Tropical geometry and its applications

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Abstract. From a formal perspective tropical geometry can be viewed as a branch of geometry manipulating with certain piecewise-linear objects that take over the rôle of classical algebraic varieties. This talk outlines some basic notions of this area and surveys some of its applications for the problems in classical (real and complex) geometry.

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1. Introduction

From a geometric point of view tropical geometry describes worst possible degenerations of the complex structure on an n-fold X. Such degenerations cause X to collapse onto an n-dimensional (over $\mathbb{R}$, i.e. $\left(\frac{\dim X}{2}\right)$-dimensional) base B which is a piecewise-linear polyhedral complex, see e.g. [13] for a conjectural picture in the special case of Calabi–Yau n-folds.

According to an idea of Kontsevich such degenerations can be useful, in particular, for computations of the Gromov–Witten invariants of X as holomorphic curves degenerate to graphs $\Gamma \subset B$. A similar picture appeared in the work of Fukaya (see e.g. [5]) where graphs come as degenerations of holomorphic membranes.

Tropical geometry formalizes the base B as an ambient variety so that the graphs Γ become curves in B. Some problems in complex and real geometry then may be reduced to problems of tropical geometry which are often much easier to solve, thanks to the piecewise-linear nature of the subject. Local considerations in tropical geometry correspond to some standard models in classical geometry while the combinatorial tropical structure encodes the way to glue these models together. In this sense it may be viewed as an extension of the Viro patchworking [32].

This talk takes a geometric point of view on tropical geometry and surveys its basic notions as well as some of its applications to classical algebro-geometrical problems. The author is indebted to Ya. Eliashberg, M. Kontsevich, A. Losev, B. Sturmfels and O. Viro for many useful conversations on tropical geometry.

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2. Tropical algebra

2.1. Tropical vs. classical arithmetics. The term *tropical semirings* was reputedly invented by a group of French computer scientists to commemorate their Brazilian colleague Imre Simon. For our purposes we use just one of these semirings, the tropical semifield $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ equipped with the operations of addition and multiplication defined by

$$\text{“}a+b\text{”} = \max\{a,b\}, \quad \text{“}ab\text{”} = a+b.$$

We use the quotation marks to distinguish the tropical operations from the classical addition and multiplication on $\mathbb{R}\cup\{-\infty\}$. It is easy to check that the tropical operations are commutative, associative and satisfy the distribution law.

There is no tropical subtraction as the operation “ $+$ ” is idempotent, but we have the tropical division “$\frac{a}{b}$” $= a-b$, $b \neq -\infty$. This makes $\mathbb{T}$ an idempotent semifield. Its additive zero is $0_{\mathbb{T}} = -\infty$, its multiplicative unit is $1_{\mathbb{T}} = 0$.

Remark 2.1. A classical example of a semifield is the semifield $\mathbb{R}_{\geq 0}$ of nonnegative numbers with classical addition and multiplication. We have the map $\log_t\colon \mathbb{R}_{\geq 0} \to \mathbb{T}$ between semifields $\mathbb{R}_{\geq 0}$ for any $t > 1$. The larger t, the closer is the map $\log_t$ to a homomorphism. More specifically, $\log_t$ is an isomorphism up to the error of $\log_t(2)$ while $\lim_{t\to+\infty} \log_t 2 = 0$. Indeed, we have $\log_t(ab) =$ “ $\log_t(a)\log_t(b)$” and

$$\text{“}\log_t(a) + \log_t(b)\text{”} \leq \log_t(a+b) \leq \text{“}\log_t(a)+\log_t(b)\text{”} + \log_t 2$$

by elementary considerations. Thus $\mathbb{T}$ can be considered as the $t \to +\infty$ limit semifield of a family of (mutually isomorphic) semifields $\mathbb{R}\cup\{-\infty\}$ equipped with arithmetic operations induced from $\mathbb{R}_{\geq 0}$ by $\log_t$. In this sense $\mathbb{T}$ is considered to be the result of *dequantization* of classical arithmetics by Maslov et al., see e.g. [15] or [33] for a geometric version of this dequantization.

Remark 2.2. The semifield $\mathbb{T}$ is closely related to non-Archimedean fields K with (real) valuations $\mathrm{val}\colon K \to \mathbb{T}$. Indeed, val is a valuation if for any $z, w \in K$ we have $\mathrm{val}(z+w) \leq$ “ $\mathrm{val}(z) + \mathrm{val}(w)$”, $\mathrm{val}(zw) =$ “ $\mathrm{val}(z)\,\mathrm{val}(w)$” and $\mathrm{val}^{-1}(-\infty) = 0$. In this sense val is a sub-homomorphism. This non-Archimedean point of view on tropical geometry is taken e.g. in [4], [25], [29] and [30].

2.2. Polynomials, regular and rational functions. There is no subtraction in the semifield $\mathbb{T}$, but we do not need it to form polynomials. A tropical polynomial is a tropical sum of monomials. For a polynomial f in n variables we get $f\colon \mathbb{T}^n \to \mathbb{R}$,

$$f(x) = \text{“}\sum_j a_j x^j\text{”} = \max(a_j + \langle j, x\rangle), \tag{1}$$

where $x = (x_1,\dots,x_n) \in \mathbb{T}^n$, $j = (j_1,\dots,j_n) \in \mathbb{Z}^n$, $x^j = x_1^{j_1}\dots x_n^{j_n}$, $\langle j, x\rangle = j_1x_1\dots j_nx_n$ and $a_j \in \mathbb{T}$.

Remark 2.3. Not all monomials in f are essential. It may happen that for some element j' we have $a_{j'} + \langle j', x\rangle \le a_j + \langle j, x\rangle$ for any $j \neq j'$. Then for any $b_{j'} \le a_j$ we have "$\sum_j a_j x^j$" = "$b_{j'} + \sum_{j\neq j'} a_j x^j$" = "$\sum_{j\neq j'} a_j x^j$". Thus the presentation of a function f as a tropical polynomial is not, in general, unique. Nevertheless, it is very close to being unique.

Note that (1) defines the Legendre transform of a (partially-defined) function on $\mathbb{R}^n$, $-a\colon j \mapsto -a_j$. The function $-a$ is defined only on a finite set, namely the set J which consists of the powers of the monomials in f. We can use involutivity of the Legendre transform to recover the coefficients a_j from the function f. Take the Legendre transform L_f of f and set $\bar{a}_j = -L_f(j)$, $j \in \mathbb{Z}^n$. It is easy to see that if f were a tropical polynomial then $a_j = -\infty = 0_{\mathbb{T}}$ for all but finitely many $j \in \mathbb{Z}^n$. Furthermore, if the function $-a$ is convex (i.e. if it is a restriction of a convex function on $\mathbb{R}^n$) then $\bar{a}_j = a_j$. Thus every function $f\colon \mathbb{T}^n \to \mathbb{T}$ obtained from a tropical polynomial has a "maximal" presentation as a tropical polynomial (clearly, $\bar{a}_j \ge a_j$).

Once we have polynomials we may form rational functions as the tropical quotients, i.e. the differences of two polynomials. The tropical quotient of two monomials is called a $\mathbb{Z}$-*affine* function. Clearly a $\mathbb{Z}$-affine function is an affine-linear function $\mathbb{R}^n \to \mathbb{R}$ with an integer slope.

A rational function $h =$ "$\frac{f}{g}$" is defined for every $x \in \mathbb{T}^n$ such that $g(x) \neq -\infty$. At such x h is called *finite* (more precisely, h is called finite at $x \in \mathbb{T}^n$ if it *can be presented* as "$\frac{f}{g}$" with $g(x) \neq -\infty$). A rational function h is called *regular* at $x \in \mathbb{T}^n$ if it is finite at x and there exist a $\mathbb{Z}$-affine function ϕ finite at x, an open neighborhood $U \ni x$ and a tropical polynomial $p\colon \mathbb{T}^n \to \mathbb{T}$ such that $h(y) = p(y) + \phi(y)$ for all $y \in U$. (Here we consider the Euclidean topology on $\mathbb{T}^n = [-\infty, +\infty)^n$.)

All functions $U \to \mathbb{T}$ that are restrictions of rational functions on $\mathbb{T}^n$ that are regular at every point of U form a semiring $\mathcal{O}(U)$. Constant functions give a natural embedding $\mathbb{T} \subset \mathcal{O}(U)$ and thus $\mathcal{O}(U)$ is a *tropical algebra*. In this way we get a sheaf $\mathcal{O}$ of tropical algebras on $\mathbb{T}^n$ called *the structure sheaf*.

3. Geometry: tropical varieties

3.1. Hypersurfaces. To every tropical polynomial f one may associate its hypersurface $V_f \subset \mathbb{T}^n$ that, by definition, consists of all points x where "$\frac{1_{\mathbb{T}}}{f}$" is not regular. This is a piecewise-linear object in $\mathbb{T}^n$.

Example 3.1. Figures 1 and 2 depict some hypersurfaces in $\mathbb{T}^2$ (i.e. planar curves).

The left-hand side of Figure 1 is given by the polynomial "$1 + 0x + 0y$". It is a line in the tropical plane $\mathbb{T}^2$. The right-hand side of Figure 1 is given by the polynomial "$10 + 5.5x + 0x^2 + 8.5y + 6.5y^2 + 4.5xy$", it is a conic. The line and the conic here intersect at two distinct points.

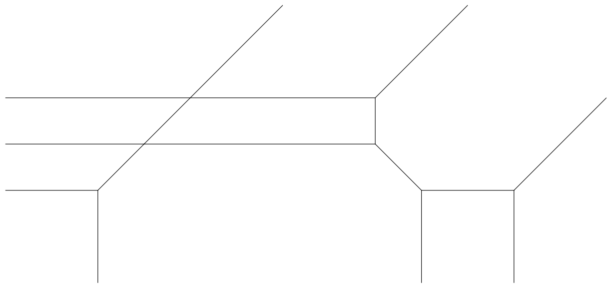

Figure 1. Tropical line and conic.

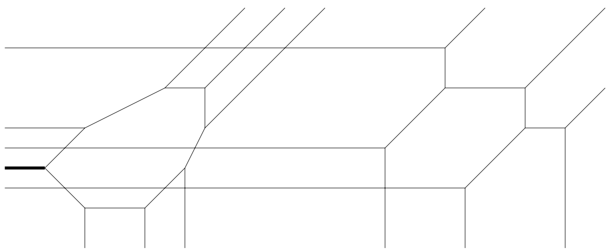

Figure 2. Two tropical cubic curves.

The left-hand side of Figure 2 is given by the polynomial

$$\text{“}5 + 4x + 2.25x^2 + 0x^3 + 4y + 2.5xy + 1x^2y + 3y^2 + 1.5xy^2 + 1.5y^3\text{”}. \quad (2)$$

The right-hand side is given by the polynomial

$$\text{“}17.5 + 12.25x + 7x^2 + 0x^3 + 16.75y + 12xy + 5.5x^2y + 15.5y^2 + 10xy^2 + 13y^3\text{”}.$$

It is easy to see that a tropical hypersurface V_f is a union of convex $(n-1)$-dimensional polyhedra called the facets of V_f. Each facet has *integer slope*, i.e. is parallel to a hyperplane in $\mathbb{R}^n$ defined over $\mathbb{Z}$. For every facet P there exist two monomials $a_{j_1}x^{j_1}$ and $a_{j_2}x^{j_2}$ of f that are equal along P and such that they are greater than any other monomial of f in the (relative) interior of P. The greatest common divisor of the components of the vector $j_2 - j_1$ is called *the weight* of P. E.g. the horizontal edge in Figure 2 adjacent to the leftmost vertex has weight 2 while all other edges have weight 1.

The intersection of any pair of facets is the face of both facets. Furthermore, at every $(n-2)$-dimensional face Q we have the following *balancing property*.

Property 3.2. *Let $P_1, \dots, P_l$ be the facets adjacent to Q and let v_j be the integer covectors whose kernels are parallel to P_j and whose gcd is equal to the weight of P_j.*

In addition we require that the orientation of $\nu_1, \dots, \nu_l$ *is consistent with a choice of direction around* Q. *Then we have*

$$\sum_{j=1}^{l} \nu_j = 0.$$

3.2. Integer polyhedral complexes. This balancing property may also be generalized to some piecewise-linear polyhedral complexes X of arbitrary codimension in $\mathbb{T}^n$. A *integer convex polyhedron* in $\mathbb{T}^n$ is the set defined by a finite number of inequalities of the type

$$\langle j, x \rangle \leq c,$$

where $x \in \mathbb{T}^n$, $j \in \mathbb{Z}^n$ and $c \in \mathbb{R}$. Here the expression $\langle j, x \rangle$ stands for the scalar product. It may happen that $\langle j, x \rangle$ is an indeterminacy (for some $x \in \mathbb{T} \smallsetminus \mathbb{T}^\times$), we include such x into the polygon. Equivalently, an integer convex polyhedron in $\mathbb{T}^n$ is the closure of a convex polyhedron (bounded or unbounded) in $\mathbb{R}^n$ such that the slopes of all its faces (including the polyhedron itself) are integers. The dimension of an integer convex polyhedron is its topological dimension.

An integer piecewise-linear polyhedral complex X of dimension k in $\mathbb{T}^n$ is the union of a finite collection of integer convex polyhedra of dimension k called *the facets of* X such that the intersection $\bigcap_{j=1}^{l} P_j$ of any finite number of facets is the common face of P_j. We may equip the facets of X with natural numbers called the weights.

The complex X is called balanced if the following property holds.

Property 3.3. *Let* Q *be a face of dimension* $k-1$ *and* $P_1, \dots, P_l$ *be the facets adjacent to* Q. *The affine-linear space containing* Q *defines a linear projection* $\lambda \colon \mathbb{R}^n \to \mathbb{R}^{n-k+1}$. *The image* $q = \lambda(Q)$ *is a point while the images of* $p_j = \lambda(P_j)$ *are intervals in* $\mathbb{R}^{n-k+1}$ *adjacent to* q. *Let* $v_j \in \mathbb{Z}^n$ *be the primitive integer vector parallel to* p_j *in the direction outgoing from* q *multiplied by the weight of* P_j. *We have*

$$\sum_{j=1}^{l} v_j = 0.$$

It is easy to see that if $k = n - 1$ then Properties 3.2 and 3.3 are equivalent.

3.3. Contractions. Let $f : \mathbb{T}^n \to \mathbb{T}$ be a polynomial. We define its *full graph*

$$\Gamma_f \subset \mathbb{T}^n \times \mathbb{T}$$

to be the hypersurface defined by "$y + f(x)$". Note that Γ_f can be obtained from the set-theoretical graph of f by attaching the intervals $[(x, -\infty), (x, f(x))]$ for all x from the hypersurface V_f (i.e. those x where "$\frac{0}{f}$" is not regular), cf. Figure 3 for the full graphs of "$x + 0$" and "$x^2 + x + 1$".

Thus, unlike the classical situation, the full graph of a map is different from the domain of the map. We define the *principal contraction*

$$\delta_f : \Gamma_f \to \mathbb{T}^n \tag{3}$$

associated to f to be the projection onto $\mathbb{T}^n$.

To get a general contraction one iterates this procedure. Suppose that a contraction $\gamma : V \to \mathbb{T}^n$ is already defined. We have $V \subset \mathbb{T}^N$.

If $g : V \to \mathbb{T}$ is a regular function (in N variables) then we can define the full graph $\Gamma_g \subset V \times \mathbb{T}$ as the union of the set-theoretical graph of g with all intervals $[(x, -\infty), (x, g(x))]$ such that "$\frac{0}{g}$" is not regular at x (i.e. at a neighborhood of x in V). The map $\delta_g : \Gamma_g \to V$ is the projection onto V.

The map δ_g is called a principal contraction to V. A general contraction is a composition of principal contractions.

Furthermore, one may associate *the weights* to the new facets of Γ_g by setting them equal to the *orders* of the pole of "$\frac{0}{g}$". Here we say that the order of a pole of a rational function is at least n at x if it can be locally presented as a tropical product of n rational functions that are not regular at x. Then we define the order of the pole to be the largest n with this property.

Definition 3.4. The *order of zero* of g at x is the order of the pole of "$\frac{0}{g}$" at x.

This definition is consistent with the definition of tropical hypersurfaces.

The facets of Γ_g contained in the set-theoretical graph of g inherit their weights from the weights of the corresponding facets of V.

Remark 3.5. Contractions may be used to define counterparts of Zariski open sets in tropical geometry. We define the complements of tautological embeddings $\mathbb{T}^k \to \mathbb{T}^n$, $k \le n$, to be Zariski open.

To define the principal open set corresponding to a polynomial f we consider the contraction $\delta_f : \Gamma_f \to \mathbb{T}^n$ and take

$$D_f = \Gamma_f \cap (\mathbb{T}^n \times \mathbb{T}^\times).$$

This together with the map $\delta_f|_{D_f}$ is the principal open set associated to f.

Example 3.6. The principal open set associated to "$x + a$" $D_{\text{“}x+a\text{”}} \to \mathbb{T}$ of "$x + a$", $a \in \mathbb{T}^\times$, can be interpreted as the result of puncturing $\mathbb{T}$ at a finite point $a \subset \mathbb{T}$. Figure 3 depicts $D_{\text{“}x+0\text{”}}$ and $D_{\text{“}x^2+x+1\text{”}}$. The corresponding maps are projections onto the x-axis.

Proposition 3.7. *If* $\delta : \mathbb{T}^N \supset V \to \mathbb{T}^n$ *is a contraction then* $V \subset \mathbb{T}^N$ *satisfies Property* 3.3.

Remark 3.8. In fact Proposition 3.7 can be used to define the weights of the facets of V and thus the orders of the zeroes of polynomials on V.

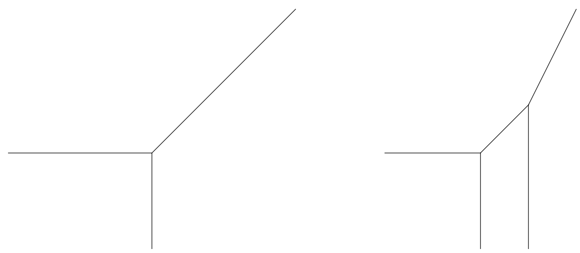

Figure 3. Once and twice punctured affine lines $\mathbb{T}$.

3.4. Tropical varieties and tropical morphisms. A map $\mathbb{R}^N \to \mathbb{R}^M$ is called *integer affine-linear* if it is a composition of a linear map defined over $\mathbb{Z}$ and a translation by an arbitrary vector in $\mathbb{R}^M$. Such a map can be extended to a partially defined map $\mathbb{T}^N \to \mathbb{T}^M$ by taking the closure. Note that integer affine-linear maps leave the class of integer piecewise-linear polyhedral complexes invariant. Furthermore, such maps take facets to facets (at least for some presentation of the image as an integer piecewise-linear polyhedral complex) and respect Property 3.3.

Let X be a topological space together with an atlas $\{U_\alpha\}$, $\phi_\alpha : U_\alpha \to \mathbb{T}^n$, $U_\alpha \subset X$.

Definition 3.9. We say that X is a *tropical variety* of dimension n if the following conditions hold.

1. Each ϕ_α is a contraction to an open subset of $\mathbb{T}^n$. More precisely, there is a contraction $\delta_\alpha : V_\alpha \to \mathbb{T}^n$, $V_\alpha \subset \mathbb{T}^{N_\alpha}$ and an open embedding $\Phi_\alpha : U_\alpha \to V_\alpha$ such that $\phi_\alpha = \delta_\alpha \circ \Phi_\alpha$.
2. The overlapping maps $\Phi_\beta \circ \Phi_\alpha^{-1}$ are induced by the integer affine-linear maps $\mathbb{R}^{N_\alpha} \to \mathbb{R}^{N_\beta}$.
3. For every point $x \in X$ in the interior of a facet of X there exists a chart ϕ_α such that $x \in U_\alpha$ and ϕ_α embeds some neighborhood of x into $\mathbb{T}^n$.
4. There exist a finite covering of X by open sets $W_j \subset X$ such that for every j there exists α such that $W_j \subset U_\alpha$ and the closure of $\Phi_\alpha(W_j)$ in $\mathbb{T}^{N_\alpha}$ is contained in $\Phi_\alpha(U_\alpha)$.

The last condition ensures *completeness* of the tropical structure on X.

A function on a subset W of X is called regular at a point $x \in W$ if it is locally a pull-back of a regular function in a neighborhood of $\Phi_\alpha(x) \in \mathbb{T}^{N_\alpha}$. In this way we get the structure sheaf $\mathcal{O}_X$ of regular functions on X.

A point $x \in X$ is called *finite* if it is mapped to $\mathbb{R}^{N_\alpha} \subset \mathbb{T}^{N_\alpha}$. Note that finiteness does not depend on the choice of a chart. At finite points of x we have the notion of an *integer tangent* vector to X. This comes from tangent vectors to $\Phi_\alpha(U_\alpha)$ at $\Phi_\alpha(x) \in \mathbb{R}^{N_\alpha}$ after identification with the corresponding counterparts for different charts $U_\beta \ni z$ under the differential of the overlapping maps.

Example 3.10. The space $\mathbb{T}^n$ is a n-dimensional tropical variety tautologically. The *projective n-space* $\mathbb{TP}^n$ is defined as the quotient of $\mathbb{T}^{n+1} \setminus (-\infty, \dots, -\infty)$ by the equivalence relation $(x_0, \dots, x_n) \sim (``\lambda x_0", \dots, ``\lambda x_n") = (\lambda + x_0, \dots, \lambda + x_n)$, where $\lambda \in \mathbb{T}^\times$.

The space $\mathbb{TP}^n$ is a tropical variety since it admits (as in the classical case) $n+1$ affine charts to $\mathbb{T}^n$ by dividing all coordinates by x_j as long as $x_j \neq -\infty$. This is an example of a compact tropical variety. There is a well-defined notion of a hypersurface of degree d in $\mathbb{TP}^n$. It is given by a homogeneous polynomial of degree d in $n+1$ variables. This polynomial can be translated to an ordinary polynomial in every affine chart of $\mathbb{TP}^n$.

In a similar way, one may construct tropicalizations of other toric varieties. The finite part of all tropical toric varieties is $(\mathbb{T}^\times)^n \approx \mathbb{R}^n$.

Proposition 3.11. *If X and Y are tropical varieties then $X \times Y$ has a natural structure of tropical variety of dimension* $\dim X + \dim Y$.

Definition 3.12. Let $f\colon X \to Y$ be a map between tropical varieties (not necessarily of the same dimension). We say that f is a *linear tropical morphism* if for every $x \in X$ there exist charts $U^X_\alpha \ni x$ and $U^Y_\beta \ni f(x)$ such that $\Phi^Y_\beta \circ f \circ (\Phi^X_\alpha)^{-1}$ is induced by an integer affine-linear map $\mathbb{R}^{N_\alpha} \to \mathbb{R}^{N_\beta}$.

The map f is called a *regular tropical morphism* if $\Phi^Y_\beta \circ f \circ (\Phi^X_\alpha)^{-1}$ is given by N_β rational functions on $\mathbb{T}^{N_\alpha}$ that are regular on $\Phi^X_\alpha(U^X_\alpha)$.

Clearly any linear morphism is a regular morphism. A regular morphism $f\colon X \to Y$ defines a map $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$. This map can be interpreted as a homomorphism of tropical algebras (defined over the semifield $\mathbb{T}$).

Proposition 3.13. *If $f\colon X \to Y$ is a linear tropical morphism then its* (*set-theoretical*) *graph is a* $(\dim X)$*-dimensional tropical variety.*

3.5. Equivalence of tropical varieties. Different tropical varieties may serve as different models for essentially the same variety. To identify such tropical varieties we globalize the notion of contraction that was so far defined only for $V \subset \mathbb{T}^n$.

Let $f\colon X \to Y$ be a tropical morphism between tropical varieties of the same dimension.

Definition 3.14. The map f is called a *contraction* if for every $y \in Y$ there exist a chart $U^Y_\beta \ni y$ with $\Phi^Y_\beta(U^Y_\beta) \subset V_\beta \subset \mathbb{T}^{N_\beta}$, a contraction $\delta\colon W \to V_\beta$ and an isomorphism of polyhedral complexes

$$h\colon f^{-1}(U^Y_\beta) \approx \delta^{-1}(\Phi^Y_\beta(U_\beta))$$

such that $\Phi^Y_\beta \circ f = \delta \circ h$.

Note that a composition of contractions is again a contraction. A contraction generates an equivalence relation on the class of tropical varieties: tropical manifolds X

and Y are called equivalent if they can be connected by a sequence of contractions or operations inverse to contractions.

Example 3.15. The cubic curve given by (2) and pictured on the left-hand side of Figure 2 is equivalent to the circle S^1 equipped with the tropical structure coming from $\mathbb{R}/4.5\mathbb{Z}$ (as $\mathbb{R} = \mathbb{T}^\times$ is a tropical variety and the translation by 4.5 is a tropical automorphism there is a well-defined tropical structure on the quotient).

In the same time the real number 4.5 is an inner invariant of this cubic curve. It is a tropical counterpart of the J-invariant of elliptic curves.

It is convenient to identify equivalent tropical varieties. This allows to present an arbitrary regular morphism $f : X \to Y$ in Definition 3.12 by a linear tropical morphism.

Proposition 3.16. *If $f : X \to Y$ is a regular tropical morphism then there exists a contraction $\delta : \tilde{X} \to X$ such that $f \circ \delta : \tilde{X} \to Y$ is a linear tropical morphism.*

Example 3.17. The map $\mathbb{T} \to \mathbb{T}$ defined by $x \mapsto f(x) =$ "$x^2 + x + 1$" is a regular morphism which is not linear. However, the map $f \circ \delta_f : \Gamma_f \to \mathbb{T}$ is a linear morphism as it is given by the projection of the full graph Γ_f onto the vertical axis (cf. the left-hand side of Figure 3).

There is a well-defined notion of a k-form on a tropical variety that is preserved by the tropical equivalence.

Definition 3.18. A k-form on a tropical variety X is an exterior real-valued k-form of the integer tangent vectors at every finite point $x \in X$ such that for every chart U_α this form is induced from a (constant) linear k-form on $\mathbb{R}^{N_\alpha}$.

A k-form ω on X is called globally defined if the following condition holds for every non-finite point $x \in X$. If $x \in U_\alpha$ and $\Phi_\alpha(x) \in \mathbb{T}^{N_\alpha}$ has its jth coordinate $-\infty$ then the form ω in $\mathbb{R}^{N_\alpha}$ vanishes on the kernel of the projection onto the jth coordinate hyperplane.

E.g. the only globally defined k-form on $\mathbb{TP}^n$ is the zero form. But there might be globally defined k-forms on other compact varieties. An easy example is provided by taking X to be a *tropical torus*, the quotient of $\mathbb{R}^n$ by translation from some integer lattice $\Lambda \subset \mathbb{R}^n$ of rank n. Such X is a compact tropical variety while the globally defined k-forms on X are in 1-1 correspondence with constant linear k-forms on $\mathbb{R}^n$.

If $\delta : \tilde{X} \to X$ is a contraction then a globally defined k-form must vanish at all vectors in the kernel of $d\delta$, thus there is a 1-1 correspondence between forms on $\tilde{X}$ and X.

4. Tropical intersection theory

4.1. Cycles in X. The notion of an integer piecewise-linear polyhedral complex may be extended to include not only complexes in $\mathbb{T}^n$, but also complexes in an arbitrary tropical variety X. We say that $B \subset X$ is an integer piecewise-linear polyhedral complex if for every chart $U_\alpha \subset X$ there exists an integer piecewise-linear polyhedral complex $B_\alpha \subset V_\alpha \subset \mathbb{T}^{N_\alpha}$ such that $\Phi_\alpha(B) \subset B_\alpha$.

As the overlapping maps preserve convex polyhedra we have a well-defined notion of a maximal facet on B and thus can consider *weighted* integer piecewise-linear polyhedral complexes in X.

Definition 4.1. A *k-cycle* B in a tropical variety X is a k-dimensional integer piecewise-linear polyhedral complex weighted by integer (possibly negative) numbers that satisfies Property 3.3 in every chart $\Phi_\alpha \colon U_\alpha \to V_\alpha \subset \mathbb{T}^{N_\alpha}$ of X. Accordingly, the codimension of B is $n-k$.

All k-cycles in X form a group by taking unions.

Remark 4.2. In this definition we excluded k-cycles with boundary components. Indeed, by our definition, every k-dimensional convex polyhedron in $\mathbb{T}^n$ is the closure of a convex polyhedron in $\mathbb{R}^n$. Thus we do not have components lying totally in $\mathbb{T}^n \smallsetminus \mathbb{R}^n$. While it is useful also to consider cycles with boundary components, their intersection theory is more elaborate.

Proposition 4.3. *The image $f_*(B) \subset Y$ of a k-cycle B under a linear tropical morphism $f \colon X \to Y$ is a k-cycle.*

In particular, a tropical subvariety is a cycle. An important example is the *fundamental cycle* of the n-dimensional tropical variety X. It is the n-cycle where each facet of X is taken with its own weight.

4.2. Cycle intersections. One very useful feature of tropical varieties is the possibility to intersect cycles there.

Let B_1, B_2 be two cycles of codimension k_1 and k_2 in the same tropical variety X. The goal of this section is to define their *product-intersection* $B_1.B_2$ as a cycle of codimension k_1+k_2 in X.

We start from an easier case when the ambient variety X is $\mathbb{T}^n$. The set-theoretical intersection $B_1 \cap B_2$ is naturally stratified by convex polyhedra that are intersections of the (convex) faces of B_1 and B_2. It might happen that the dimension of some of these polyhedra is greater than $n-(k_1+k_2)$.

Definition 4.4. We define $B_1.B_2$ as the closure of the union of the strata of dimension $n-(k_1+k_2)$ in $B_1 \cap B_2$ equipped with certain weights which we define as follows.

- Suppose that an $(n-(k_1+k_2))$-dimensional stratum $S \subset B_1 \cap B_2$ is the intersection of two facets $F_1 \subset B_1$ and $F_2 \subset B_2$. Let $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^n$ be the subgroups consisting of all integer vectors parallel to F_1 and F_2 respectively. Since by assumption S is of codimension k_1+k_2 the sublattice $\Lambda_1 + \Lambda_2 \subset \mathbb{Z}^n$ is of finite index. We set the weight of S equal to the product of this index and the weights of F_1 and F_2.

- Suppose that the $(n-(k_1+k_2))$-stratum $S \subset B_1 \cap B_2$ is the intersection of perhaps smaller-dimensional faces $G_1 \subset B_1$ and $G_2 \subset B_2$. We choose a small vector $\vec{v} \in \mathbb{R}^n$ in the generic (non-rational with non-rational projections) direction and denote by $\tau_v \colon \mathbb{T}^n \to \mathbb{T}^n$ the translation by v (which, clearly, extends from $\mathbb{R}^n$ to $\mathbb{T}^n$). The face G_1 is adjacent to a finite number of facets $F_1^{(\alpha)}$ of B_1 while the face G_2 is adjacent to a finite number of facets $F_2^{(\beta)} \subset B_2$. As v is chosen to be generic the facets $F_1^{(\alpha)}$ and $\tau_v(F_2^{(\beta)})$ intersect transversely along a convex polyhedron parallel to S. Thus the weight of their intersection is already defined. We assign to S the weight equal to the sum of the weights of all such intersections (where the weight equals zero if $F_1^{(\alpha)}$ and $\tau_v(F_2^{(\beta)})$ are disjoint). Proposition 4.5 asserts that this total sum does not depend on the choice of v.

This definition is essentially the same as the definition of *stable intersection* from [25], note also similarities with the *Minkowski weights* from [6].

In the general case one can use in a certain way the contraction charts $\phi_\alpha : U_\alpha \to \mathbb{T}^n$ and the product-intersections $\phi_\alpha(B_1).\phi_\alpha(B_2)$ to define $B_1.B_2$ for an arbitrary tropical variety X.

Proposition 4.5. *The product-intersection $B_1.B_2$ is well defined. It gives an $(n-(k_1+k_2))$-dimensional cycle in X. The operation of taking the product-intersection is commutative and associative.*

4.3. Pull-backs, deformations, linear equivalence. Proposition 4.3 allows us to take the push-forward of a cycle under a linear tropical morphism. Using the product-intersection we can also define a pull-back.

Let $f\colon X \to Y$ be a linear tropical morphism and $B \subset Y$ is a k-cycle. The product $X \times B$ is a $(\dim X + k)$-cycle while the (set-theoretical) graph Γ_f of f is a $(\dim X)$-cycle in $X \times Y$. Their product-intersection $(X \times B).\Gamma_f$ is thus a $(k + \dim X - \dim Y)$-cycle in $X \times Y$. We define the pull-back of B by

$$f^*(B) = \pi^X_*((X \times B).\Gamma_f) \subset X,$$

where $\pi^X\colon X \times Y \to X$ is the projection onto X.

Any cycle $B \subset X \times Y$ can be considered as a family of cycles in X. Indeed, every $y \in Y$ defines a cycle

$$B_y = \pi^X_*(B.(X \times \{y\})) \subset X.$$

Definition 4.6. We call such a family *algebraic* and two cycles that appear in the same family with a connected B results of *deformation* of each other. Two cycles are called *linearly equivalent* if they appear in the same family with $Y = \mathbb{TP}^1$.

Proposition 4.7. *If $B_1, B_2 \subset X$ are two cycles and $B_1', B_2' \subset X$ are results of their deformation then $B_1'.B_2'$ is a result of deformation of $B_1.B_2$.*

The deformations are especially interesting in the case when X is compact as the following proposition shows. Note that a 0-cycle in X (as by our assumption X is covered by a finite collection of charts) is a finite union of points. We define the *degree* of a 0-cycle $B \subset X$ to be the sum of the weights of all points of B.

Proposition 4.8. *The degree of a* 0*-cycle in a compact tropical variety is a deformation invariant.*

We may define the *intersection number* of a collection of cycles of total codimension n as the degree of their product-intersection.

Example 4.9. All hypersurfaces of the same degree in $\mathbb{TP}^n$ (see Example 3.10) can be obtained from each other by deformation. Furthermore, they are linearly equivalent. We have the tropical Bezout theorem: the intersection number of n hypersurfaces of degree $d_1, \dots, d_n$ is equal to $\prod_{j=1}^{n} d_j$, cf. [31].

We can see the illustration to this theorem in Figures 1 and 2. In Figure 1 we have a line and a conic and they intersect in two distinct points. The weight of each of these intersection points is 1 as the primitive integer vectors parallel to the edges at the points of intersection form a basis of $\mathbb{Z}^2$. In Figure 2 we have two cubics that intersect at eight points. Out of these eight points seven have weight 1 while one, the point of intersection of a horizontal edge with an edge of slope 2, has weight 2. Thus the total intersection number of these two cubics equals 9.

4.4. Intersection with divisors. Taking the product-intersection simplifies in the case when one of the cycles is a (Cartier) divisor.

Definition 4.10. A *divisor* $D \subset X$ is a finite formal linear combination with integer coefficients of $(\dim X - 1)$-cycles given by an open covering $\{U_\alpha\}$ and regular functions $f_\alpha \colon \mathcal{U}_\alpha \to \mathbb{T}$ that defines $D \cap U_\alpha$ as its hypersurface.

Not every cycle of codimension 1 in X is a divisor.

Example 4.11. Let $X \subset \mathbb{T}^3$ be the hypersurface given by "$x+y+z+0$" (see Figure 4). Let $B \subset X$ be the line $\{(t, t, 0) \mid t \in \mathbb{T}\}$. It is a 1-cycle in the 2-dimensional tropical variety X. Yet it cannot be presented as a hypersurface near the point $(0, 0, 0)$ which is the vertex of X.

If $B \subset X$ is a k-cycle and $D \subset X$ is a divisor then in each U_α the product-intersection $D.B$ coincides with the hypersurface defined by f_α on B (i.e. with the points where "$\frac{1_{\mathbb{T}}}{f_\alpha}$" is not regular weighted by the order of poles of "$\frac{1_{\mathbb{T}}}{f_\alpha}$").

Similarly, we can define a pull-back of a divisor $D \subset Y$ under a regular morphism $h \colon X \to Y$ by taking the pull-backs of the functions f_α. As the pull-backs $(f_\alpha) \circ h$ are regular on $h^{-1}(U_\alpha)$ they define a divisor $h^*(D) \subset X$.

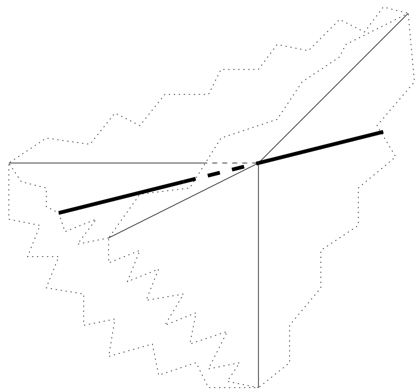

Figure 4. A cycle of codimension 1 which is not a (Cartier) divisor.

5. Tropical curves

5.1. Tropical curves as metric graphs. As in the classical case the easiest varieties to understand are curves. A tropical structure on a curve (which is topologically a graph) can be expressed by introducing a metric. Such presentation is not unlike the presentation of complex curves of negative Euler characteristic with hyperbolic surfaces. For simplicity we restrict our attention to compact tropical curves.

Recall that a *leaf* of a graph is an edge adjacent to a 1-valent vertex. We call an edge which is not a leaf a *finite edge*. Denote the set of all 1-valent vertices by Vert_1.

Definition 5.1 (cf. e.g. [1]). A *metric graph* is a finite graph Γ such that every finite edge has a prescribed positive real length. The length of all leaves is set to be $+\infty$.

This makes $\Gamma \setminus \mathrm{Vert}_1(\Gamma)$ a complete metric space (equipped with an inner metric). We denote the resulting metric by d_Γ. A homeomorphism between metric graphs is an isomorphism if it is an isometry on $\Gamma \setminus \mathrm{Vert}_1(\Gamma)$. Note that a presentation of a topological space as a graph is not unique, at our will we may introduce or erase 2-valent vertices.

Proposition 5.2. *There is a* 1-1 *correspondence between compact tropical curves and metric graphs.*

A primitive integer tangent vector at a point of a tropical curve (in every chart) has the unit length in the corresponding metric.

Proposition 5.3. *A map $f : \Gamma \to \Gamma'$ between tropical curves is a regular morphism if there exists a presentation of Γ as a graph so that for every edge $E \subset \Gamma$ there exists $n(E) \in \mathbb{N} \cup \{0\}$ such that*

$$d_{\Gamma'}(f(x), f(y)) = n(E) d_\Gamma(x, y)$$

for any $x, y \in E$.

Presentation as a metric graph is a convenient way to specify a tropical structure on a curve. The *genus* of a tropical curve Γ is its first Betti number $b_1(\Gamma)$. (This term is justified by Proposition 5.5.)

Example 5.4. Figure 5 depicts all tropical curves of genus 2. Varying the length of the edges we vary the tropical structure on the curve. For the curves in the left and in the right we have three lengths of edges, a, b and c, to vary. The middle curve is an intermediate case when one of these lengths becomes zero.

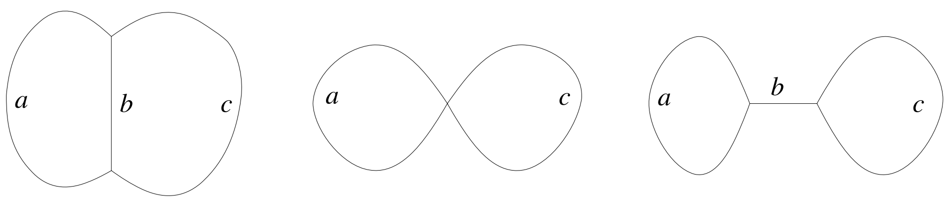

Figure 5. Tropical curves of genus 2.

This shows that the space of tropical curves of genus 2 is 3-dimensional. Furthermore, one may observe that these curves are hyperelliptic as there exists a non-trivial isometry involution for each of the three types in the picture. (The fixed points of this isometry are the midpoints of the edges for the left picture; the midpoint of the edges and the vertex for the center picture; the edge connecting the vertices and the midpoints of the other edges for the right picture.)

5.2. Jacobian varieties and the Riemann–Roch inequality

Proposition 5.5. *All* 1*-forms on a compact tropical curve* Γ *form a real vector space* $\Omega(\Gamma)$ *of dimension* $b_1(\Gamma)$*, i.e. the genus of* Γ.

Note that any 0-cycle on a curve Γ is a divisor in the sense of Definition 4.10. The divisors of degree 0 form a group Div_0 while the divisors of degree d form a homogeneous space Div_d over Div_0. We define the *Picard group* Pic_0 by taking the quotient group of Div_0 by the linear equivalence (see Definition 4.6). Similarly we define Pic_d.

A divisor is called *principal* if it is linearly equivalent to zero. Clearly, the degree of a principal divisor is 0. The classical Mittag-Leffler problem to determine whether a divisor of degree 0 is principal is answered by the Abel–Jacobi theorem. A similar answer exists also in the tropical set-up.

Note that given a 1-form ω and a path $\gamma : [0,1] \to \Gamma$ there is a well-defined integral $\int_\gamma \omega$. Clearly, the value of this integral depends only on the relative homology class of γ. Let $\Omega^*(\Gamma)$ be the (real) dual vector space to $\Omega(\Gamma)$. Its dimension is equal to the genus g of Γ by Proposition 5.5. Each element $a \in H_1(\Gamma;\mathbb{Z})$ determines a point of $\Omega^*(\Gamma)$. This point is given by the functional

$$\Lambda(a)\colon \omega \mapsto \int_a \omega.$$

The *Jacobian* of a tropical curve Γ is defined by

$$\mathrm{Jac}(\Gamma) = \Omega^*(\Gamma)/\Lambda(H_1(\Gamma;\mathbb{Z})).$$

The Jacobian is an example of a tropical torus. It is homeomorphic to $(S^1)^g$ and carries the structure of a tropical variety (which depends on the lattice $\Lambda(H_1(\Gamma;\mathbb{Z}))$.

To define the *Abel–Jacobi* map

$$\alpha\colon\ \mathrm{Pic}_0(\Gamma) \to \mathrm{Jac}(\Gamma) \tag{4}$$

we take any 1-chain C whose boundary is a divisor in the equivalence class $p \in \mathrm{Pic}_0$ and set $\alpha(p)$ to be the functional $\omega \mapsto \int_C \omega$.

Proposition 5.6 (Tropical Abel–Jacobi theorem). *The Abel–Jacobi map* (4) *is a well-defined bijection.*

This gives the structure of a tropical variety on Pic_0 as well as on Pic_d (which is a homogeneous space over Pic_0). We have the tautological map

$$\mathrm{Sym}^d(\Gamma) \to \mathrm{Pic}_d(\Gamma)$$

defined by taking the equivalence class.

As in the classical case this map is especially interesting in the case $d = g - 1$. The image of this map in this case (as well as all its translates in $\mathrm{Pic}_0 = \mathrm{Jac}$) is called the Θ-*divisor*. There is a tropical counterpart of the Riemann theorem stating that the Θ-divisor is given by a θ-function, see [23].

Figure 6 sketches the Θ-divisor in the case of genus 2. In this case it is isomorphic to the curve Γ itself. The Jacobian in Figure 6 is obtained by identifying the opposite sides of the dashed parallelogram.

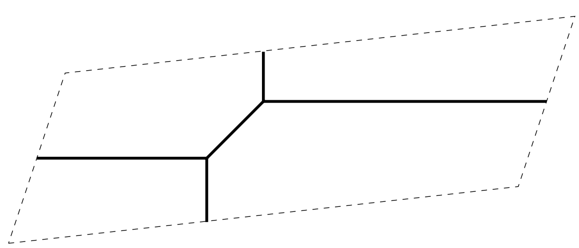

Figure 6. Θ-divisor for genus 2.

The divisor is called *effective* if the weights at all its points are positive. The Riemann–Roch theorem allows to find the dimension of all effective divisors linearly equivalent to a given one. (Alternatively, this number plus one can be interpreted as the dimension of the space of sections of the line bundle corresponding to a given divisor.) Tropically, we have the Riemann–Roch theorem in form of an inequality.

Proposition 5.7 (The tropical Riemann–Roch inequality). *The dimension of the space of effective divisors in the equivalence class $p \in \mathrm{Pic}_d$ is at least $d - g$.*

5.3. The canonical class, regular and superabundant curves in X. We have the Chern classes for compact tropical varieties which are easy to compute (as a tropical variety is already parallelized by the integer affine structure on its facets).

Let X be an n-dimensional compact tropical variety. We have the natural stratification of the faces of X by their dimension. Furthermore, there is the *boundary stratification* of X. We say that a face $F \subset X$ is in the kth boundary stratum if there is a chart $U_\alpha \ni x$ where $\Phi_\alpha(F)$ is contained in an intersection of k out of N_α tropical coordinate hyperplanes of $\mathbb{T}^{N_\alpha}$ and not contained in the intersection of any $(k+1)$ coordinate hyperplanes.

A face in the 0th boundary stratum is called a *finite* face. Clearly a face in the kth boundary stratum has dimension at most $(n-k)$.

The kth Chern class of X is an $(n-k)$-cycle that is the linear combination of all $(n-k)$-dimensional faces of X. Here the strata of boundary codimension k are taken with the weight equal to $(-1)^k$ times some positive number depending only on the local geometry near a point x in the relative interior of F (this weight can be computed inductively from a local presentation of a neighborhood of x by a contraction $\delta\colon V \to \mathbb{T}^n$).

Here we give the computation of weights only in the case of $-c_1$. This is the $(n-1)$-cycle in X that consists of all finite $(n-1)$-faces F taken with the weight equal to the number of adjacent facets to F minus 2 and all $(n-1)$-faces in the 1-boundary stratum taken with the weight -1.

Definition 5.8. The *canonical class* K is the $(n-1)$-cycle in X equal to $-c_1$.

A *parameterized tropical curve* in X is a linear tropical morphism $h\colon \Gamma \to X$. One may use Proposition 5.7 to compute the dimension in which the map h varies (we allow to deform both h and the tropical structure on Γ).

The adjunction formula exists also in the tropical geometry. As in this talk we have not defined tropical vector bundles in general and, in particular, the normal bundles, we state it only in the case when h is an embedding and X is a compact surface. We have

$$h_*([\Gamma]).h_*([\Gamma]) = 2g - 2 - K.h_*([\Gamma]). \tag{5}$$

Here $[\Gamma]$ stands for the fundamental 1-cycle in Γ.

Example 5.9. Consider the compactification of Example 4.11. Let $X \subset \mathbb{TP}^3$ be the closure of the hypersurface $x + y + z + 0$ and $h\colon \mathbb{TP}^1 \to X$ be the linear tropical morphism whose image is the closure of B. The two infinite ends of B contribute $+1$ each to its self-intersection, thus the self-intersection contribution of B at $(0, 0, 0)$ is -1 (see Figure 4)

Proposition 5.10. *The map $h\colon \Gamma \to X$ varies in a family of dimension at least*

$$\mathrm{vdim}_{\mathrm{h}} = K.h_*([\Gamma]) + (n-3)(1-g)$$

if we allow to deform both h and Γ. The number vdim *is called the* virtual dimension *of the deformations of h.*

Definition 5.11. The parameterized curve h is called *regular* if the local dimension of all its deformations is equal to vdim_h. Otherwise, if it is strictly greater than vdim_h, the curve h is called *superabundant*.

E.g. the curve h will necessarily be superabundant if $h(\Gamma)$ contains a loop that is contained in a proper affine-linear subspace of a facet of X.

5.4. Tropical moduli spaces. Let Γ be a compact tropical curve of genus g. Let us choose k distinct points $x_1, \dots, x_k \in \Gamma$. We call x_j the *marked points*. By replacing Γ with an equivalent tropical curve if needed, we may assume that Γ has exactly k 1-valent vertices which coincide with $x_1, \dots, x_k$. Such presentation exists unless $g = 0$ and $k < 2$ and it is unique up to isomorphism. Thus we may restrict our attention only to such models.

We denote by $\mathcal{M}_{g,k}$ the space of all tropical curves of genus g with k marked points up to equivalence.

A tropical curve with marked points is determined by its combinatorial type and the lengths of its finite edges. E.g. the tropical curve from Figure 7 is determined by the lengths a and b. The length of the finite edges can be used to enhance $\mathcal{M}_{g,k}$ with

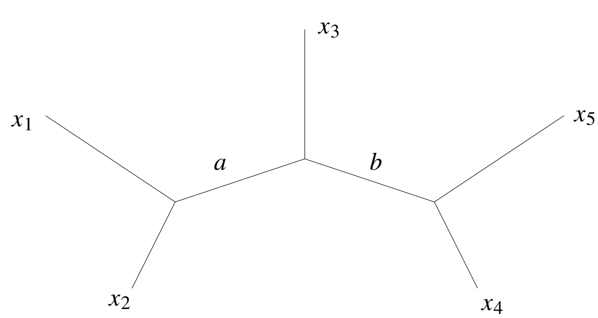

Figure 7. A rational curve with 5 marked points.

a tropical structure. The only problem is that we may only identify the edges within the same combinatorial type of the curve.

One can easily solve this problem if $g = 0$. Let us introduce a global function Z_{x_i,x_j} on $\mathcal{M}_{0,k}$ for any pair of marked points $\{x_i, x_j\}$. We set Z_{x_i,x_j} equal to the total length of finite edges between x_i and x_j. E.g. for the curve $\Gamma \in \mathcal{M}_{0,5}$ from Figure 7 we have $Z_{x_1,x_2}(\Gamma) = 0$, $Z_{x_1,x_3}(\Gamma) = a$ and $Z_{x_1,x_5}(\Gamma) = a + b$.

Proposition 5.12. *The functions Z_{x_i,x_j} define an embedding*

$$\mathcal{M}_{0,k} \subset \mathbb{R}^{\frac{k(k-1)}{2}}.$$

Its image is a tropical subvariety.

Remark 5.13. This embedding is related to the Plücker coordinates on the Grassmannian $G_{2,k}$, see [30] for a tropical version.

Thus we see that $\mathcal{M}_{0,k}$ has a natural structure of a tropical variety. We may compactify $\mathcal{M}_{0,k}$ by allowing the lengths of some (or all) finite edges of Γ to take the value equal to $+\infty$. Such a generalized curve splits into several components: the finite points of each such component are within finite distance from each other.

Proposition 5.14. *The resulting space $\bar{\mathcal{M}}_{0,k}$ is a compact tropical variety.*

This compactification is a tropical counterpart of the Deligne–Mumford compactification.

If $g > 0$ one may use similar arguments to show that $\mathcal{M}_{g,k}$ is a tropical orbifold which can be compactified to a compact tropical orbifold $\bar{\mathcal{M}}_{g,k}$.

5.5. Stable curves in X. As in the classical case we call a parameterized tropical curve $h\colon \Gamma \to X$ in a compact n-dimensional tropical variety X with compact Γ *stable* if there are no infinitesimal automorphisms of h, i.e. if the number of isomorphisms Φ of Γ such that $h = h \circ \Phi$ is finite.

All deformations of a regular stable curve h locally form a tropical variety of dimension $K.h_*([\Gamma]) + (n-3)(1-g)$. Let us fix some class β of stable curves to X that is closed with respect to regular curve deformations. The class β can be given e.g. by prescribing the intersection number with all $(n-1)$-cycles in X.

Denote by $\mathcal{M}^{\beta}_{g,k}(X)$ the space of stable curves of genus g with k marked points in the class. In many cases the space $\mathcal{M}^{\beta}_{g,k}(X)$ can be compactified to a compact tropical variety $\bar{\mathcal{M}}^{\beta}_{g,k}(X)$. This holds for instance if $g = 0$, $X = \mathbb{TP}^n$ and β is formed by curves of degree d. Another instance is if $X = \mathbb{TP}^2$, there are no restrictions on g and β is formed by topological immersions of degree d. These are the two principal cases for our enumerative applications. More generally we may assume that X is any compact toric variety, but then we have to impose the additional constraint on β that it does not contain curves with components lying totally in the boundary divisors of X.

Furthermore, in these cases we have the *evaluation* map

$$\mathrm{ev}_j\colon \bar{\mathcal{M}}^{\beta}_{g,k}(X) \to X,$$

$\mathrm{ev}_j(h) = h(x_j)$ as well as the maps $\mathrm{ft}\colon \bar{\mathcal{M}}^{\beta}_{g,k}(X) \to \bar{\mathcal{M}}_{g,k}$, $\mathrm{ft}(h) = \Gamma$, and the maps $\pi_j\colon \bar{\mathcal{M}}^{\beta}_{g,k}(X) \to \bar{\mathcal{M}}^{\beta}_{g,k-1}(X)$ "forgetting" the marked point x_j. These maps are linear tropical morphisms.

This allows to set up a tropical framework for the Gromov–Witten theory. E.g. given a collection of cycles in X we may take their pull-backs and then take their intersection number in $\bar{\mathcal{M}}^{\beta}_{g,k}(X)$.

Many reasonings in the Gromov–Witten theory can be literally repeated in this tropical set-up. A good example is the WDVV-relation, cf. [7]. As in the classical case the stable curves in $\bar{\mathcal{M}}^{\beta}_{0,k}(X) \setminus \mathcal{M}^{\beta}_{0,k}(X)$ must consist of several components.

Remark 5.15. Moduli spaces of higher-dimensional tropical varieties is a very interesting, but much more difficult subject. Already the tropical K3-surfaces form a very sophisticated geometric object, see [14] and [9].

6. Tropical curves in $\mathbb{R}^n$, their phases and amoebas

Let $V \subset (\mathbb{C}^\times)^n$ be an algebraic variety. Its *amoeba* (see [8]) is the set $\mathcal{A} = \mathrm{Log}(V) \subset \mathbb{R}^n$, where $\mathrm{Log}(z_1, \dots, z_n) = (\log|z_1|, \dots, |z_n|)$. Similarly we may consider the map

$$\mathrm{Log}_t\colon (\mathbb{C}^\times)^n \to \mathbb{R}^n$$

corresponding to taking the logarithm with the base $t > 1$.

Amoebas themselves have proved to be a very useful tool in several areas of mathematics, see e.g. [4], [16], [17], [22], [26], [27]. However, for the purposes of this talk we only use them as an intermediate link between the classical and tropical geometries.

Definition 6.1. The curve $h\colon \Gamma \to \mathbb{R}^n$ is called *classically realizable* if there exist a small regular neighborhood $U \supset B$ in $\mathbb{R}^n$, a retraction $\rho\colon U \to B$, a regular family of holomorphic maps $H_t\colon C_t \to (\mathbb{C}^\times)^n$ for a family of Riemann surfaces C_t defined for all sufficiently large positive $t \gg 1$ and smooth maps $\lambda_t\colon C_t \to \Gamma$ such that

- $h \circ \lambda_t = \rho \circ \mathrm{Log}_t \circ H_t$;
- the genus of C_t coincides with the genus of Γ;
- the number of punctures of C_t coincides with the number of ends of Γ.

The family $H_t\colon C_t \to (\mathbb{C}^\times)^n$ is called an *approximating family* of h.

Proposition 6.2. *If $h\colon \Gamma \to \mathbb{R}^n$ is a tropical curve approximated by H_t then for a sufficiently large t the inverse image $\lambda_t^{-1}(p)$ is a smooth circle for every p inside an edge of Γ while the inverse image $\lambda_t^{-1}(W)$ is diffeomorphic to a sphere with u punctures for a small connected neighborhood W of a vertex of valence u.*

In particular, if Γ is 3-valent then λ_t defines a pair-of-pants decomposition. In turn, this pair-of-pants decomposition determines a point in the boundary of the classical Deligne–Mumford space $\overline{\mathcal{M}}^{\mathbb{C}}_{g,k}$. This point is the limit of the Riemann surfaces C_t.

See [17] for a generalization of the pair-of-pants decomposition for the case of higher-dimensional hypersurfaces .

Remark 6.3. Classical realizability of tropical varieties in $\mathbb{R}^n$ is closely related to their presentation by *non-Archimedean amoebas*, the images of algebraic varieties in $(K^\times)^n$ under the coordinatewise valuations (as defined by Kapranov [10]). Here K is an algebraically closed field with a non-Archimedean valuation $\mathrm{val}\colon K^\times \to \mathbb{R}$. See [29] for an account of what is known on such presentations.

To formulate the realizability theorem in full generality we need to define the *phases* for Γ. We start from their definition in a model case.

Let $\Gamma_k \subset \mathbb{R}^k$ be the tropical curve consisting of $(k+1)$ rays emanating from $0 \in \mathbb{R}^k$ in the directions $(-1, \dots, 0), \dots, (0, \dots, -1)$ and $(1, \dots, 1)$. The tautological embedding $\Gamma_k \subset \mathbb{R}^k$ is easily realizable. For an approximating family we can take $H_t = L_k \subset (\mathbb{C}^\times)^n \subset \mathbb{CP}^k$, where L_k is a line with $(k+1)$ ends in $(\mathbb{C}^\times)^n$. The choice of L_k up to a multiplication by a point of $(\mathbb{R}_+)^k$ is called the phase of Γ_k.

Locally, near any point $x \in \Gamma$ the map h coincides with the map

$$\Gamma_k \subset \mathbb{R}^k \xrightarrow{A+c} \mathbb{R}^n$$

near $0 \in \Gamma_k$ for a linear map A defined over $\mathbb{Z}$ and $c \in \mathbb{R}^n$, where $(k+1)$ is the valence of x. The linear map A can be exponentiated to a multiplicative linear map $a\colon (\mathbb{C}^\times)^k \to (\mathbb{C}^\times)^n$. The phase σ_U of Γ at a small neighborhood $U \ni x$ is defined to be the equivalence class of $\xi a(L_k) \subset (\mathbb{C}^\times)^n$, $\xi \in (\mathbb{C}^\times)^n$, up to multiplication by an element of $(\mathbb{R}_+)^n$. Two local phases are called *compatible* if they agree on the intersection of the neighborhoods.

We say that H_t approximates Γ *with the local phase* σ_U if $H_t(C_t) \cap \mathrm{Log}_t^{-1}(U)$ converges (in the Hausdorff metric on compacts in $(\mathbb{C}^\times)^n$) to $\xi t^C a(L_k) \in \sigma_U$.

Theorem 1. *Any regular tropical curve $h\colon \Gamma \to \mathbb{R}^n$ equipped with any compatible system of phases is classically realizable.*

Cf. [19] and [28] for the special case $n = 2$ with no restriction on the genus, and [20] and [24] for the special case of genus 0 with no restriction on n. The proof in the general case (though in a somewhat different language) is contained in [2] (cf. also [29]).

Remark 6.4. Regularity is a necessary condition in Theorem 1, it is easy to construct an example of a non-realizable superabundant curve even for a topological immersion of an elliptic curve in $\mathbb{R}^3$, see e.g. [18].

7. Applications

One of the greatest advantages of tropical geometry is that most classical problems become much simpler after their tropicalization (if such a tropicalization exists!). This simplicity comes from the piecewise-linear nature of the tropical objects. Indeed, once we fix the combinatorial type of the data, a tropical problem becomes linear. E.g. if there are finitely many solutions to a problem then in every combinatorial type we will have a unique solution or no solutions at all.

This allows one to find (at least) an algorithmic answer to a tropical problem. Sometimes one can show that the answer to a classical problem and its tropicalization must coincide and obtain the answer to the classical problem in this way. In this last section we list some examples of such problems.

7.1. Complex geometry. Let $g \geq 0$ and $d \geq 1$ be integers. Fix a collection $\mathcal{Z} = \{z_j\}_{j=1}^{3d-1+g}$ of points in $\mathbb{CP}^2$ in general position. There are finitely many holomorphic curves of genus g and degree d passing through $\mathcal{Z}$. Let $N^{\mathbb{C}}_{g,d}$ be their number.

Such set-up can be almost literally repeated in the tropical framework. Fix a collection $\mathcal{X} = \{x_j\}_{j=1}^{3d-1+g}$ of points in $\mathbb{TP}^2$ in general position (see [19]). Again, it can be shown that there are finitely many tropical curves $h\colon \Gamma \to \mathbb{TP}^n$ of genus g and degree d passing through x_j. But in the tropical set-up these curves come with natural positive integer multiplicities $m(h)$ not necessarily equal to 1.

Tropical curves of genus g and degree d that pass through $x_j \in \mathbb{TP}^2$ form a codimension 1 cycle in $\bar{\mathcal{M}}^d_{g,d}(\mathbb{TP}^2)$. We set $m(h)$ to be the weight of their product-intersection at h (in other words this is their local intersection number in h). Let $N^{\mathbb{T}}_{g,d}$ be the number of the tropical curves of degree d and genus g passing through $\mathcal{X}$ counted with multiplicity m.

Theorem 2 ([19]).
$$N^{\mathbb{C}}_{g,d} = N^{\mathbb{T}}_{g,d}.$$

Of course, there are well-known ways to compute $N^{\mathbb{C}}_{g,d}$ (see [12], [3]) as they coincide with the Gromov–Witten invariants of $\mathbb{CP}^2$. Yet Theorem 2 gives another simple and visual way to do it, see [19] for details.

Example 7.1. Figure 8 depicts rational cubic curves in $\mathbb{TP}^2$ passing through a generic configuration $\mathcal{X}$ of 8 points. There are nine curves. Eight of them have multiplicity 1. The remaining one, namely the rightmost in the middle row, has multiplicity 4 (it has

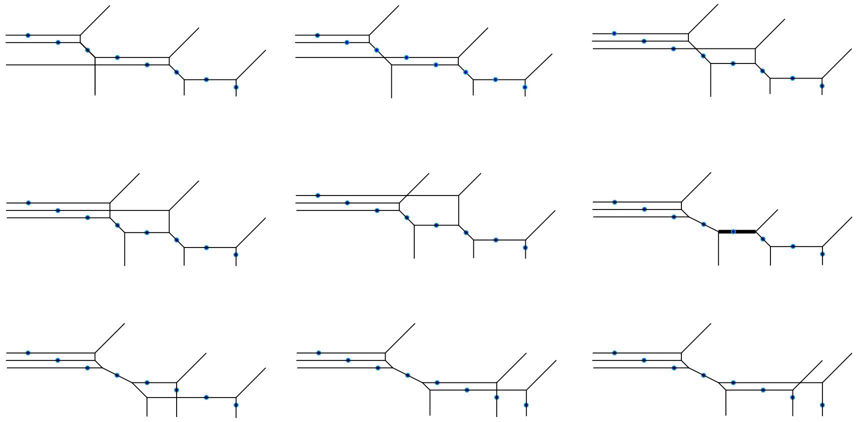

Figure 8. Rational tropical cubics via 8 generic points in the plane.

an edge of weight 2 shown by a bold line in the picture). Thus the total number of tropical curves is 12.

Should we choose a different generic configuration $\mathcal{X}$ the number of tropical curves could be different. But their total number counted with multiplicities is invariant. E.g. there exists a different choice of $\mathcal{X}$ where we have ten tropical curves, nine of multiplicity 1 and one of multiplicity 3. In fact, no other partition of 12 can appear by Example 7.3.

Remark 7.2. In [19] there is also a version of Theorem 2 for other toric surfaces. Note that if the ambient toric surface is not Fano then there is a difference between counting irreducible curves in a given homology class and computing the corresponding Gromov–Witten number (as the latter also has a contribution from curves with some boundary divisors as components). Tropical geometry has the advantage of giving a way to compute the number of irreducible curves directly without having to deal with those extra components.

The same advantage allows to apply tropical geometry for giving an algorithmic answer to another classical problem of complex geometry, namely the computation of Zeuthen's numbers, see [21]. These are the numbers of curves of degree d and genus g that pass through a collection of generic points and tangent to a collection of generic lines in $\mathbb{CP}^2$. Here the total number of points and lines in the configuration is $3d-1+g$. These Zeuthen's numbers also turn out to be equal to the corresponding tropical numbers, and the latter can be computed by a finite (though quite extensive for large genus) algorithm. As far as the author knows such computation in general is not currently accessible by non-tropical techniques.

7.2. Real geometry. The number of real curves of degree d and geometric genus g passing via a collection $\mathcal{Z} = \{z_j\}_{j=1}^{3d-1+g}$ of points in $\mathbb{RP}^2$ depends on $\mathcal{Z}$ even if we choose it to be generic. This is the feature of $\mathbb{R}$ as a non-algebraically closed field. Yet, as it was suggested in [34], one may prescribe a *sign* ± 1 to every such curve so that the sum W_d of these signs becomes invariant in the special case of $g = 0$. The number W_d is called the *Welschinger number*.

If $\mathcal{Z}$ is a generic configuration of $3d - 1$ points in $\mathbb{RP}^2$ then any real rational curve $\mathbb{R}C$ of degree d passing through $\mathcal{Z}$ is a nodal curve, i.e. all singularities of $\mathbb{R}C$ are non-degenerate double points. Over $\mathbb{R}$ there are two types of such nodes: the *hyperbolic* node, corresponding to the intersection of two real branches (given in local coordinates by $x^2 - y^2 = 0$) and the *elliptic* node, corresponding to the intersection of two complex-conjugate branches (given in local coordinates by $x^2 + y^2 = 0$). The sign of $\mathbb{R}C$ is defined as $(-1)^{e(C)}$, where $e(C)$ is the number of elliptic nodes of C.

The Welschinger number also has a tropical counterpart. To a tropical curve $h\colon \Gamma \to \mathbb{TP}^2$ of degree d and genus g passing through a configuration $\mathcal{X}$ of $3d-1+g$ points we associate its *real multiplicity* $m^{\mathbb{R}}(h)$ which is ± 1 or 0. If $h(\Gamma)$ has an edge of even multiplicity then $m^{\mathbb{R}}(h) = 0$. Otherwise we define the local real multiplicity $m_v^{\mathbb{R}}(h)$ for a vertex $v \in \Gamma$ to be $(-1)^{e_v}$, where e_v is the number of integer points in the interior of the lattice triangle such that its sides are perpendicular to the edges adjacent to v and of integer length equal to the weight of that edge. Then we define $m^{\mathbb{R}}(h) = \prod_v m_v^{\mathbb{R}}(h)$. Let $W_{g,d}^{\mathbb{T}}$ be the corresponding tropical number.

Theorem 3 ([19]).

$$W_d = W_{g,d}^{\mathbb{T}}.$$

This theorem is the only currently known way to compute the Welschinger numbers, see [11].

Example 7.3. Let us revisit Example 7.1. The real multiplicities of eight out of the nine curves in Figure 8 is 1 while the real multiplicity of the remaining one is 0. Thus we have $W_3 = 8$. For another choice of a generic configuration of eight points in $\mathbb{TP}^2$ mentioned in Example 7.1 we get nine curves with $m^{\mathbb{R}}(h) = +1$ and one with $m^{\mathbb{R}}(h) = -1$.

Note that we always have $m^{\mathbb{R}}(h) = +1$ if $m(h) = 1$; $m^{\mathbb{R}}(h) = +1$ if $m(h) = 1$ if the multiplicity is 1; and $m^{\mathbb{R}}(h) = 0$ if $m(h)$ is even. Note also that we may never get $m(h) = 2$. Since the sum of the multiplicities has to be 12 and the sum of real multiplicities has to be 8 the partitions $12 = 8 + 4 = 9 + 3$ are the only two possible partitions. In particular, there does not exist a configuration of eight points in $\mathbb{TP}^2$ such that the twelve tropical cubics will all be distinct.

Remark 7.4. There is a 3-dimensional version of the number W_d which is the number of real rational curves in $\mathbb{RP}^3$ of degree d passing through a generic configuration of $2d$ points taken with certain signs, see [35]. These numbers also have tropical counterparts and Theorem 3 extends to the 3-dimensional case providing a way to compute the real geometry numbers, see [20].

As the last application of tropical geometry we would like to mention extending the patchworking [32] to curves in real toric varieties of higher dimensions by using Theorem 1 with real phases.

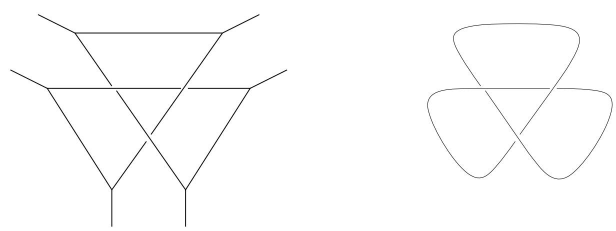

Figure 9. Real algebraic knots from tropical curves in $\mathbb{R}^3$.

Example 7.5. Let $K \subset \mathbb{R}^3$ be a knot presented as an embedded piecewise-linear circle made with k straight intervals. Then there exists an algebraic curve in $(\mathbb{R}^\times)^3$ with a unique closed component in $(\mathbb{R}_+)^3$ isotopic to K such that its complexification has genus 1 and is punctured k times. E.g. a trefoil may be presented by an elliptic curve with 6 punctures in $(\mathbb{C}^\times)^3$, see Figure 9.

To deduce this statement we perturb the broken line K to make the slopes of its intervals rational. Then we add an extra ray at every corner to get a tropical curve $\Gamma \supset K$. Finally we choose real phases for Γ so that the phases of all its bounded edges include the positive quadrant.

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