories in the ith phase dimension never converge or diverge. Similarly, if λ_i is positive or negative, this means an exponential divergence or convergence of nearby trajectories in the ith phase dimension. Of all the Lyapunov exponents of the n-dimensional dynamical system (1), the maximal Lyapunov exponent λ_1 is the most important one. In many papers it was claimed that the maximal Lyapunov exponent λ_1, if positive, means chaos. Beyond all doubt, that the maximal Lyapunov exponent λ_1 is positive is the precondition for chaos in dynamical system (1). However the computation of the maximal Lyapunov exponent only can not give a complete portray of the complicated motion. For example, there might exist more than two positive Lyapunov exponents in 4 or higher dimensional dissipative dynamical systems, where the chaotic motion is called hyperchaos. In some dissipative dynamical systems one can also find a negative maximal Lyapunov exponent which reflects the existence of a stable fixed point. Two trajectories that approach the fixed point also approach each other exponentially. If the motion settles down onto a limit cycle, the maximal Lyapunov exponent is zero and the motion is called marginally stable.

2.2 The Sum of All Lyapunov Exponents

A dynamical system described by autonomous differential equations can not be chaotic in less than three dimensions [3].

So the chaotic system has at least three Lyapunov exponents. What about the sum of them?

The definition of the Lyapunov exponent λ_i above is concerned with the convergence or divergence of nearby trajectories in each phase dimension, and hence they are 1-dimensional. In the same way, the *k*-dimensional Lyapunov exponent $\lambda(e^k)$ can also be defined according to the expanding rate of the volume of the *k*-dimensional parallelepiped in the tangent space along the orbit which starts at x_0 [4].

$$\lambda(e^k) = \lim_{t\to\infty}\frac{1}{t}\ln\frac{V(t)}{V(0)} \tag{6}$$

The *k*-dimensional Lyapunov exponent $\lambda(e^k)$ may take, at most, C_n^k distinct values, and each value is connected with a sum of *k* distinct 1-dimensional exponents. For instance, in the case *n*=3, the *k*-dimensional Lyapunov exponent $\lambda(e^k)$ may take the following values respectively:

$\lambda(e^1)$ =one of the values in { $\lambda_1, \lambda_2, \lambda_3$ },

$\lambda(e^2)$ =one of the values in {(λ_1 + λ_2), (λ_1 + λ_3), ($\lambda_2 + \lambda_3$)}

$\lambda(e^3)$ =($\lambda_1 + \lambda_2 + \lambda_3$).

According to the Liouville theorem, the phase volume of the *n*-dimensional dynamical system (1) is contracting, if it is dissipative. The contracting rate of phase space is defined as [5]

$$\Lambda(x) = \sum_{i=1}^{n}\frac{\partial f_i(x)}{\partial x_i} = divf(x) \tag{7}$$

Apparently, in the dissipative dynamical system (1)

$$\Lambda(x) < 0 \tag{8}$$

The *n*-dimensional Lyapunov exponent $\lambda(e^n)$ is also an indication of the contracting rate of the phase volume of the *n*-dimensional dissipative dynamical system (1), namely,

$$\lambda(e^n) < 0 \tag{9}$$

According to the statement above, the *n*-dimensional Lyapunov exponent $\lambda(e^n)$ equals to the sum of all of the *n* 1-dimensional Lyapunov exponents λ_i. So, it is easy to gain the conclusion that there is at least one negative exponent in any dissipative dynamical system and the sum of all of the 1-dimensional Lyapunov exponents is negative. This is the second condition Lyapunov exponents should satisfy in a dissipative dynamical system.

2.3 At Least One Lyapunov Exponent Vanishes

There is another attractive 1-dimensional Lyapunov exponent. H. Haken [6] provided a mathematically rigorous proof that for each trajectory, which does not terminate at a fixed point, at least one Lyapunov exponent vanishes, namely, zero. This 1-dimensional Lyapunov exponent corresponds to the direction tangent to the flow defined by dynamical system (1). The direction corresponding to a Lyapunov exponent zero is called marginally stable. This direction is usually useless for analysis and the Poincaré section method is just to remove this direction from the phase space.

Further more, what is called the maximal Lyapunov exponent is the greatest one of all of the 1-dimensional Lyapunov exponents except this zero one.

Up to now, the three conditions the 1-dimensional Lyapunov exponents should satisfy in a dissipative dynamical system when the chaos appears have been all derived. They are

1) At least one 1-dimensional Lyapunov exponent is positive.

2) At least one 1-dimensional Lyapunov exponent is zero.

3) At least one 1-dimensional Lyapunov exponent is negative and the sum of all of the 1-dimensional Lyapunov exponents is negative.

In order to verify the conclusion, the 1-dimensional and 3-dimensional Lyapunov exponents of a forced nonlinear vibration system were calculated.

3. 1- AND 3-DIMENSIONAL LYAPUNOV EXPONENT OF THE DUFFING SYSTEM

In this section, the explicit result of the complete set of the 1-dimensional and the 3-dimensional Lyapunov exponents for the Duffing system is presented.

The forced vibration of a single-degree-of-freedom nonlinear system with square and cubic nonlinearity is described by the following equation

$$M\frac{d^2X}{dT^2} + C\frac{dX}{dT} + K_1X + K_2X^2 + K_3X^3 = F\cos\omega T \tag{10}$$

where M is the mass of the oscillator, $C > 0$ the damping, K_i (i=1, 2, 3) the stiffness, F the excitation amplitude, and ω the frequency of the excitation.

Let $\Omega_0 = \sqrt{K_1/M}$ be the natural frequency of the corresponding linearized undamped system, $X = x\sqrt{C/N}$, and $t = T/\Omega_0$. Hence, the system (10) can be rewritten as

$$\ddot{x} + 2\xi\dot{x} + x + \delta x^2 + x^3 = f\cos(bt) \tag{11}$$

where $\xi = C/2\sqrt{MK_1}$ is the damping coefficient, $\delta = K_2/\sqrt{K_1K_3}$, $b = \omega/\Omega_0$, and $f = F\sqrt{K_3}/\sqrt{K_1^3}$.

The qualitative analysis showed that the system has no equilibrium, but that all the trajectories are attracted into a limited phase space [7].

In order to compute the Lyapunov exponents, system (11) is rewritten as an autonomous one

$$\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = -x_1 - \delta x_1^2 - x_1^3 - 2\xi x_2 + f\cos(bt)\\ \dot{t} = 1\end{cases} \tag{12}$$

Thus the contracting rate of the phase space of system (12) is

$$\Lambda(x) = \frac{\partial}{\partial x_1} x_2 + \frac{\partial}{\partial x_2}[-x_1 - \delta x_1^2 - x_1^3 - 2\xi x_2 + f\cos(bt)] + \frac{\partial}{\partial t}(1.0)$$
$$= -2\xi < 0 \quad (13)$$

That is, the contracting rate of the phase volume of system (12) is constant. This indicates that the contracting rate of the phase space of system (12) is uniform the whole phase space. Then the sum of all the 1-dimensional Lyapunov exponents is

$$\lambda_1 + \lambda_2 + \lambda_3 = -2\xi \quad (14)$$

Based on the algorithm developed by Shimada [8], a MATLAB scheme was developed for the computation of the 1-dimensional and 3-dimensional Lyapunov exponent of system (12). The property of the Duffing system presented in equation (14) can be used to verify the numerical scheme.

Let $\xi = 0.1$, $\delta = 0.2$, $b = \omega/\omega_0 = 2.5$. The 1-dimensional and 3-dimensional Lyapunov exponents for different f are shown in Table 1. Let $\xi = 0.1$, $\delta = 0.2$, $f^2 = 900$. The 1-dimensional and 3-dimensional Lyapunov exponents for different b are shown in Table 2.

The computation results showed in Table 1 and 2 verified the three conditions derived in section 3. Especially, the sum of 1-dimensional Lyapunov exponents λ_i equals to -2ξ.

Table 1 1- and 3-dimensional Lyapunov exponents for different *f* where $\xi = 0.1$, $\delta = 0.2$, and $b = 2.5$

f^2	$\lambda(e^1)$			$\lambda(e^3)$
	λ_1	λ_2	λ_3	
1200	0.0236	0.0000	-0.2236	-0.2000
1300	0.1198	-0.0002	-0.3196	-0.2000
1390	0.0222	0.0001	-0.2223	-0.2000
2370	0.0041	-0.0003	-0.2038	-0.2000
2426	0.0279	-0.0001	-0.2278	-0.2000
2630	0.0054	-0.0000	-0.2054	-0.2000

Table 2 1- and 3-dimensional Lyapunov exponents for different *b* where $\xi = 0.1$, $\delta = 0.2$, and $f^2 = 900$

b	$\lambda(e^1)$			$\lambda(e^3)$
	λ_1	λ_2	λ_3	
0.6	0.0468	0.0001	-0.2469	-0.2000
0.9	0.0663	0.0000	-0.2663	-0.2000
1.1	0.0070	-0.0001	-0.2069	-0.2000
1.3	0.0231	-0.0002	-0.2229	-0.2000
1.5	0.1065	0.0000	-0.3065	-0.2000
1.8	0.0279	0.0003	-0.2282	-0.2000

4. CONCLUSION

In order to characterize the chaotic behavior of the dissipative dynamical systems, the Lyapunov exponent spectrum was analyzed comprehensively and three conditions the 1-dimensional Lyapunov exponents should satisfy in the dissipative dynamical system when the chaotic motion appears were derived.

The most striking feature of chaos is the unpredictability of the future despite a deterministic time evolution, namely, the sensitive dependence on the initial conditions. This is quantified by one positive 1-dimensional Lyapunov exponent. It is the precondition of chaos that at least one 1-dimensional Lyapunov exponent is positive which reflects a "direction" in which two trajectories diverge exponentially.

Any continuous dynamical system without a fixed point will have at least one zero exponent, corresponding to the slowly changing magnitude of a principal axis tangent to the flow defined by the dynamical system.

The sum of the 1-dimensional Lyapunov exponents is the time-averaged divergence rate of the phase volume: Hence any dissipative dynamical system has at least one negative exponent and the sum of all of the 1-dimensional Lyapunov exponents is negative. Especially, for some dissipative dynamical systems with a constant divergence rate of the phase volume, i.e., $\Lambda(x)$ =*constant*, the sum of the 1-dimensional Lyapunov exponents takes a fairly simple form. The maximal positive Lyapunov exponent and the negative sum of all of the 1-dimensional Lyapunov exponents shows that the dynamical system experiences the repeated stretching and folding.

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