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Exact Derivation of the Nonlinear Negative-Resistance Oscillator Pulling Figure

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Abstract — **Only approximate relations are so far available for the pulling figure of an oscillator. An exact derivation of the pulling figure is presented here, taking fully into account the nonlinearity of the oscillator admittance. Effect of the oscillator nonlinearity on the asymmetry of the pulling range is presented.**

I. Introduction

Oscillator frequency variation with the load changes is often represented by its pulling figure. The pulling figure has so far been calculated either by neglecting the oscillator admittance variation with the RF voltage [1] or has been calculated by approximately taking into account the transferred admittance in the oscillator plane for a small-load perturbation [2]. We present here an exact derivation of the pulling figure taking fully into account the nonlinear behavior of the oscillator admittance. The relation between the asymmetry of the oscillator pulling range and the nonlinearity of the oscillator admittance has been derived. Pulling figures for certain particular cases are also presented.

II. Frequency Variation with the Load Changes

The oscillation condition at the oscillator-output plane without any load perturbation is represented by

$$Y_{T0} = Y_T + Y_0 = 0 \tag{1}$$

where Y_T is the oscillator nonlinear output admittance and Y_0 is the load admittance.

If the oscillations exist with a load perturbation of ΔY_L in the oscillator output plane and writing $\Delta\omega$ and ΔV as the corresponding frequency and RF voltage changes, the oscillation condition can be represented by

$$Y_{T0} + \Delta Y_L + \frac{dY_{T0}}{d\omega}\cdot\Delta\omega + \frac{dY_{T0}}{dV}\Delta V = 0. \tag{2}$$

From (1) and (2)

$$\Delta Y_L + \frac{dY_{T0}}{d\omega}\cdot\Delta\omega + \frac{dY_{T0}}{dV}\Delta V = 0 \tag{3}$$

separating into real and imaginary parts

$$\Delta G_L + \frac{dG_{T0}}{d\omega}\Delta\omega + \frac{dG_{T0}}{dV}\Delta V = 0 \tag{4}$$

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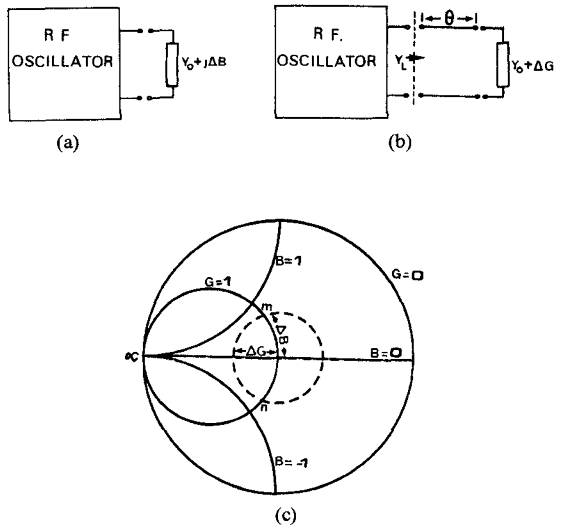

Fig. 1. Simulation of load variation.

and

$$\Delta B_L + \frac{dB_{T0}}{d\omega}\Delta\omega + \frac{dB_{T0}}{dV}V = 0 \tag{5}$$

where

$$\Delta Y_L = \Delta G_L + \Delta B_L$$

and

$$Y_{T0} = G_{T0} + B_{T0}.$$

From (4) and (5)

$$\Delta\omega = \frac{\Delta G_L \cdot \dfrac{dB_{T0}}{dV}}{\dfrac{dG_{T0}}{dV}\cdot\dfrac{dB_{T0}}{d\omega} - \dfrac{dG_{T0}}{d\omega}\cdot\dfrac{dB_{T0}}{dV}} - \frac{\Delta B_L \cdot \dfrac{dG_{T0}}{dV}}{\dfrac{dG_{T0}}{dV}\dfrac{dB_{T0}}{d\omega} - \dfrac{dG_{T0}}{d\omega}\cdot\dfrac{dB_{T0}}{dV}} \tag{6}$$

and

$$\Delta V = \frac{\Delta B_L \cdot \dfrac{dG_{T0}}{d\omega}}{\dfrac{dG_{T0}}{dV}\cdot\dfrac{dB_{T0}}{d\omega} - \dfrac{dG_{T0}}{d\omega}\cdot\dfrac{dB_{T0}}{dV}} - \frac{\Delta G_L \cdot \dfrac{dB_{T0}}{d\omega}}{\dfrac{dG_{T0}}{dV}\cdot\dfrac{dB_{T0}}{d\omega} - \dfrac{dG_{T0}}{d\omega}\cdot\dfrac{dB_{T0}}{dV}}. \tag{7}$$

III. Load Variation Simulation

From Fig. 1 it can be noted that any reactive load perturbation of value $j\Delta B$ can be represented by a nonreactive load perturbation of ΔG by suitably selecting the reference plane in the output line between the oscillator and the perturbation. For the purposes of the exact derivation of the pulling figure, we simulate the load perturbation by ΔG (Fig. 1(b)) with the transmission-line length l variable between 0 and $\lambda/2$.

For any value of $\theta = \beta l$, the transferred load admittance Y_L at the oscillator output plane can be written as

$$Y_L = Y_0 \cdot \frac{Y_0 + \Delta G + jY_0 \tan\theta}{Y_0 + j(Y_0 + \Delta G)\tan\theta} \tag{8}$$

$$= Y_0 \cdot \frac{S(1+\tan^2\theta)}{1+S^2\tan^2\theta} + jY_0 \cdot \frac{(1-S^2)\tan\theta}{1+S^2\tan^2\theta} \tag{9}$$

where S is the VSWR of the perturbating admittance in the output line given by

$$S = 1 + \frac{\Delta G}{Y_0}. \tag{10}$$

The change in the load admittance, at the oscillator output plane, $\Delta Y_L (= \Delta G_L + j\Delta B_L) = Y_L - Y_0$, can now be represented by the following two equations:

$$\Delta G_L = Y_0 \frac{(S-1)(1-S\tan^2\theta)}{1+S^2\tan^2\theta} \tag{11}$$

$$\Delta B_L = Y_0 \frac{(1-S^2)\tan\theta}{1+S^2\tan^2\theta}. \tag{12}$$

IV. Derivation of the Pulling Figure

Substituting (11) and (12) into (6) the frequency variation $\Delta\omega$ can be written as

$$\Delta\omega = Y_0 \cdot \frac{\frac{dB_{T0}}{dV}}{\frac{dG_{T0}}{dV}\cdot\frac{dB_{T0}}{d\omega} - \frac{dG_{T0}}{d\omega}\cdot\frac{dB_{T0}}{dV}} \cdot \frac{(S-1)(1-S\tan^2\theta)}{1+S^2\tan^2\theta} - Y_0 \cdot \frac{\frac{dG_{T0}}{dV}}{\frac{dG_{T0}}{dV}\cdot\frac{dB_{T0}}{d\omega} - \frac{dG_{T0}}{d\omega}\cdot\frac{dB_{T0}}{dV}} \cdot \frac{(1-S^2)\tan\theta}{1+S^2\tan^2\theta}. \tag{13}$$

Now maximum frequency derivation as a function of θ can be calculated from

$$\frac{d\Delta\omega}{d\theta} = 0 \tag{14}$$

giving us

$$S^2\tan^2\theta + 2S\alpha\tan\theta - 1 = 0 \tag{15}$$

with

$$\alpha = \frac{dB_{T0}/dV}{dG_{T0}/dV} \tag{16}$$

where α is a nonlinear constant of the oscillator. (15) has the following two solutions of θ which correspond to the extreme values of $\Delta\omega_1$ and $\Delta\omega_2$:

$$\theta_1 = \tan^{-1}\frac{\sqrt{\alpha^2+1} - \alpha}{S} \tag{17}$$

and

$$\theta_2 = \tan^{-1}\frac{\sqrt{\alpha^2+1} + \alpha}{-S}. \tag{18}$$

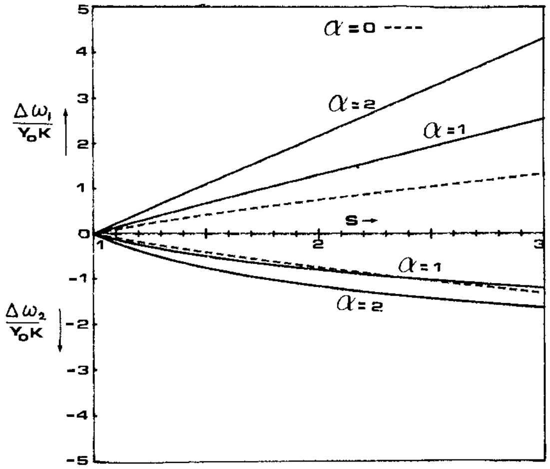

Fig 2 Maximum frequency variation as a function of load VSWR S for various values of oscillator nonlinearity constant α.

Substituting (17) and (18) into (13)

$$\Delta\omega_1 = Y_0 K \frac{S-1}{2S} \frac{\sqrt{\alpha^2+1}\,(2\alpha^2+s+1) - 2\alpha(1+\alpha^2)}{\alpha^2 - \alpha\sqrt{\alpha^2+1} + 1} \tag{19}$$

and

$$\Delta\omega_2 = Y_0 K \frac{1-S}{2S} \frac{\sqrt{\alpha^2+1}\,(2\alpha^2+S+1) + 2\alpha(1+\alpha^2)}{\alpha^2 + \alpha\sqrt{\alpha^2+1} + 1} \tag{20}$$

where

$$K = \frac{dG_{T0}/dV}{\frac{dG_{T0}}{dV}\cdot\frac{dB_{T0}}{d\omega} - \frac{dG_{T0}}{d\omega}\cdot\frac{dB_{T0}}{dV}}. \tag{21}$$

The maximum total variation $\Delta\omega_M = \Delta\omega_1 - \Delta\omega_2$ can now be found to be

$$\Delta\omega_M = Y_0 K \sqrt{\alpha^2+1}\left\{S - \frac{1}{S}\right\}. \tag{22}$$

This is the relation which represents rigourously the pulling figure of a nonlinear negative-resistance oscillator.

An interesting feature of the above (19) and (20) results, is that it brings into evidence the asymmetry of the pulling range, around the free-running oscillator frequency, as a function of nonlinearity constant α of the oscillator. Fig. 2 represents $\Delta\omega_1$ and $\Delta\omega_2$ as a function of VSWR S of the perturbating admittance (10) for various values of α. It may be noted that for $\alpha = 0$ the pulling range is symmetrical and becomes more and more asymmetrical with increasing values of α.

In the same way, from (7) the RF voltage variation ΔV can be shown to be a function of

$$\beta = \frac{dG_{T0}/d\omega}{dB_{T0}/d\omega}$$

and asymmetrical in the pulling range.

The unknown constants α and K of relation (22) can be found by dividing (19) and (20) and arranging

$$\frac{\Delta\omega_1}{\Delta\omega_2} = \frac{\sqrt{\alpha^2+1}\,(S+1) + \alpha(S-1)}{\alpha(S-1) - \sqrt{\alpha^2+1}\,(S+1)} \tag{23}$$

or

$$\sqrt{\alpha^2+1}\left\{1+\frac{\Delta\omega_1}{\Delta\omega_2}\right\}+\alpha\Gamma\left\{1-\frac{\Delta\omega_1}{\Delta\omega_2}\right\}=0 \qquad (24)$$

where Γ represents the reflection coefficient of perturbed load equals $S-1/S+1$. $\Delta\omega_1$ and $\Delta\omega_2$ being opposite in sign, $\Delta\omega_1/\Delta\omega_2$ is always negative.

Substituting

$$x=\left|\frac{\Delta\omega_1}{\Delta\omega_2}\right|=-\frac{\Delta\omega_1}{\Delta\omega_2} \qquad (25)$$

we have x always positive.

From (24)

$$\sqrt{\alpha^2+1}\,(1-x)+\alpha\Gamma(1+x)=0. \qquad (26)$$

From (26), α can now be calculated for the three possible cases as

$$\begin{aligned} &\text{for } x=1, && \alpha=0\\ &\text{for } \frac{1}{S}<x<1, && \alpha=-\frac{1}{\sqrt{A^2-1}}\\ &\text{for } 1<x<S, && \alpha=\frac{1}{\sqrt{A^2-1}} \end{aligned} \qquad (27)$$

where

$$A=\frac{\Gamma(1+x)}{(1-x)}.$$

Knowing α, S, $\Delta\omega_1$, and $\Delta\omega_2$, the value of K can be determined from (19) or (20).

A frequency deviation $\Delta\omega_{M0}$ can be defined [3] for the condition when the real part of the perturbed load admittance at the oscillator plane is Y_0, i.e., $\Delta G_L=0$. This corresponds to $\omega_m-\omega_n$ where m and n are the common points on $G=1$ and the perturbed load impedance locus (Fig. (1c)). From (11), this condition gives

$$\theta_0=\pm\tan^{-1}\left(\frac{1}{\sqrt{S}}\right). \qquad (28)$$

From (13), the total frequency deviation $\Delta\omega_{M0}$ is now given by

$$\Delta\omega_{M0}=2Y_0K\left(\frac{S-1}{\sqrt{S}}\right). \qquad (29)$$

TABLE I

	without approximation	with $\frac{dG_{To}}{d\omega}=0$	with $\frac{dG_{To}}{d\omega}=\frac{dB_{To}}{dV}=0$
$\Delta\omega_M$	$Y_oK\sqrt{\alpha^2+1}.\ \{S-\frac{1}{S}\}$	$\frac{\omega_o}{2Q_{ext}}\sqrt{\alpha^2+1}\ \{S-\frac{1}{S}\}$	$\frac{\omega_o}{2Q_{ext}}\ \{S-\frac{1}{S}\}$
$\Delta\omega_{M_o}$	$2Y_oK\ \{\frac{S-1}{\sqrt{S}}\}$	$\frac{\omega_o}{Q_{ext}}\{\frac{S-1}{\sqrt{S}}\}$	$\frac{\omega_o}{Q_{ext}}\{\frac{S-1}{\sqrt{S}}\}$

From (22) and (29)

$$\Delta\omega_M=\Delta\omega_{M0}\frac{S+1}{\sqrt{S}}\sqrt{\alpha^2+1}. \qquad (30)$$

Knowing $\Delta\omega_M$, S, and α $\Delta\omega_{M0}$ can be determined.

A number of particular cases can be derived from the general solution presented above. Table I lists expressions for the pulling figure with and without approximations where

$$Q_{ext}=\frac{\omega_0}{2Y_0}\frac{dB_{T0}}{d\omega}$$

and α and K are defined by (16) and (21), respectively.

It may be noted that

$$S-\frac{1}{S}=\frac{4|\Gamma|}{1-|\Gamma|^2}$$

with $\Gamma^2\ll 1$

$$S-\frac{1}{S}=4\left\{\frac{S-1}{S+1}\right\}$$

and $\Delta\omega_M$, for example, for the approximation $dG_{T0}/d\omega=0$, can be written as given in [2]

$$\Delta\omega_M=\frac{2\omega_0}{Q_{ext}}\sqrt{\alpha^2+1}\left\{\frac{S-1}{S+1}\right\}.$$

Acknowledgment

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