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WAVE MOTION OF A NONLINEAR ELASTIC BAR SUBJECTED TO AXIAL EXCITATION

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ABSTRACT

This research intends to investigate the wave motion in a nonlinear elastic bar with large deflection subjected to an axial external exertion. A nonlinear elastic constitutive relation governs the material of the bar. General form of the nonlinear wave equations governing the wave motion in the bar is derived. With a modified complete approximate method, the asymptotic solution of solitary wave is developed for theoretical and numerical analyses of the wave motion. Various initial conditions and system parameters are considered for investigating the shape and propagation of the nonlinear elastic wave. With the governing equation of the wave motion of the bar and the solution developed, the characteristics of the nonlinear elastic wave of the bar are analyzed theoretically and numerically. Properties of the wave propagation and the effects of the system parameters of the bar and the influences of the initial conditions to the characteristics of the wave motion are investigated in details. Based on the theoretical analysis as well as the numerical simulations, it is found that the nonlinearity of the elastic bar may cause solitary wave in the bar. The velocity of the solitary wave propagating in the bar is related to the initial condition of the wave motion. This exhibits an obvious different characteristic between the nonlinear wave and that of the linear wave of an elastic bar. It is also found in the research that the solitary wave is a pulse wave with stable propagation. If the stability of the wave propagation is destroyed, the solitary wave will no longer exist. The results of the present research may provide guidelines for the wave motion analysis of nonlinear elastic solid elements.

1. INTRODUCTION

In solid mechanics, many researchers have studied the bifurcation and chaotic motion of the nonlinear elastic elements or systems with/without large deflection [1-3]. There are two main concerns in the systems of nonlinear differential equations for the waves in a nonlinear elastic element: namely, solitons and chaos of the solutions of the governing equations. A significant contribution was made by Luo et al [4] on predicting the chaotic response of a nonlinear rod. Chaotic and bifurcation conditions were provided. Investigations on the nonlinear behavior of a pinned Sine-Gordon soliton and the nonlinear Schr ö dinger systems are also reported [5]. Though more archival publications dealt with the propagation of the nonlinear wave in the structural elements taking into considerations of physical nonlinearity and material nonlinearity are found in the literature, systematic and thorough analyses on the behavior of the wave in highly nonlinear structural elements are still in need.

This paper investigates the wave motion in a nonlinear elastic structural element with large deflection subjected to an external exertion. The KdV-mKdV equation is to be established for the wave motion. An asymptotic solution of solitary wave is to be derived with utilization of a modified complete approximate method.

With the governing equation of the wave motion in the element and the solution derived, the characteristics of the nonlinear elastic wave of the element can be analyzed theoretically and numerically. Properties of the wave propagation and the effects of the system parameters of the elastic element and the influences of the initial conditions to the characteristics of the wave motion are investigated in details. For the wave propagations of nonlinear elastic solid elements, it is anticipated that the results of the present research can be furthered to the analytical and numerical investigations.

2.DEVELOPMENT OF GOVERNING EQUATION

A uniform nonlinear elastic circular structural element of infinite length as shown in Fig.1 is considered in this paper. In the figure, ρ is the mass per unit length and R is the radius of the element. The structural element so defined is axial-symmetric, cylindrical coordinates are therefore convenient for implementation.

Assume that the plane section hypothesis holds true in the case that the element is exerted by an axial compression or an extension impulse. Additionally, for the sake of clarification in developing the equations of motion of the element, the following fundamental hypotheses stand true.

a) The structural element is in one dimensional stress state during loading and wave propagating, i.e., $\sigma_r = \sigma_\theta = 0$.

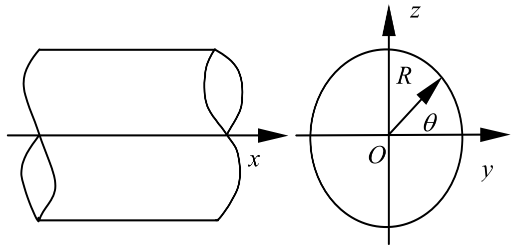


Fig.1 The structural element with infinite length

b) The lateral inertia is taken into consideration, i.e. $\varepsilon_r = -\nu\varepsilon_x$. Utilizing the geometric relations, one may have $u_r = r\varepsilon_r = -\nu r\dfrac{\partial u}{\partial x}$.

c) The material of the structural element obeys a nonlinear elastic constitutive relation; such that $\sigma_x = E\varepsilon_x + E\sum_{i=2}^{n}\alpha_i\varepsilon_x^i$, in which the linear portion is clearly distinguished from the rest items that describe the nonlinear properties of the element's material. In the constitutive relation, E is the elastic modulus, α_i and n are the material constants. The $\sigma_x - \varepsilon_x$ curves orresponding to the constitutive relation defined are as shown in Fig.2, where $n = 3$.

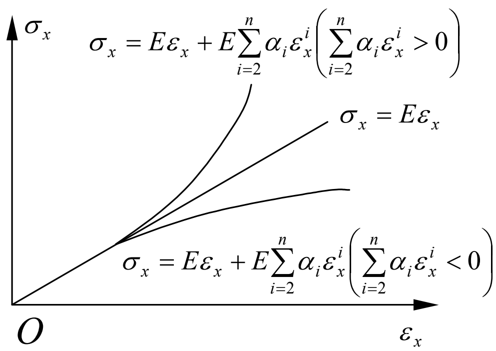


Fig.2 Material constitutive relation

Taking into consideration of the lateral inertia, the kinetic energy per unit length of the elastic element involves both the longitudinal and the radial energy and the total kinetic energy can be expressed in the following form:

$$\begin{aligned} T &= \int_0^R \frac{1}{2}\rho(2\pi r\,dr)\left(\frac{\partial u}{\partial t}\right)^2 + \int_0^R \frac{1}{2}\rho(2\pi r\,dr)\left(-\nu r\frac{\partial^2 u}{\partial t\partial x}\right)^2 \\ &= \frac{1}{2}\rho\omega\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{4}\rho\omega R^2\nu^2\left(\frac{\partial^2 u}{\partial t\partial x}\right)^2 \end{aligned} \tag{1}$$

where $\omega = \pi R^2$, is the area of the cross section of the structural element. Implementing the one-dimensional stress assumption as defined previously, the strain energy per unit length of the element can be expressed by the following equation.

$$\begin{aligned} W &= \int_0^R\left(\int_0^\varepsilon \sigma\,d\varepsilon\right)(2\pi r\,dr) = \omega\int_0^\varepsilon\left(E\varepsilon_x + E\sum_{i=2}^{n}\alpha_i\varepsilon_x{}^i\right)d\varepsilon_x \\ &= \frac{1}{2}\omega E\left(\frac{\partial u}{\partial x}\right)^2 + \omega E\sum_{i=2}^{n}\frac{1}{i+1}\alpha_i\left(\frac{\partial u}{\partial x}\right)^{(i+1)} \end{aligned} \tag{2}$$

Using Hamilton principle and taking $H = \delta\int_{t_1}^{t_2}\left(\int_{x_1}^{x_2}(T-W)dx\right)dt = 0$ together with the new energy expression of $F = T - W$, one may obtain

$$\begin{aligned} F &= \frac{1}{2}\rho\omega\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{4}\rho\omega R^2\nu^2\left(\frac{\partial^2 u}{\partial t\partial x}\right)^2 \\ &\quad - \frac{1}{2}\omega E\left(\frac{\partial u}{\partial x}\right)^2 - \omega E\sum_{i=2}^{n}\frac{1}{i+1}\alpha_i\left(\frac{\partial u}{\partial x}\right)^{(i+1)} \end{aligned} \tag{3}$$

According to Euler equation, if $F(u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, \cdots) = 0$, one has

$$F_u - \frac{\partial}{\partial x}F_{u_x} - \frac{\partial}{\partial y}F_{u_y} + \frac{\partial^2}{\partial x^2}F_{u_{xx}} + \frac{\partial^2}{\partial y^2}F_{u_{yy}} + \frac{\partial^2}{\partial x\partial y}F_{u_{xy}} - \cdots = 0 \tag{4}$$

the nonlinear governing equation for the elastic structural element is thus obtained as follows.

$$\frac{\partial^2 u}{\partial t^2} - \frac{E}{\rho}\left(1 + \sum_{i=2}^{n} i\,\alpha_i\left(\frac{\partial u}{\partial x}\right)^{i-1}\right)\left(\frac{\partial^2 u}{\partial x^2}\right) - \frac{1}{2}\nu^2R^2\frac{\partial^4 u}{\partial t^2\partial x^2} = 0 \tag{5}$$

where $c_0{}^2 = \dfrac{E}{\rho}$ and c_0 is the linear elastic longitudinal wave velocity. Let $n = 3$, Eq. (5) can be rewritten in the following form

$$\frac{\partial^2 u}{\partial t^2} - \frac{E}{\rho}(1 + 2\alpha_2 u_x + 3\alpha_3 u_x{}^2)\left(\frac{\partial^2 u}{\partial x^2}\right) - \frac{1}{2}\nu^2R^2\frac{\partial^4 u}{\partial t^2\partial x^2} = 0 \tag{6}$$

Introduce the new variables such that

$$\alpha' = 3\alpha_3, \quad \beta' = \frac{1}{2}\nu^2R^2 \quad , \tag{7}$$

and

$$\xi = x - c_0 t, \quad \tau = \alpha' t \quad , \tag{8}$$

and substitute them into Eq. (6) to obtain

$$\begin{aligned} & c_0{}^2u_{\xi\xi} - 2c_0\alpha' u_{\xi\tau} + \alpha'^2u_{\tau\tau} - c_0{}^2[1 + 2\alpha_2u_\xi + \alpha' u_\xi{}^2]u_{\xi\xi} \\ & - \beta'[c_0{}^2u_{\xi\xi\xi\xi} - 2c_0\alpha' u_{\xi\xi\xi\tau} + \alpha'^2u_{\xi\xi\tau\tau}] = 0 \end{aligned} \tag{9}$$

Let $\mu' = \dfrac{4\beta'}{\alpha' c_0{}^2}$ to have

$$\beta' = \frac{1}{4}\mu'\alpha' c_0{}^2, \quad (\alpha'^2 << 1) \quad , \tag{10}$$

Eq. (9) can then be further reduced into the following form:

$$2c_0\alpha' u_{\xi\tau} + c_0{}^2(2\alpha_2 u_\xi + \alpha' u_\xi{}^2)u_{\xi\xi} + \frac{1}{4}\mu'\alpha' c_0{}^4 u_{\xi\xi\xi\xi} = 0 \qquad (11)$$

Making a transformation of

$$v = \frac{\partial u}{\partial \xi}, \quad y = \frac{2}{c_0}\xi \quad , \qquad (12)$$

the KdV-mKdV equation can thus be obtained as,

$$v_\tau + \frac{2\alpha_2}{\alpha'} v v_y + v^2 v_y + \mu' v_{yyy} = 0 \qquad (13)$$

With the introduction of the following parameters,

$$\alpha'' = \frac{2\alpha_2}{\alpha'} = \frac{2\alpha_2}{3\alpha_3}, \qquad (14a)$$

$$\beta'' = 1, \qquad (14b)$$

$$\gamma = \mu' = \frac{2\nu^2 R^2}{3\alpha_3 c_0{}^2} \qquad (14c)$$

Eq. (13) can be rewritten as follows

$$v_\tau + \alpha'' v v_y + \beta'' v^2 v_y + \gamma v_{yyy} = 0 \,. \qquad (15)$$

Since $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}$ and $v = \frac{\partial u}{\partial \xi}$, v is the strain in the element.

Eq. (15) or Eq. (13) thus developed is the governing equation for the motion of the structural element and satisfies the strain conditions of the element.

3. SOLITARY SOLUTION

Introduce the following transformations

$$v \to 0.1w, \quad y \to y, \text{ and } \tau \to \tau \,. \qquad (16)$$

Additional, let $\alpha = 0.1\alpha''$, and β be a small constant such that $\beta = 0.01$; where $\alpha = \frac{\alpha_2}{15\alpha_3}$ and $\gamma = \frac{2\nu^2 R^2}{3\alpha_3 c_0{}^2}$. Substitute Eq. (16) into Eq. (13) to obtain

$$w_\tau + \alpha w w_y + \beta w^2 w_y + \gamma w_{yyy} = 0 \qquad (17)$$

Note that w in this equation is still the strain in the element. Assume that the w takes the following form:

$$w(y,\tau) = a\phi(\eta), \qquad (18a)$$

where

$$\eta = \sqrt{\frac{\lambda}{\gamma}}(y - \lambda\tau) \qquad (18b)$$

λ in this equation is the wave speed.

Let the initial condition be

$$w(y,\tau)\big|_{y=0,\tau=0} = a \,, \qquad (19)$$

where a is actually the maximum value of the strain when $\eta = 0$, and $\phi(0) = 1$. With the initial condition shown in Eq. (19) and the assumption indicated in Eq. (18), Eq. (17) can be written as follows

$$-\lambda\phi''' + \alpha a \phi\phi' - \lambda\phi' + \beta a^2 \phi^2 \phi' = 0 \qquad (20)$$

Let

$$\lambda = \frac{\alpha a}{3} + \beta\lambda_1 + O(\beta^2) \qquad (21)$$

Eq. (20) can be given as

$$\phi''' + 3\phi\phi' - \phi' + \beta(\frac{3a}{\alpha}\phi^2\phi' + \frac{3\lambda_1}{\alpha a}(\phi''' - \phi')) + O(\beta^2) = 0 \qquad (22)$$

By the following transformations

$$\begin{cases} \phi = \phi(\varsigma) + O(\beta^2) \\ \varsigma = \eta + \beta F[w(\eta)] + O(\beta^2), \qquad F[w(0)] = 0 \\ \frac{dF}{d\eta}[w(\infty)] < \infty \\ \frac{d^2F}{d\eta^2}[w(\infty)] < \infty \end{cases} , \qquad (23)$$

Eq. (22) becomes

$$\begin{aligned} &\phi'''(\varsigma) + 3\phi(\varsigma)\phi'(\varsigma) - \phi'(\varsigma) + \beta\Bigg[\phi'\frac{d^3F}{d\varsigma^3} + 3\phi''\frac{d^2F}{d\varsigma^2} \\ &+ (3\phi''' + 3\phi\phi' - \phi')\frac{dF}{d\varsigma} + \frac{3a}{\alpha}\phi^2\phi' + \frac{3\lambda_1}{\alpha a}(\phi''' - \phi')\Bigg] = O(\beta^2) \end{aligned} \qquad (24)$$

In the case that the coefficient β in Eq. (24) becomes zero, following two equations can be derived.

$$\begin{aligned} &\phi'\frac{d^3F}{d\varsigma^3} + 3\phi''\frac{d^2F}{d\varsigma^2} + (3\phi''' + 3\phi\phi' - \phi')\frac{dF}{d\varsigma} \\ &= -\frac{3a}{\alpha}\phi^2\phi' - \frac{3\lambda_1}{\alpha a}(\phi''' - \phi') \end{aligned} \qquad (25)$$

and

$$\phi'''(\varsigma) + 3\phi(\varsigma)\phi'(\varsigma) - \phi'(\varsigma) = 0 \,. \qquad (26)$$

Eq. (26) is one of the well-known nonlinear wave equations and has soliton solution [6,7]. The corresponding soliton solution is expressible as

$$\phi = \sec h^2 \frac{\varsigma}{2} \quad . \qquad (27)$$

Combining Eq. (26) with Eq. (25), one may have

$$\frac{d}{d\varsigma}\left[\frac{1}{\phi'}\frac{d}{d\varsigma}\left(\phi'^2\frac{dF}{d\varsigma}\right)\right] = -\frac{3a}{\alpha}\phi^2\phi' - \frac{3\lambda_1}{\alpha a}(\phi''' - \phi') \qquad (28)$$

in which

$$\phi'^2\frac{dF}{d\varsigma} = -\frac{a}{4\alpha}\phi^4 + \frac{3\lambda_1}{3\alpha a}\phi^3 \quad . \qquad (29)$$

Consider

$$\phi'^2 = \phi^2 - \phi^3, \quad \phi'(0) = 0\ , \qquad (30)$$

and let

$$\lambda_1 = \frac{1}{6}a^2 \quad , \qquad (31)$$

One may have

$$\frac{dF}{d\varsigma} = \frac{a}{4\alpha}\phi \quad . \qquad (32)$$

This leads to

$$F = \frac{a}{4\alpha}\int_0^{\varsigma}\phi\, d\varsigma = \frac{a}{4\alpha}\tanh\frac{\varsigma}{2} \quad . \qquad (33)$$

Using the following equation

$$\begin{aligned} w &= a\phi(\varsigma) + O(\beta^2) \\ &= a\phi(\eta + \beta F(a\phi)) + O(\beta^2) \\ &= a\phi(\eta) + a\beta\phi'(\eta)F\left[a\phi(\eta)\right] + O(\beta^2) \end{aligned} \qquad (34)$$

one may obtain

$$\begin{aligned} w &= a\sec h^2\frac{\eta}{2} - \frac{a^2\beta}{2\alpha}\sec h^2\frac{\eta}{2}\tanh^2\frac{\eta}{2} + O(\beta^2) \\ \eta &= \sqrt{\frac{\lambda}{\gamma}}(y - \lambda\tau), \\ \lambda &= \frac{\alpha\, a}{3} + \frac{1}{6}a^2\,\beta\,. \end{aligned} \qquad (35)$$

The solution of Eq. (17) may therefore be expressed in the following form.

$$w = \frac{a\,\sec h^2\left(\frac{1}{2}\sqrt{\frac{\lambda}{\gamma}}(y - \lambda\tau)\right)}{1 + \frac{a\beta}{2\,\alpha + a\,\beta}\tanh^2\left(\frac{1}{2}\sqrt{\frac{\lambda}{\gamma}}\,(y - \lambda\tau)\right)} \quad , \qquad (36)$$

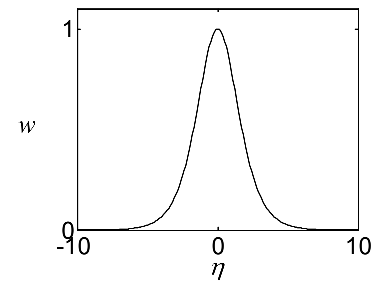
where

$$\lambda = \frac{\alpha\, a}{3} + \frac{1}{6}a^2\beta\ .$$

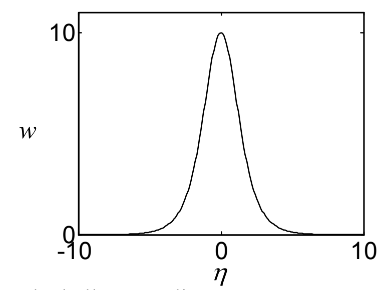
This implies that the wave width of the structural element is $2\pi\sqrt{\gamma/\lambda}$.

4. COMPUTATIONAL RESULTS AND DISCUSSIONS

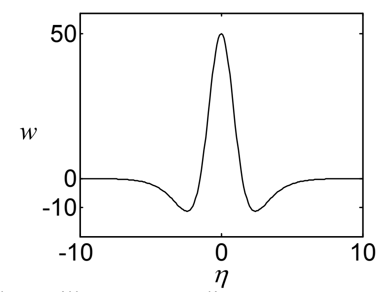
Computational results based on the solutions provided in Section 3 are illustrated in Fig. 3 corresponding to different values of α and a . It should be noted that w and η in the figures included in Fig. 3 are two non-dimensional parameters. As can be seen from the figures, there are two different appearances of the solitary waves. One is the bell-type solitary wave and another is the oscillatory-type solitary wave as indicated in the figures. With a constant α value, as illustrated in figures (a), (b), (c), (d) and the figures (e), (f), (g), (h) of Fig. 3, the appearance of the solitary wave changes from the bell-type to oscillatory-type as the initial strain in the structural element increases, provide that the initial strain in the element is small. Oscillatory-type wave will not occur when the value of α becomes large.



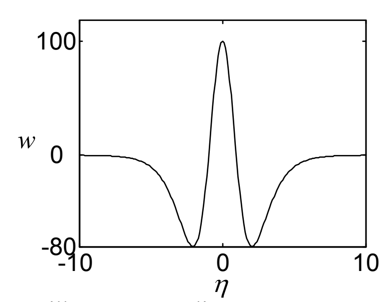
a) The bell-type solitary wave; $\alpha = 0.1, a = 1$



b) The bell-type solitary wave; $\alpha = 0.1, a = 10$

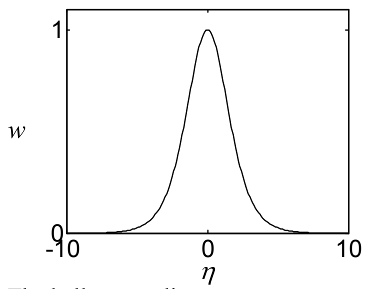


c) The oscillatory-type solitary wave $\alpha = 0.1,\ a = 50$

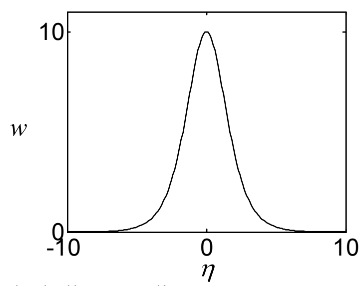


d) The oscillatory-type solitary wave; $\alpha = 0.1,\ \ a = 100$

Fig.3 Solitary waves in a nonlinear elastic structural element



e) The bell-type solitary wave; $\alpha = 0.5, a = 1$



f) The bell-type solitary wave $\alpha = 0.5,\ a = 10$

Fig.3 Solitary waves in a nonlinear elastic structural element (Continued)

5. CONCLUSION

The wave motion in an elastic structural element consisting of the material governed by a nonlinear elastic constitutive relation is studied in this paper. General form of the nonlinear wave equations governing the wave motion in the nonlinear elastic element is derived. With a modified complete approximate method, the asymptotic solution of solitary wave is developed for theoretical and numerical analyses of the wave motion. Shapes of the nonlinear elastic wave are studied with various initial conditions and system parameters. Findings of this research may provide guidelines for the wave motion analysis of nonlinear elastic solid elements.

Theoretical and numerical analyses both indicate that the material nonlinearity may cause solitary wave of the elastic structural element. As described in Section 3, the velocity of the solitary wave propagating in the element, λ , is related to the initial condition of the wave motion. The larger the initial wave amplitude is; the bigger is the wave velocity propagating along the structural element. This is a different characteristic of the nonlinear wave from that of the linear wave. It is also described in Section 3 that the initial condition controls the wave amplitude of the element. The wave width is related to the wave velocity and the value of γ which is related to the dispersion effect. As indicated in the context, the wave width is inverse proportional to the wave velocity, i.e., the larger the wave amplitude is; the narrower is the wave width. This implies that the solitary wave is directly related to the nonlinearity and the lateral properties of the material of the element.

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