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Yield Surface Characteristics Arising From Orthorhombic Symmetry

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Introduction

One of the fundamental problems of the mathematical theory of plasticity consists of a description of the change of the yield surface induced by plastic flow, accurately enough to reproduce the behaviour of real materials. Theoretical predictions of plastic flow in members subject to yield is commonly based on isotropic yield functions, notably the quadratic one proposed by von Mises. In certain situations, however, metals exhibit pronounced directional properties as regards yield. An obvious situation is when the current state has been attained by processes like rolling. If such an operation has commenced from an isotropic state it will induce three mutually preferred directions in the material resulting in orthorhombic symmetry.

When modeling the behavior of real materials by aid of experimental data, the existence of preferred (or symmetry) directions at any generic instant is frequently assumed, knowingly or not. It seems though that the explicit consequences of such an imposition have not always been fully appreciated and it is the present purpose to elucidate some implications in particular for the case of plane stress.

Orthorhombic Yield Surface Models

Suppose that a yield state is defined by a scalar condition

$$f(\boldsymbol{\sigma}, \mathbf{A}) = 0 \tag{1}$$

at any generic instant where, referred to a fixed Cartesian coordinate system x_i, $\boldsymbol{\sigma}$ denotes Cauchy stress and $\mathbf{A}$ any set of scalar or tensor-valued constitutive variables defining the current material state. Several popular models fall within this formulation.

In an early proposal by Hill [1], exhibiting orthorhombic symmetry, f is quadratic in the stress components and $\mathbf{A}$ stands for six independent scalars being functionals of the deformation history. Recently, [2], this yield condition has been generalized to allow for noninteger powers of the stress components.

When attempting to account for the (uniaxial) Bauschinger effect in more general circumstances, Prager [3] proposed the concept of kinematic hardening by introducing a symmetric second rank tensor determining a translation of the yield locus induced by plastic flow. Subsequently second rank tensors have been introduced in a more elaborate way implying rotation and distortion of the yield surface in stress space. For a recent proposal and an account cf. [4]. Still though whatever intricate way a symmetric second rank tensor is introduced, it possesses three orthogonal principal directions and an immediate consequence is that orthorhombic symmetry prevails for the material state at any generic state of deformation.

A more general approach to the use of state variables in order to represent the internal structure of materials has for some time been advocated by Onat. Several ingenious methods to determine the number and rank of internal variables necessary to obtain realistic representations have been proposed by Onat and Fardshisheh [5]. Subsequently in a constitutive theory for elastic-plastic solids based on a set of even rank irreducible tensors as state variables, symmetry requirements arising in particular from a single symmetric second rank tensor have been discussed by these writers [6].

Explicit Symmetry Properties

As in the present circumstances the material symmetry operations consist of 180° rotations around the preferred directions, $\bar{x}_i$ (say), then on introducing an appropriate integrity basis, the yield condition is expressible as

$$f(\bar{\sigma}_{11}, \bar{\sigma}_{22}, \bar{\sigma}_{33}, \bar{\sigma}_{12}^2, \bar{\sigma}_{23}^2, \bar{\sigma}_{31}^2, \bar{\sigma}_{12}\bar{\sigma}_{23}\bar{\sigma}_{31}, \bar{\mathbf{A}}) = 0 \tag{2}$$

in obvious notation.

It is immediately clear then that, when referred to the preferred directions, the yield function is invariant with respect to pair-wise changes of sign of shear stresses and eventually single changes when the triple product in (2) is absent. Experimental evidence of the second event indicates that a quadratic yield function might suffice to model material properties.

In the degenerate case of transverse isotropy, with respect to $\bar{x}_3$, (2) reduces to

$$g(\bar{\sigma}_{11} + \bar{\sigma}_{22}, \bar{\sigma}_{11}\bar{\sigma}_{22} - \bar{\sigma}_{12}^2, \bar{\sigma}_{33}, \bar{\sigma}_{23}^2 + \bar{\sigma}_{31}^2, |\boldsymbol{\sigma}|, \bar{\mathbf{A}}) = 0 \tag{3}$$

by aid of an appropriate basis, [7].

In the particular case when $\mathbf{A}$ stands for a symmetric second rank tensor, $\boldsymbol{\alpha}$, the arguments of the yield function reduce to the single and mutual invariants of $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ and the yield condition (1) is expressible as

$$h(\operatorname{tr}\boldsymbol{\sigma}, \operatorname{tr}\boldsymbol{\sigma}^2, \operatorname{tr}\boldsymbol{\sigma}^3, \operatorname{tr}\boldsymbol{\alpha}, \operatorname{tr}\boldsymbol{\alpha}^2, \operatorname{tr}\boldsymbol{\alpha}^3, \operatorname{tr}(\boldsymbol{\sigma\alpha}), \operatorname{tr}(\boldsymbol{\sigma\alpha}^2), \operatorname{tr}(\boldsymbol{\sigma}^2\boldsymbol{\alpha}), \operatorname{tr}(\boldsymbol{\sigma}^2\boldsymbol{\alpha}^2)) = 0 \tag{4}$$

with respect to any coordinate system.

In such models it is common that the yield function also depends on a number of scalars transforming in an invariant manner and representing, in a geometric formalism, for instance expansion or shrinkage of the corresponding yield surface in stress space. In the present context, however, there is no loss of generality in suppressing the presence of these.

In the fully degenerate case when $\boldsymbol{\alpha}$ is spherical, naturally the yield function is isotropic. Note though that a common assumption is that $\boldsymbol{\alpha}$ is deviatoric and then this case must be excluded unless $\boldsymbol{\alpha}$ vanishes.

When investigating the applicability of different hardening models to real material behavior, it is customary to subject material specimens to various stress histories and from measurements determine the associated yield surfaces in stress space. Due to obvious practical difficulties, almost invariably subsequent yield surfaces are determined by experiments in the subspace of plane stress and the explicit consequences of (2), or the particular form (4), are then perhaps best visualized in such a space. It must be remembered, however, that in a general situation the preferred material directions are unknown and only the components of the stress tensor are conceptually measurable.

Suppose then that a state of plane stress is prevailing in a material plane perpendicular to a preferred direction as might be the case for instance when dealing with rolled sheet metal. Accordingly the yield condition (2) reduces to

$$f(\bar{\sigma}_{11}\, \bar{\sigma}_{22}, \bar{\sigma}_{12}^2, \bar{\mathbf{A}}) = 0 \tag{5}$$

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As a consequence

$$\bar{\sigma}_{12} = 0 \tag{6}$$

represents a symmetry plane in the (three-dimensional) stress space.

In order to see the implication of (6) when the preferred directions are unknown and the plane stress tensor $\sigma_{\alpha\beta}$ is referred to an arbitrary coordinate system x_α, $\alpha = 1, 2$, it proves advantageous to introduce the transformation

$$\sigma_{12}^* = \sqrt{2}\sigma_{12} \tag{7}$$

As explained by Hahn [8], a rotation by an angle θ of a Cartesian coordinate frame x_1x_2 in a physical plane corresponds, as regards an associated plane second rank symmetric tensor **T** (say), to a pure rigid rotation by an angle 2θ around the line $T_{11} - T_{22} = 0$, $T_{12} = 0$, in a coordinate frame of a reduced space spanned by T_{11}, T_{22}, $\sqrt{2}T_{12}$. Thus, utilizing (7) in the present case, for any basic coordinate system chosen, the yield surface will retain its shape; only the coordinate frame in the reduced stress space will be rotated.

When referred to a frame x_α, rotated an (unknown) angle θ with respect to the preferred material in-plane directions, the symmetry plane, *SP* (say), given by (6) is in the associated transformed stress space represented by

$$\frac{1}{\sqrt{2}}(\sigma_{22} - \sigma_{11}) \sin 2\theta + \sqrt{2}\sigma_{12} \cos 2\theta = 0 \tag{8}$$

as illustrated in Fig. 1.

It is evident then that, for the particular case of anisotropy being discussed, points of the intersection between this plane and the yield surface must represent stress states having common principal axes and, in particular, coinciding with those of the preferred material directions.

Thus, when inspecting the shapes in stress space of experimentally determined yield surfaces, under the restrictions previously introduced, it might immediately be established whether subsequent yielding may, with some accuracy, be theoretically predicted on the assumption of orthorhombic symmetry and as a special case by means of a single symmetric second rank tensor apart from scalars. If such circumstances prevail, the rotation of the preferred material directions may be determined at subsequent plastic states. Plausibly such information as regards the evolution of the directions of principal axes is of value when judging the relevance of different growth laws for state variable tensors as for instance those discussed by Ziegler [9] for kinematic hardening.

Results from a two-dimensional normal stress-shear stress subspace, as commonly obtained from combined tension and torsion tests, will in general not suffice for the proposed type of investigation as has been indicated in a preliminary investigation of the tensor approach by Fardshisheh [10]. In the present framework this is immediately obvious from Fig. 1 as the mirror images of points in the vertical coordinate planes with respect to the symmetry plane *SP*, will in general be located in the same planes only for the fortuitous case $\theta = 0$.

An extensive and careful investigation of the hardening properties of pure aluminum under complex stress paths has been carried out by Phillips and associates in the 3-space of plane stress. A visual inspection of some subsequent yield surfaces depicted by Phillips and Kasper [11] is, however, discouraging as regards desired symmetries except for very simple stress histories. It must be underlined though that the shapes of yield surfaces are indeed sensitive to the definition of yield, [12, 13]. Phillips and Kasper's results are based on a proportional limit definition of yield (3μ plastic probing strain). The outcome might have been quite different had any other common yield definition been adopted. Furthermore it must be remembered that the present argument rests upon the prerequisite condition that one preferred direction is perpendicular to the material plane. Such circumstances, however, seem plausible in the experiments reported by Phillips and coworkers, as the specimens used had enjoyed a stress-free isotropic state. In this context it is interesting to note that, when interpreting experimental results for aluminum, copper, and brass, with apparent success, Phillips and Weng, [14], have been led to introduce a yield surface representation based on two second rank tensors and a scalar.

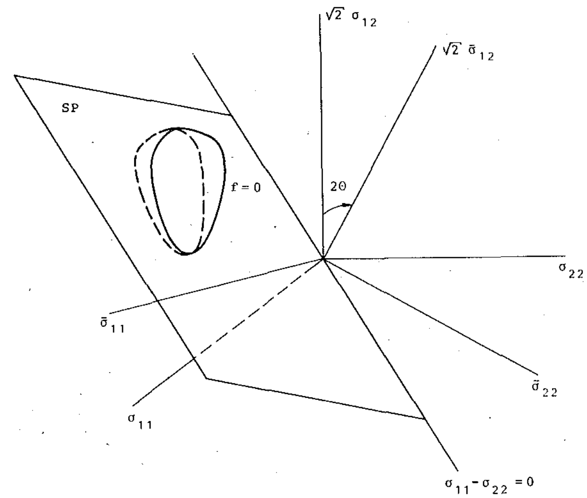

Fig. 1 Yield surface and symmetry plane (*SP*) in reduced space of plane stress

It is beyond the present scope to scan the literature for further experimental results or to explore the restrictions imposed on other types of constitutive functions. The foregoing arguments may, however, for instance be applied to potential surfaces of constant dissipation rate for creeping materials.

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