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Quasi-Harmonic Friction-Induced Vibration

A theoretical and experimental investigation of quasi-harmonic friction-induced vibration is reported. The vibration is of near-sinusoidal form and is solely governed by dynamic friction forces. However, the friction-velocity curve must be of a particular shape for the vibration to occur. The amplitude of the quasi-harmonic vibration is shown to increase with sliding velocity until oscillation ceases at some upper velocity boundary. The introduction of suitable damping will quench the vibration entirely. The vibration can exist at high sliding velocities and as a consequence may influence the operation of automatic transmissions, brakes, and clutches.

Introduction

Friction-induced vibration has been observed in a wide variety of systems. For example, discontinuous motion may occur during the positioning of large masses in machine tools and servomechanisms. Alternatively, brake squeal and vibration in automatic transmission elements are manifestations of self-excitation produced by friction. The Froude pendulum and the motion of a violin string under the action of a bow are frequently cited as examples of friction-induced vibration.

Two forms of autonomous friction-induced vibration may be classified: stick-slip vibration and quasi-harmonic oscillation. The stick-slip or relaxation oscillation is characterized by the sawtooth displacement-time waveform of Fig. 1. The regimes of stick and slip constitute the complete vibration cycle. The stick phase is dependent on the static friction forces established during stationary contact. At the end of stick, sudden relaxation occurs, and during this movement the system is governed by dynamic friction forces. A theory recently developed by Brockley, Cameron, and Potter [1][1] permits the prediction of the conditions necessary for the existence and decay of the stick-slip oscillation. In another recent paper by Brockley and Davis [2], a theoretical and experimental study of the time-dependence of static friction is reported. The findings of this paper are directly applicable to the theory of the stick-slip oscillation. A list of references concerned with stick-slip vibration is given in [1].

The quasi-harmonic vibration, Fig. 2, has a waveform which is approximately sinusoidal. Comparatively little work has been devoted to the mechanics of this form of vibration. Papenhuyzen [3] accurately classifies friction-induced vibration into the two general types but he does not present a satisfactory theoretical analysis of the quasi-harmonic form. The present work extends the knowledge concerning this particular type of friction-induced vibration.

Theory

The differential equation for the system as shown in Fig. 3 is:

$$m\ddot{x} + r\dot{x} + kx = F_f \tag{1}$$

where F_f is the friction force.

[1] Numbers in brackets designate References at end of paper.

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Nomenclature

A = amplitude of friction-induced vibration
C_0, C_1, etc. = coefficients of friction force functions
F_f = friction force
I_0, I_1, I_2 = modified Bessel functions of the first kind
k = stiffness of support system
m = equivalent mass of vibratory system
r = damping coefficient of vibratory system
r_c = critical damping coefficient for extinction of friction-induced vibration
t = time
x = displacement of slider
$\dot{x}, y$ = absolute velocity of slider
$\ddot{x}$ = acceleration of slider
ω = damped natural frequency of vibratory system
γ = coefficient of nonlinear velocity terms
ϕ = phase angle

It has been indicated earlier that in the case of stick-slip oscillation, F_f is time-dependent during stick and velocity-dependent during slip. However, in the case of quasi-harmonic oscillation the motion is governed by velocity-dependent friction forces only. The existence of the quasi-harmonic oscillation is critically dependent on the particular shape of the dynamic friction curve.

Actual friction couples display a variety of forms of dynamic friction curve. Fig. 4 illustrates a possible form of the curve. Alternatively, there is evidence to support the type of curve illustrated by Fig. 5. Kragelskii [4] demonstrates that this form of curve is found for metals in dry sliding contact. Recently, the study of the friction behavior of rubber [5] and of polymers [6] has shown that similar humped friction-velocity curves exist for nonmetallic materials. Experimental results [7] for the edge contact of lubricated rotating disks reveal the existence of friction-torque versus velocity curves of the form of Fig. 5. It is of interest to note that the humped friction-velocity curve exists for metals in dry and lubricated sliding contact as well as for non-metallic surfaces. Further work is required in order to determine whether or not a common factor is responsible for the humped characteristic curve which is found for the various combinations.

The friction force F_f in equation (1) is considered as a function of sliding velocity. Various mathematical expressions have been used to represent the friction force function. In general, the function can be expressed in the form of an exponential function or as an nth-order polynomial, such as

$$F_f = [C_1 + C_2(v - \dot{x})]e^{-C_3(v-\dot{x})} + C_4(v - \dot{x}) + C_5 \quad (2)$$

$$F_f = C_n(v - \dot{x})^n + C_{n-1}(v-\dot{x})^{n-1} + \ldots\ldots + C_0 \quad (3)$$

where $(v - \dot{x})$ is the relative sliding velocity and C_0, C_1, etc., are constants which may be adjusted to fit the equation to measured friction values.

The existence or nonexistence of self-excited vibration of the quasi-harmonic form may be investigated for the various dynamic friction curves proposed. The phase-plane graphical method of Liénard [8] provides a useful technique for a semi-qualitative investigation. In order to apply the method, equation (1) is modified by letting $\dot{x} = y$ and by replacing F_f by a function $F(v - y)$, which depends on the relative sliding velocity. After manipulation, these modifications yield

$$\frac{dy}{dx} = \frac{F(v - y) - ry - kx}{my} \quad (4)$$

If $\frac{dy}{dx}$ is set to zero in (4), then it is found that

$$x = \frac{F(v - y) - ry}{k} \quad (5)$$

Equation (5) describes the locus of all points of zero slope on a phase-plane diagram. It is evident that the zero-slope isocline is simply a modified friction-velocity characteristic curve. Accordingly, employing this equation and Liénard method, the diagrams of Figs. 6, 7, and 8 were prepared using Figs. 4 and 5. Fig. 6 illustrates the case of a limit cycle of the stick-slip type

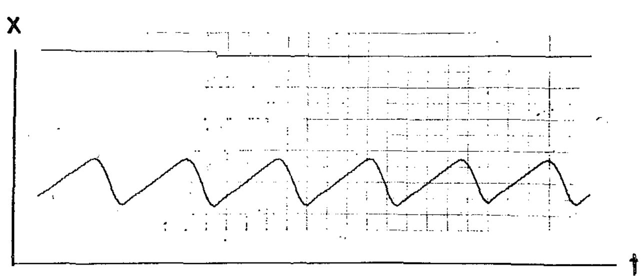

Fig. 1 Displacement-time recording of typical stick-slip vibration

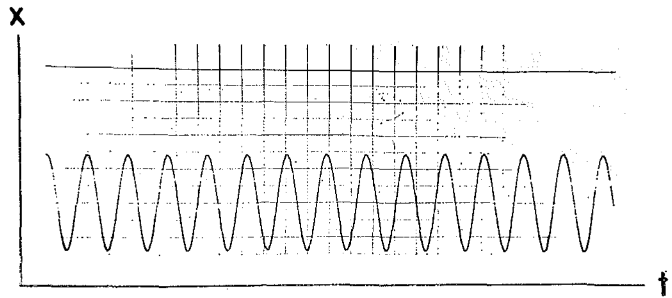

Fig. 2 Displacement-time recording of typical quasi-harmonic vibration

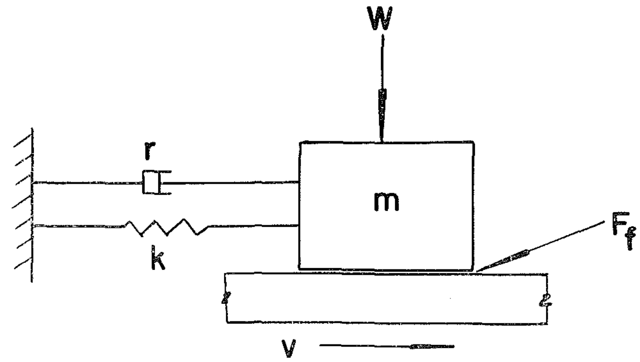

Fig. 3 Spring-mass-damper system showing lower surface moving at constant velocity, *v*

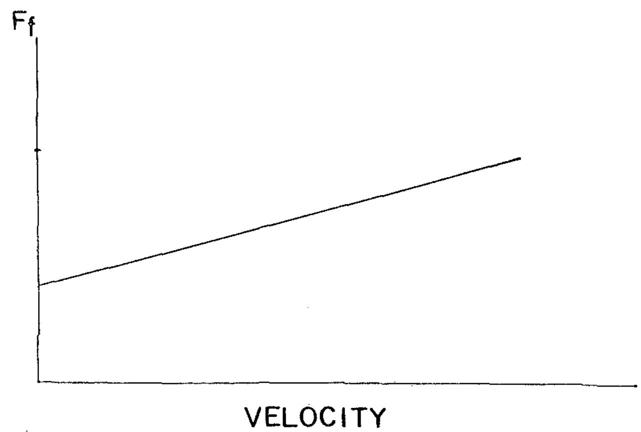

Fig. 4 Linearized dynamic friction-velocity relationship

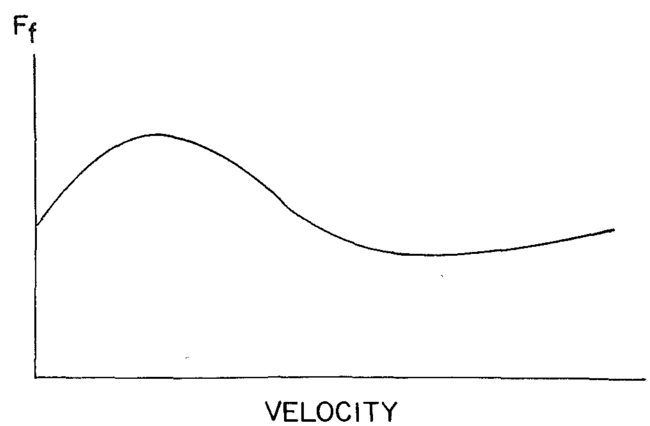

Fig. 5 Typical humped friction-velocity curve which gives rise to quasi-harmonic vibration

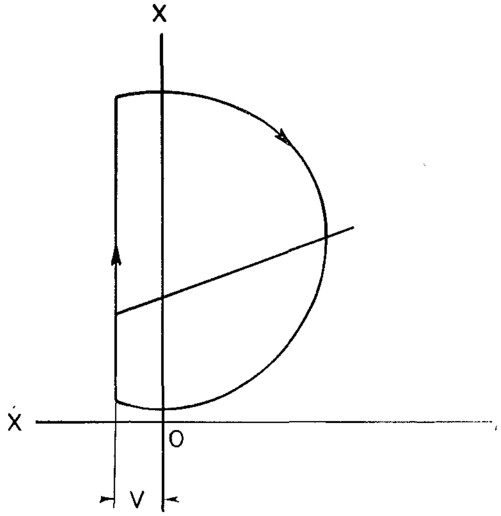

Fig. 6 Typical phase-plane diagram for stick-slip vibration associated with a linear dynamic friction-velocity curve

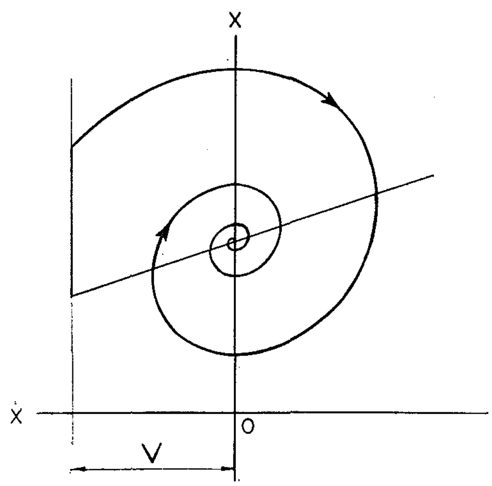

Fig. 7 Phase-plane diagram illustrating a phase trajectory which leads to stable dynamic equilibrium

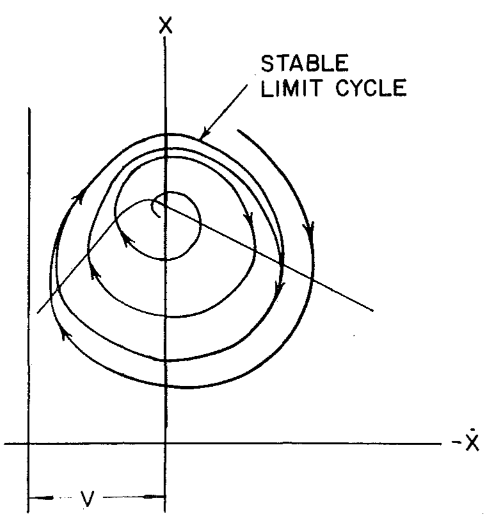

Fig. 8 Phase-plane diagram of quasi-harmonic vibration showing a stable limit cycle

produced by entrainment of the phase trajectory into the static friction axis. Fig. 7 shows a situation whereby the mass achieves a position of stable dynamic equilibrium and limit cycle motion does not occur. An extended analysis is given in the earlier paper [1]. In general, it may be observed that the friction characteristic of Fig. 4 will give rise to stick-slip vibration or stable displacement, depending on the system parameters. In any event, oscillations of the quasi-harmonic form do not occur for this particular friction-velocity relationship. However, limit cycle motion is possible in the case of the humped friction-velocity curve of Fig. 5, and the phase-plane solution of Fig. 8 illustrates that near-harmonic oscillation occurs. Hence, the hump in the friction-velocity curve appears to be one of the conditions necessary for the existence of this form of oscillation.

A singular point analysis provides further information concerning the conditions necessary for the existence of the oscillation [9]. The analysis reveals that for the case where the friction function is represented by expression (2), the condition for a stable system is

$$[C_1C_3 + C_2C_3v - C_2] < (r + C_4)e^{C_3v} \tag{6}$$

From this condition it is possible to define the damping r_c required for a completely stable system. In [10] it is shown that

$$r_c = C_2 e\left(\frac{C_1C_3 - 2C_2}{C_2}\right) - C_4 \tag{7}$$

A solution of the nonlinear differential equation (1) is found using the method of Kryloff and Bogoliuboff [11]. Substitution of expression (2) into equation (1), gives after manipulation

$$\ddot{x} + \frac{r + C_4}{m}\dot{x} - \frac{C_1 + C_2v}{me^{C_3v}} e^{C_3v} + \frac{C_2}{me^{C_3v}}\dot{x}e^{C_3\dot{x}} + \omega^2 x = \frac{C_4v + C_5}{m} \tag{8}$$

The constant term on the right-hand side of equation (8) constitutes the static displacement of the vibration. Since we are only interested in the amplitude of the oscillation, the right-hand term is omitted in the analysis. Equation (8) can be written as

$$\ddot{x} + \gamma G(\dot{x}) + \omega^2 x = 0 \tag{9}$$

where $\omega^2 = k/m$; and

$$\gamma G(\dot{x}) = \frac{r + C_4}{m}\dot{x} - \frac{C_1 + C_2v}{me^{C_3v}} e^{C_3\dot{x}} + \frac{C_2}{me^{C_3v}}\dot{x}e^{C_3\dot{x}} \tag{10}$$

If $\gamma G(\dot{x}) \ll 1$, then a solution of the form

$$x = A(t)\sin[\omega t + \phi(t)] \tag{11}$$

may be proposed. Here $A(t)$ and $\phi(t)$ are considered to be functions which vary slowly with time. The application of the method specified in [11] gives rise to the integrals

$$\frac{dA}{dt} = -\frac{\gamma}{2\pi\omega}\int_0^{2\pi} G(A\omega\cos\psi)\cos\psi d\psi \tag{12}$$

$$\frac{d\psi}{dt} = \omega + \frac{\gamma}{2\pi\omega A}\int_0^{2\pi} G(A\omega\cos\psi)\sin\psi d\psi \tag{13}$$

where $\psi = \omega t + \phi$.

In a practical sense, equation (12) is of particular interest since amplitude variations with time may be measured, whereas phase variations are difficult to detect. Accordingly, substitution of equation (10) into (12) yields

$$\frac{dA}{dt} = -\frac{(r + C_4)A}{2m} + \frac{C_1 + C_2v}{m\omega e^{C_3v}} I_1(C_3A\omega) - \frac{C_2A}{2me^{C_3v}}[I_2(C_3A\omega) + I_0(C_3A\omega)] = \Phi(A) \tag{14}$$

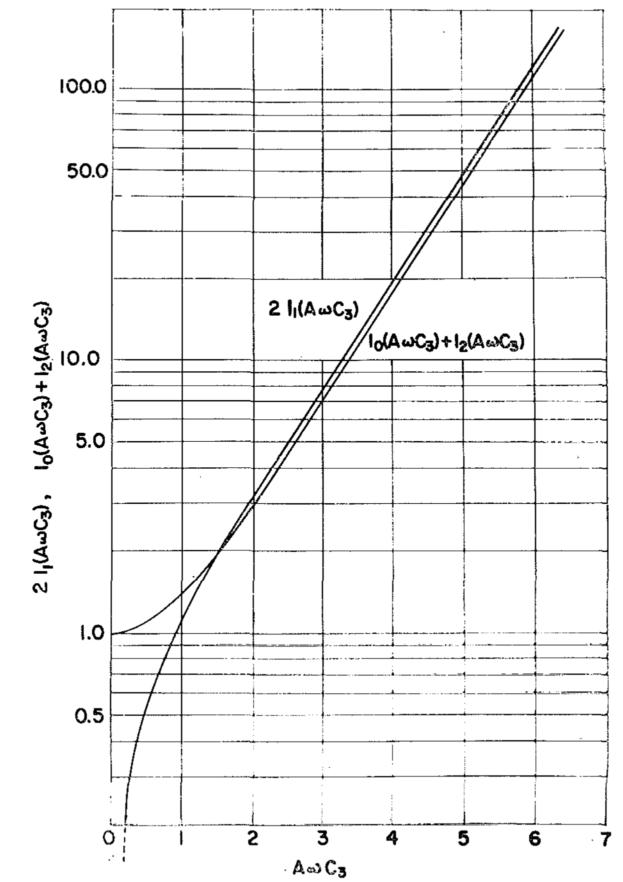

Fig. 9 $I_1(C_3A\omega)$ and $I_2(C_3A\omega) + I_0(C_3A\omega)$ plotted as functions of $C_3A\omega$

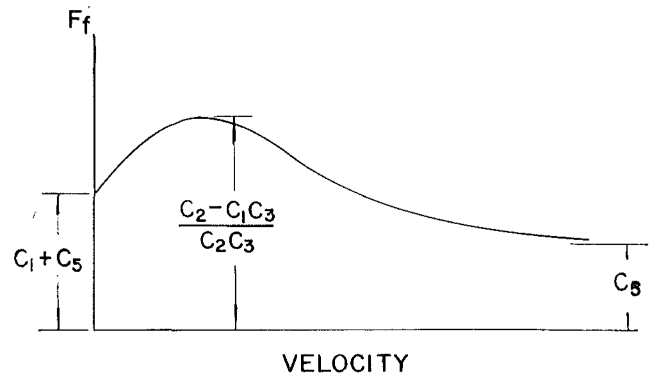

Fig. 10 Friction-velocity function for $C_4 = 0$

where $I_1(C_3A\omega)$, $I_2(C_3A\omega)$ and $I_0(C_3A\omega)$ are Bessel functions.

Generally, interest centers on stationary values for A. If $\frac{dA}{dt} = 0$ in equation (14), then

$$\frac{2(C_1 + C_2v)}{\omega e^{C_3v}} I_1(C_3A\omega) - \frac{C_2A}{e^{C_3v}} [I_2(C_3A\omega) + I_0(C_3A\omega)] - (r + C_4)A = 0 \quad (15)$$

This last equation can be solved by a computer for discrete v values if the system constants are known. Fig. 9 illustrates $I_1(C_3A\omega)$ and $I_2(C_3A\omega) + I_0(C_3A\omega)$ plotted as functions of $C_3A\omega$.

For the stability of the amplitude of vibration, according to Kryloff and Bogoliuboff [11], $\frac{d\Phi(A_1)}{dA} < 0$ is the condition for a stable limit cycle with amplitude A_1.

Carrying out the derivation of equation (15), we have

$$\frac{d\Phi(A)}{dA} = -(r + C_4) + \frac{2(C_1C_3 + C_2C_3v)}{e^{C_3v}} I_0(C_3A\omega) - \frac{2[C_1C_3 + C_2C_3v + C_2(C_3A\omega)^2 + C_2]}{C_3A\omega e^{C_3v}} I_1(C_3A\omega) \quad (16)$$

Thus the stability of the amplitude obtained from equation (15) can be investigated by substituting it into equation (16).

Equation (15) can be further analyzed by omitting the last term $(r + C_4)A$. This represents a system with a friction characteristic as shown in Fig. 10 and with negligible damping. For this condition, equation (15) reduces to

$$C_3v + \frac{C_3C_1}{C_2} = C_3A\omega \frac{I_2(C_3A\omega) + I_0(C_3A\omega)}{2I_1(C_3A\omega)} \quad (17)$$

Application of values for $I_1(C_3A\omega)$, $I_2(C_3A\omega)$, and $I_0(C_3A\omega)$ derived from Fig. 9 permits the construction of the solution displayed by Fig. 11. This solution indicates that vibration will commence at a velocity corresponding to the peak of the friction-velocity curve (see Fig. 10). In the undamped case the vibration amplitude increases without limit as the lower surface velocity increases. However, actual systems possess some damping which suggests that amplitude limitation will exist. Indeed, the stability analysis referred to earlier shows that with damping present the vibration will be limited at some upper velocity boundary.

It has been indicated earlier that the friction force function can be expressed in the form of an nth-order polynomial. Substituting expression (3) in equation (1) we have after some manipulation

$$\ddot{x} + \frac{1}{m} [r\dot{x} + B_1\dot{x} - B_2\dot{x}^2 + B_3\dot{x}^3 - \ldots\ldots - (-1)^n B_n\dot{x}^n] + \omega^2 x = \frac{B_0}{m} \quad (18)$$

where

$$B_0 = C_0 + C_1v + C_2v^2 + \ldots\ldots + C_nv^n$$

$$B_1 = C_1 + 2C_2v + \ldots\ldots\ldots\ldots + nC_nv^{n-1}$$

$$B_2 = C_2 + 3C_3v + \ldots\ldots\ldots\ldots + \frac{n(n-1)}{2} C_nv^{n-2}$$

and

$$B_k = C_k + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{n!}{(n-k)!k!} C_nv^{n-k}$$

$$k = 0, 1, \ldots, n.$$

The constant term on the right-hand side of equation (18) can be omitted in the amplitude analysis. The application of the method specified in [11] gives the integral

$$\frac{dA}{dt} = -\frac{1}{2m\omega}\left[A\omega r + \sum_{k=1}^{k\leq\frac{n+1}{2}} \frac{{}_{2k-1}C_k}{2^{(2k-2)}} B_{2k-1}(A\omega)^{2k-1}\right] \quad (19)$$

or

$$\Phi(A) = -\frac{A}{2m}\left[r + \sum_{k=1}^{k\leq\frac{n+1}{2}} \frac{{}_{2k-1}C_k}{2^{(2k-2)}} B_{2k-1}(A\omega)^{2k-2}\right] \quad (20)$$

where ${}_pC_q = \frac{p}{(p-q)!q!}$ is the expression for the binomial coefficients.

The condition for a stationary oscillation is $\Phi(A) = 0$. $\frac{d\Phi(A_1)}{dA}$

< 0 is the condition for a stable limit cycle with amplitude A_1. Accordingly, if $\frac{d\Phi(0)}{dA} > 0$ we have an unstable singularity, in other words, self-excitation will start from rest.

If $\Phi(A) = 0$ in equation (20), we have

$$r + \sum_{k=1}^{k \leq \frac{n+1}{2}} \frac{{}_{2k-1}C_k}{2^{(2k-2)}} B_{2k-1}(A\omega)^{2k-2} = 0 \qquad (21)$$

Differentiating equation (20) with respect to A we have

$$\frac{d\Phi(A)}{dA} = \frac{1}{2m}\left[r + \sum_{k=1}^{k \leq \frac{n+1}{2}} \frac{(2k-1)_{2k-1}C_k}{2^{(2k-2)}} B_{2k-1}(A\omega)^{2k-2}\right] \qquad (22)$$

Equation (21) is a polynomial which can be easily solved with the aid of a computer. The amplitudes obtained can be substituted in equation (22) for the stability investigation.

Experimental

Apparatus and Instrumentation. Apparatus of the pin and disk type was employed for the experimental investigation. The slider was attached to a spherical, shaped mount which, when fitted into a hemispherical-shaped retaining cup in the specimen holder, provided the self-aligning action for the slider, thus insuring uniform contact at all times. The specimen holder was attached to a cantilever beam which formed the elastic supporting system of the slider. The beam assembly was pivoted and load was applied through a pulley system. The driving unit consisted of a variable-speed d-c motor and a double worm gear speed reducer.

Instrumentation was designed to measure the displacement, velocity, and acceleration of the slider motion during vibration. Strain gauges and a bridge amplifier were used for the displacement measurement. The velocity transducer consisted of a coil of very fine enameled wire which vibrates in the gap between two horseshoe-shaped permanent magnets. A seismic-type accelerometer which weighed 3 oz was attached at the top of the specimen holder. A more detailed description of the apparatus and instrumentation is given in another paper by the authors [12].

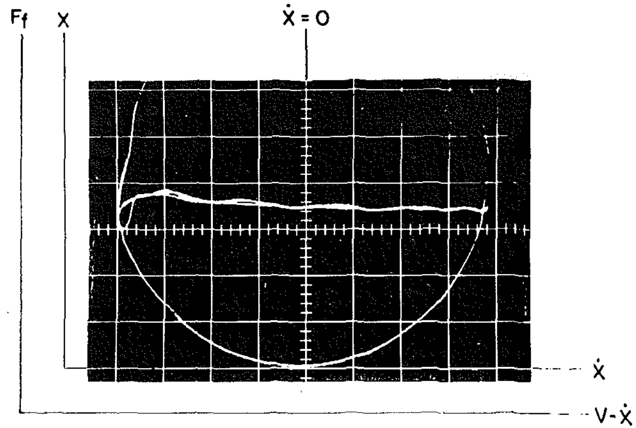

Fig. 12 Typical phase-plane oscilloscope trace of x versus ẋ for quasi-harmonic vibration. *The inner trace in the photograph represents the variation of friction force, F_f, with velocity.* Scales:

F_f—0.236 lb/major division
x—0.004 in/major division
x—0.553 in/sec/major division

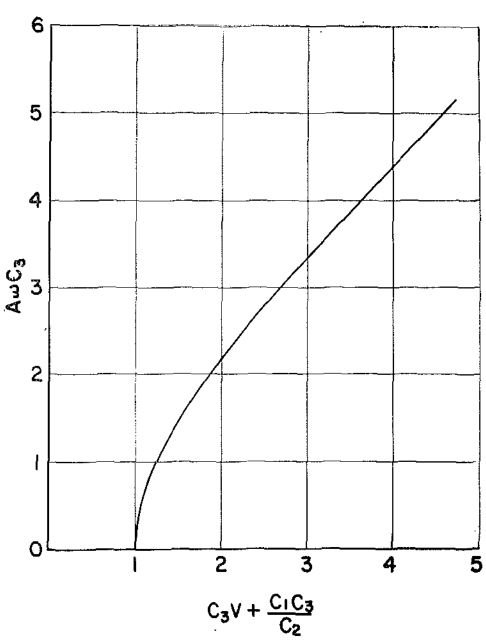

Fig. 11 Simplified solution (equation (17)) for zero damping and C_4 = 0. For known values of v, ω, C_1, C_2, and C_3 the amplitude of vibration can be found directly from the plot of $A\omega C_3$ versus $C_3 v + \frac{C_1 C_3}{C_2}$

Specimens. Preliminary investigation indicated that several material and lubricant combinations gave the desired quasi-harmonic vibration characteristics. The combinations employed for experimental verification of the theory were:

(*a*) Blotting paper on the slider surface running on a C 1020 steel disk with automatic transmission fluid as lubricant gave satisfactory results. This particular combination simulates the behavior of some automatic transmissions.

(*b*) A steel slider specimen running on the same steel disk as (*a*) with petrolatum (U.S.P.) as the lubricant gave quasi-harmonic vibration after a run-in period. It was observed that the vibration started after the formation of a fine black deposit on the disk track.

The steel surfaces of (*a*) and (*b*) were prepared by grinding and then lapping to a final finish of 25 microin. A. A. All surfaces were cleaned with hexane prior to use.

Results

Fig. 12 illustrates a typical phase-plane oscilloscope trace of displacement x versus vibration velocity $\dot{x}$ for blotting paper on steel at a load of 5.4 lb. A plot of friction force versus velocity is displayed in the same diagram. The foregoing results were obtained during one cycle of the vibration. Similar traces were obtained for a sequence of disk velocities, and the results of Fig. 13 were obtained by plotting vibration amplitude versus disk velocity. In the low-velocity region the system had a propensity to execute stick-slip oscillation. However, above a certain velocity near-circular phase-plane diagrams were obtained on the oscilloscope which were indicative that the quasi-harmonic form of vibration existed. A plot of vibration frequency as a function is also shown in Fig. 13.

Similar results were obtained for the steel on steel combination although some inconsistencies in vibration behavior existed which were found to be critically associated with the extent of the black deposit on the disk track.

For both systems the equivalent vibrating mass was 1.2 lb, the equivalent beam stiffness was 59 lb/in., and the viscous damping coefficient was 0.01 lb/in/sec. The foregoing values gave a damped natural frequency of 138 rad/sec for the slider support system.

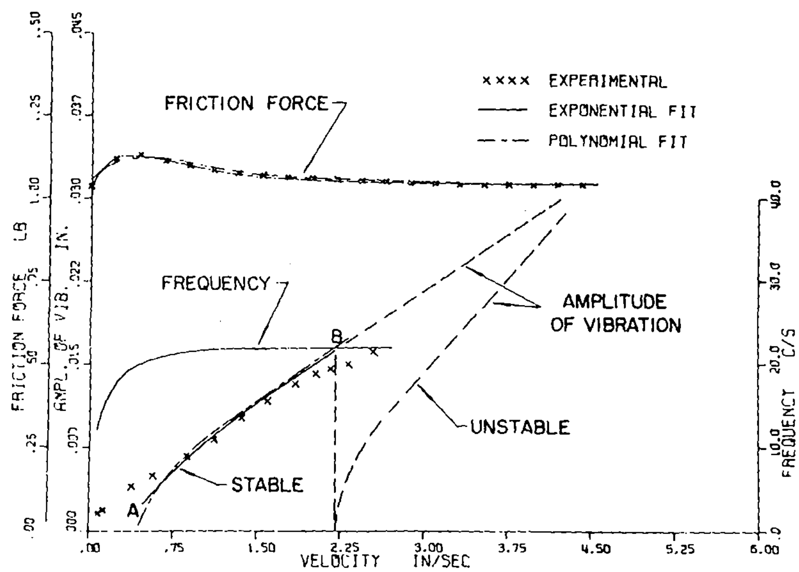

Fig. 13 *(a) Experimental friction-force versus velocity compared with the exponential and polynomial functions*

(b) Comparison of experimental results and theoretical predictions for vibration amplitude versus disk velocity

(c) Vibration frequency versus disk velocity

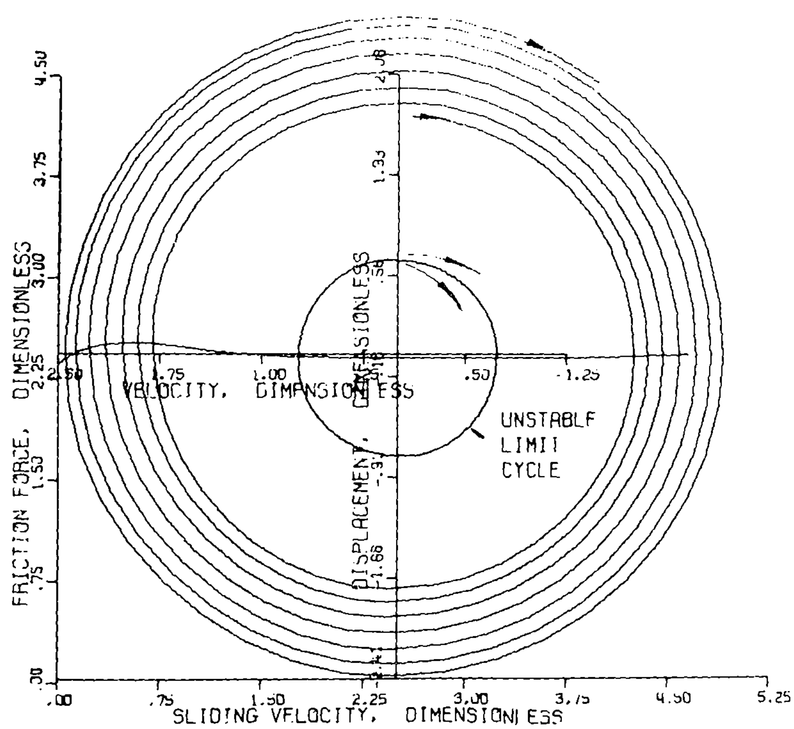

Fig. 14 *Phase-plane diagram (computer solution) of the hard excitation case of quasi-harmonic vibration*

Discussion

The exponential and polynomial functions (equations (2) and (3)) were computer-fitted to the experimental friction-velocity curve of Fig. 13. The exponential equation was applied to the theory developed earlier in the paper (equation (15)) to give the theoretical amplitude-velocity curves displayed in Fig. 13. Theoretical amplitude values derived from equation (15) were inserted into equation (16) in order to check the stability. A similar procedure was followed for the polynomial approximation employing equations (21) and (22). In Fig. 13 the amplitude-velocity curve for the polynomial theory is shown for the region A to B only.

The amplitude-velocity curves of Fig. 13 illustrate that the experimental results and the predictions by the exponential and polynomial theories are in reasonable agreement. It should be noted that the experimental points to the left of A represent stick-slip amplitude values. In theory and by experiment the quasi-harmonic oscillation commences at a discrete velocity A, and the amplitude of vibration increases in a near-linear fashion with increasing velocity until vibration suddenly decays at B. Theory predicts that the vibration is stable between the velocity limits A and B. To the right of B instability is possible, and in fact two vibration amplitudes are predicted at each velocity. The vibration of larger amplitude is stable whereas the lower curve represents an unstable condition. If the system is perturbed to an amplitude between the two curves, the upper amplitude will be adopted for steady-state vibration. Alternatively, if the perturbation displacement is below the lower curve, vibration will not occur. Vibration in the region A to B corresponds to soft excitation, whereby the system departs from an unstable singularity and arrives at a state of steady-state vibration with a stable amplitude [12]. To the right of B the singularity is stable and hard excitation prevails. Under these conditions the barrier presented by an unstable limit cycle must be crossed before a stable vibration can exist. Fig. 14 illustrates a phase-plane diagram of the hard excitation case obtained by computer for the blotting paper-steel system. Experimentally, the vibration tended to diminish at point B largely because inconsistencies in friction between the disk and slider permitted a reduction in vibration amplitude. However, once the vibration had decayed to a small amplitude value (below the lower amplitude curve), it could not return to the original steady-state vibration under hard excitation conditions.

The constants C_1 to C_5 for the exponential function were determined to be

$$C_1 = -0.0225;\ C_2 = 0.5929;\ C_3 = 2.7385;\ C_4 = -0.00435;\ C_5 = 1.059.$$

Substitution of these values into equation (7) gave an estimate of the damping coefficient required for complete extinction of vibration over the entire range of sliding velocities. The damping coefficient obtained from this calculation was 0.077. A test was performed using a permanent magnet as damper which had a damping coefficient of approximately 0.08. With this damper only stick-slip oscillation was observed in the low-velocity region and quasi-harmonic oscillation did not occur. Hence, it is possible to introduce controlled damping into the system in order to prohibit quasi-harmonic oscillation. This finding could be utilized in the design of practical systems where vibration is undesired.

In all tests, the frequency of quasi-harmonic vibration was of the order 137 rad/sec which almost coincided with the damped natural frequency of the slider support system. In the majority of systems subjected to self-induced vibration, the frequency of quasi-harmonic vibration is very nearly equal to the system natural frequency.

It is of some interest to observe that a comparatively small hump in the friction-velocity curve can lead to substantial vibration amplitudes. Normally, friction-induced oscillation is associated with slow-speed sliding whereas the present work demonstrates that vibration may exist over a large velocity range. The presence of vibration of the quasi-harmonic form could influence the performance of practical machine elements such as automatic transmissions, brakes, and clutches.

Conclusion

A theoretical and experimental study of quasi-harmonic vibration has been described. The results of the investigation may be summarized as follows:

1 Quasi-harmonic friction-induced vibration may exist provided the dynamic friction curve is of a particular form.

2 If the necessary friction-velocity curve exists, then the vibration commences at some lower velocity boundary. As the sliding velocity increases, the amplitude of vibration increases until the motion suddenly ceases at an upper critical velocity.

3 Damping and the form of the dynamic friction curve play a role in determining the range of velocity for which the vibration exists. Sufficient increase in damping will lead to the complete extinction of the vibration over the entire velocity range.

4 The theoretical predictions and the experimental results are in reasonable agreement for the friction materials and lubricants employed in the present work.

5 In general, quasi-harmonic vibration may exist at comparatively high sliding velocities and as a consequence may be detrimental in the operation of frictional machine elements such as automatic transmissions, brakes, and clutches.

Acknowledgment

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