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Nonlinear Contractions and Semigroups in Complete Fuzzy Metric Spaces

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Abstract

Our main purpose in this paper is to introduce the notion of ϕ-contractive mapping in fuzzy metric spaces and to present two new results on the existence and the approximation of fixed point of nonlinear contractions mappings and semigroups in fuzzy metric spaces. These results are of interest in view of analogous results in metric spaces (see for example [1], [7])

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1 Introduction

Our terminology and notation for fuzzy metric spaces conform of that George et al. [5, 6]. Recently Gregori et al. [4] have showed that the study of the intuitionistic fuzzy metric space (IFMS) $(X, M, N, *, \diamondsuit)$ can be reduced to the study of the fuzzy metric space (FMS) $(X, M, *)$. More exactly, the topology $\tau_{(MN)}$ of an IFMS $(X, M, N, *, \diamondsuit)$ coincides with the topology $\tau_{(M)}$ generated by the FMS $(X, M, *)$, which has as a base the family of open sets. So, our study is limited to FMSs.

Definition 1.1 *A subset B of FMS $(X, M, *)$ is called fuzzy bounded if for each $t > 0$ there exists $\lambda \in (0, 1)$ such that $M(x, y, t) \geq \lambda$ for all $x, y \in B$.*

Remark 1.2 *Let (X, d) be a metric space. Denote $a * b = \min\{a, b\}$ for all $a,\ b \in [0, 1]$ and let M_d be a fuzzy set on $X^2 \times (0, \infty)$ defined as follows:*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

It easy to check is that $(X, M_d, *)$ is a fuzzy metric space, and

$$B(x, r, t) = \{y \in :\ d(x, y) < \frac{rt}{1 - r}\}$$

So $(X, M_d, *)$ is a complete fuzzy metric space if and only if (X, d) is a complete metric space. Moreover, a nonempty subset A of $(X, M_d, *)$ is fuzzy bounded if and only if A is bounded in (X, d)

Lemma 1.3 *Every convergence sequence of a FMS is fuzzy bounded.*

Proof. Let $\{x_n\}$ be a convergence sequence of a FMS $(X, M, *)$ then there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$, which implies, for each $t > 0$ that $M(x_n, x, \frac{t}{2}) \to 1$, then there exists $N \in \mathbb{N}$ such that $M(x_n, x, \frac{t}{2}) > \frac{1}{2}$, $\forall n \geq N$. It follows

$$M(x_n, x, \frac{t}{2}) \geq R, \quad \forall n \in \mathbb{N}$$

where $R = \min\left\{\frac{1}{2}, \min_{\{n \leq N\}} M(x_n, x, \frac{t}{2})\right\}$. Then

$$M(x_n, x_p, t) \geq M(x_n, x, \frac{t}{2}) * M(x_p, x, \frac{t}{2}) \geq R * R.$$

For all $p, n \in \mathbb{N}$. This completes the proof of the Lemma

Definition 1.4 *Let $(X, M, *)$ be a FMS. Let $\phi : [0, \infty) \to [0, \infty)$ is an upper semi-continuous from the right function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$.*
We will say the sequence $\{x_n\}$ in X is fuzzy ϕ-contractive if for each $t > 0$,

$$\frac{t}{M(x_{n+2}, x_{n+1}, t)} - t \leq \phi\Big(\frac{t}{M(x_{n+1}, x_n, t)} - t\Big)$$

for all $n \in \mathbb{N}$.
A selfmap f on $(X, M, *)$ is called fuzzy ϕ-contractive if for each $t > 0$,

$$\frac{t}{M(f(x), f(y), t)} - t \leq \phi\Big(\frac{t}{M(x, y, t)} - t\Big)$$

for all $x, y \in X$.

Remark 1.5 *It is not hard to prove that every ϕ-contractive selfmap f on a metric (X, d) is fuzzy ϕ-contractive on $(X, M_d, *)$ ($*$ a t-norm such that $(X, M_d, *)$ is a fuzzy metric space). As it very easy to check that a fuzzy contractive mapping of contractive constant k is a ϕ-contractive with $r \to \phi(r) = rk$*

Lemma 1.6 *Every fuzzy bounded ϕ-contractive sequence $\{x_n\}$ of a FMS $(X, M, *)$ is a fuzzy Cauchy sequence.*

Proof. Let $\{x_n\}$ be a fuzzy bounded contractive sequence. Since $\{x_n\}$ is bounded then for $t > 0$ there exists $\lambda > 0$ such that $M(x_n, x_p, t) \geq \lambda$. On the other hand, since $\{x_n\}$ is ϕ-contractive then for $n > p \geq 0$ we have

$$\frac{t}{M(x_n, x_p, t)} - t \leq \phi^p\Big(\frac{t}{M(x_{n-p}, x_0, t)} - t\Big)$$

it follows

$$\frac{t}{M(x_n, x_p, t)} - t \leq \phi^p\Big(\frac{t}{\lambda} - t\Big)$$

Letting $p \to \infty$. Then we obtain

$$\frac{t}{M(x_n, x_p, t)} - t \to 0$$

which implies that $\{x_n\}$ is a Cauchy sequence.

2 Main Results

Theorem 2.1 *Let $(X, M, *)$ be a complete fuzzy metric space and f be a ϕ-contractive self-map on X. If there exists x_0 such that the sequence $\{f^n(x_0)\}$ is fuzzy bounded. Then f has a unique fixed point in X. Furthermore, the Picard iterates associate to each point of X converge to the fixed point*

Proof. Let $x_0 \in X$, such that $x_n = f^n(x_0),\ n \in \mathbb{N}$ is bounded. Since f is ϕ-contractive therefore $\{x_n\}$ is fuzzy bounded ϕ-contractive sequence. Hence from the Lemma 1.2 $\{x_n\}$ is a Cauchy sequence in a complete fuzzy metric space. Then there exists a point $x \in X$ such that $x_n \to x$ as $n \to \infty$. Now we shall show that $fx = x$.

Since f is ϕ-contractive then

$$\frac{t}{M(f(x_n), f(x), t)} - t \leq \phi\Big(\frac{t}{M(x_n, x, t)} - t\Big) \to 0$$

as $n \to \infty$, then $\lim_{n\to\infty} M(f(x_n), f(x), t) = 1$ for each $t > 0$. Therefore, $\lim_{n\to\infty} x_n = f(x)$ that is $f(x) = x$. Hence f has a fixed point $x \in X$. Next, let $z \in X$ then

$$\frac{t}{M(f^n(z), x, t)} - t = \frac{t}{M(f^n(z), f^n(x), t)} - t \leq \phi^n\Big(\frac{t}{M(z, x, t)} - t\Big) \to 0$$

as $n \to \infty$, then $\lim_{n\to\infty} M(f^n(z), x, t) = 1$ for each $t > 0$. Therefore, $\lim_{n\to\infty} f^n(z) = x$. We claim that x is the unique fixed point of f. For this suppose that y ($x \neq y$) is another fixed point of f in X. Then

$$\frac{t}{M(x, y, t)} - t = \frac{t}{M(f(x), f(y), t)} - t \leq \phi^n\Big(\frac{t}{M(x, y, t)} - t\Big)$$

Letting $n \to \infty$ we obtain $M(x, y, t) = 1$ for each $t > 0$, that is $x = y$, which is a contradiction. This completes the proof of the theorem.

As a consequence of Remark 1.2, Lemma 1.1 and Theorem 1.1, we can obtain the following

Corollary 2.2 [1] *Suppose that T is a ϕ-contractive selfmap on a complete metric space (X, d). Then T has a unique fixed point z and, moreover, for any $x \in X$, the sequence of iterates $\{T^n(x)\}$ converges to z.*

Corollary 2.3 [3] *Let $(X, M, *)$ be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let $f : X \to X$ be an fuzzy contractive mapping. Then f has a unique fixed point*

3 Common Fixed Point Theorem

Let S be a semigroup of selfmaps on $(X, M, *)$. For any $x \in X$, the orbit of x under S starting at x is the set $\mathcal{O}(x)$ defined to be $\{x\} \cup Sx$, where Sx is the set $\{g(x) : \ g \in S\}$. We say that S is left reversible if, for any f, g in S, there are $a, \ b$ such that $fa = gb$. It is obvious that left reversibility is equivalent to the statement that any two right ideals of S have nonempty intersection.
The next theorem can be seen as fuzzy version of a result of Huang et al. [5] on metric space

Theorem 3.1 *Suppose S is a left reversible semigroup of ϕ-contractive selfmaps on a complete fuzzy metric space $(X, M, *)$. If there exists a point x_0 in X such that $\mathcal{O}(x_0)$ is fuzzy bounded, then S has a unique common fixed point z and, moreover, for any $f \in S$ and $x \in X$, the sequence of iterates $\{f^n(x)\}$ converges to z.*

Proof. It follows from the Theorem 1.1 that each $f \in S$ has a unique fixed point z_f in X and for any $x \in X$, the sequence of iterates $\{f^n(x)\}$ converges to z_f. So, to complete the proof it suffices to show that $z_f = z_g$ for any $f, g \in S$. Let n be a positive integer, the left reversibility of S shows that are a_n and b_n in S such that $f^n a_n = g^n b_n$, and so

$$M(z_f, z_g, t) \geq M(z_f, f^n a_n(x_0), \frac{t}{2}) * M(z_g, g^n b_n(x_0), \frac{t}{2}). \tag{1}$$

Also, since f is ϕ-contractive we then have

$$\frac{t}{M(f^n(x_0), f^n a_n(x_0), s)} - t \leq \phi^n\Big(\frac{t}{M(x_0, a_n(x_0), s)} - t\Big)$$

Letting $n \to \infty$ in the last inequality and using the fact that $\mathcal{O}(x_0)$ is bounded we obtain

$$M(f^n(x_0), f^n a_n(x_0), s) \to 1. \tag{2}$$

Since

$$M(z_f, f^n a_n(x_0), t) \geq M(z_f, f^n(x_0), \frac{t}{2}) * M(f^n(x_0), f^n a_n(x_0), \frac{t}{2}). \tag{3}$$

Letting $n \to \infty$ in (3) and using $f^n(x_0) \to z_f$, we get

$$f^n a_n(x_0) \to z_f.$$

Likewise, we also have $g^n b_n(x_0) \to z_g$ which implies that, as $n \to \infty$ in (1) we obtain

$$M(z_f, z_g, t) \to 1$$

This completes the proof of the Theorem.

Corollary 3.2 [7] Suppose S is a left reversible semigroup of selfmaps on a complete metric space (X, d) such that the following conditions (i) and (ii) are satisfied

i. There is x_0 in X, its orbit $\mathcal{O}(x_0)$ is bounded;
ii. There exists a gauge function $\varphi : [0, \infty) \to [0, \infty)$ such that $d(fx, fy) \leq \varphi(d(x, y))$ for any f in S and x, y in X.

Then S has a unique common fixed point z and, moreover, for any $f \in S$ and $x \in X$, the sequence of iterates $\{f^n(x)\}$ converges to z.

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