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Quantum Cohomology of Minuscule Homogeneous Spaces III Semi-Simplicity and Consequences

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Abstract. We prove that the quantum cohomology ring of any minuscule or cominuscule homogeneous space, specialized at $q = 1$, is semisimple. This implies that complex conjugation defines an algebra automorphism of the quantum cohomology ring localized at the quantum parameter. We check that this involution coincides with the strange duality defined in our previous article. We deduce Vafa–Intriligator type formulas for the Gromov–Witten invariants.

1 Introduction

This paper is a sequel to [CMP1] and [CMP2] where we developed a unified approach to the quantum cohomology of (co)minuscule homogeneous manifolds $X = G/P$.

Recall that a $\mathbb{Z}$-basis for the ordinary cohomology ring $H^*(X)$ (or for the Chow ring $A^*(X)$) of X is given by the Schubert classes $\sigma(w)$, where $w \in W_X$ belongs to the set of minimal lengths representatives of W/W_P, the quotient of the Weyl group W of G by the Weyl group W_P of P. The Schubert classes are also a basis over $\mathbb{Z}[q]$ of the (small) quantum Chow ring $QA^*(X)$, whose associative product is defined in terms of 3-point Gromov–Witten invariants. If R is a ring, we denote by $QA^*(X,R)$ the tensor product $QA^*(X) \otimes_{\mathbb{Z}} R$ and $QA^*(X,R)_{loc}$ its localization at q, that is, $QA^*(X,R)_{loc} = QA^*(X,R) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q,q^{-1}]$. The main result of [CMP2], *strange duality*, was that one could define, for any $w \in W_X$, a nonnegative integer $\delta(w)$, and an algebraic number $\zeta(w)$ in such a way that the endomorphism ι of $QA^*(X,\mathbb{R})_{loc}$, defined by

$$\iota(q) = q^{-1} \quad \text{and} \quad \iota(\sigma(w)) = q^{-\delta(w)}\zeta(w)\sigma(\iota(w)),$$

would be a ring involution.

In this paper we give a natural explanation of the existence of such an involution, which turns out to be directly related to the semisimplicity of the quantum cohomology ring specialized at $q = 1$. For the classical cases this semisimplicity has been known for some time, as it can readily be read off the explicit presentations that have been found. We complete the picture by checking the semisimplicity in the exceptional cases.

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To this end, we recall that for $X = G/P$, a minuscule homogeneous space, the rational quantum cohomology ring can be described as

$$QH^*(G/P) = \mathbb{Q}[\mathfrak{t}]^{W_P}[q]/(I_{d_1}, \dots, I_{d_m} - q),$$

where $\mathfrak{t}$ is a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G, and $I_{d_1}, \dots, I_{d_m}$ are homogeneous generators of the W-invariants; the maximal degree d_m is the Coxeter number of $\mathfrak{g}$.

Let $Z(\mathfrak{g})$ denote the subscheme of $\mathfrak{t}$ defined by the equations $I_{d_1} = \cdots = I_{d_{m-1}} = I_{d_m} - 1 = 0$. Observe that it does not depend (up to homothety) on the choice of the invariants. Moreover, it is a finite scheme of length $\#W$, since if we replace $I_{d_m} - 1$ by I_{d_m} in this set of equations, we get the spectrum of the Chow ring of the full flag variety of G, which is a finite scheme of length $\#W$ supported at the origin. The following result indicates a fundamental difference between the classical and the quantum settings.

Proposition 1.1 *For any simple Lie algebra $\mathfrak{g}$ (except possibly $\mathfrak{f}_4$ and $\mathfrak{e}_8$), the scheme $Z(\mathfrak{g})$ is reduced and is a free W-orbit.*

In fact, this result is relevant for quantum cohomology only in type A, D, E, with E_8 excepted. For the classical types it is in fact very easy to check, but the cases of E_6 and E_7 are somewhat more involved. It implies the "minuscule" part of our next statement.

Corollary 1.2 *For any minuscule or cominuscule rational homogeneous space G/P, the algebra $QA^*(G/P)_{q=1} \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C}[W/W_P]$ is semisimple.*

More intrisically, the spectrum $Z(G/P)$ of the quantum algebra $QA^*(G/P)_{q=1}$ is $Z(\mathfrak{g})/W_P$, at least in the minuscule case.

Now, any commutative, semisimple, finite-dimensional algebra H is a product of fields, hence over $\mathbb{R}$ it decomposes as $H = \mathbb{R}^n \oplus \mathbb{C}^p$. By conjugating the complex factors we get a canonical algebra automorphism, the complex involution.

We therefore get an algebra involution of $QA^*(G/P, \mathbb{R})_{q=1}$. We point out that, because $Z(G/P)$ is reduced, this involution lifts to an algebra involution of $QA^*(G/P)_{loc}$, mapping q to q^{-1} and any class of degree d to a class of degree $-d$ (see Theorem 2.1). This leads to a new interpretation of strange duality.

Theorem 1.3 *For any minuscule or cominuscule homogeneous space $X = G/P$, complex conjugation and strange duality define the same involution.*

The proof is case by case. In fact, this result has already been observed by Hengelbrock for the case of Grassmannians with a different method [H]. What we will check is that strange duality and the complex involution coincide on a set of generators of the quantum cohomology ring. In the classical cases we will also provide a direct check that the complex involution is given by the same expressions as in [CMP2]. In particular, this will explain the occurrence of the irrationalities introduced by the function ζ (which one can eventually get rid of by rescaling q).

The advantage of this approach to strange duality through complex conjugation is that, besides being conceptually enlightening, the fact that it is an algebra automorphism becomes completely obvious—while in [CMP2] this was the result of painful computations, especially in the exceptional cases. What is not clear a priori is that the complex conjugate of a Schubert class is again (a multiple of) a Schubert class, while in [CMP2] this was given by the very definition of strange duality. It would be interesting to have a conceptual explanation of that phenomenon (which no longer holds true for non minuscule or cominuscule spaces).

We stress that the smoothness of the finite scheme $Z(G/P)$ plays an essential role here. In Section 7, we consider the case of $G_\omega(2,6)$, the Grassmannian of isotropic planes in a six-dimensional symplectic complex vector space. This is the simplest example of a homogeneous space with Picard number one that is neither minuscule nor cominuscule. We check that its quantum cohomology ring is not semisimple. In fact, the scheme $Z(G_\omega(2,6))$ is made of ten simple points and one double point. Moreover, the existence of this double point prevents the complex conjugation from being lifted to an involution of the quantum cohomology ring reversing degrees.

Finally, we use the schemes $Z(G/P)$ to obtain Vafa–Intriligator type formulas for the Gromov–Witten invariants. We express these formulas in a uniform way in terms of a *quantum Euler class* $e(X)$ introduced in [A] for any projective manifold X. Abrams proved that the invertibility of that class is equivalent to the semisimplicity of the quantum cohomology ring $QH^*(X)$ (after specialization of the quantum parameters). In the (co)minuscule setting, we prove that the quantum Euler class is simply given by the product of the positive roots of $\mathfrak{g}$ that are not roots of $\mathfrak{p} = \mathrm{Lie}(P)$. The Vafa–Intriligator type formulas that we obtain are expressed in terms of that class. They are equivalent to the formulas obtained in [ST] and [R] for Grassmannians, but they are simpler than the formulas given in [Ch] for the other classical cases.

2 The Complex Involution

Let X be a smooth projective variety with Picard number one. Let $QH^*(X,\mathbb{C}) = H^*(X,\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[q]$ be its small quantum cohomology ring over the complex numbers, and $QH^*(X,\mathbb{C})_{loc}$ the algebra obtained by inverting the quantum parameter q.

Theorem 2.1 *Suppose that the spectrum of the finite dimensional algebra $QH^*(X)_{q=1}$ is a reduced finite scheme. Then there exists an algebra automorphism of $QH^*(X,\mathbb{C})_{loc}$ mapping q to q^{-1}.*

Proof The inclusion $\mathbb{C}[q,q^{-1}] \hookrightarrow QH^*(X,\mathbb{C})_{loc}$ yields a finite morphism $\pi\colon C \to D$ of curves over $\mathbb{C}$. We consider the involution i of D given by $q \mapsto q^{-1}$; the theorem states that we can lift i to C under the hypothesis that C is a smooth curve.

Let us consider a homogeneous presentation

$$H^*(X,\mathbb{R}) = \mathbb{R}[X_1,\dots,X_n]/(r_1,\dots,r_k)$$

of $H^*(X,\mathbb{R})$. The quantum cohomology ring can be presented as

$$\mathbb{R}[X_1,\dots,X_n,q]/(R_1,\dots,R_k),$$

where R_i is again a homogeneous relation that specialises to r_i when $q = 0$.

If $(x_i, q) \in C(\mathbb{C})$ is a complex point of C, then $(\overline{x}_i, \overline{q}) \in C(\mathbb{C})$ because C is defined over $\mathbb{R}$, and thus, by homogeneity, $(\overline{x}_i/\|q\|^{2\deg(q)/\deg(x_i)}, \overline{q}/\|q\|^2)$ also belongs to $C(\mathbb{C})$. This is a complex point of C over $\overline{q}/\|q\|^2 = q^{-1}$. We claim that the map

$$j\colon (x_i, q) \mapsto (\overline{x}_i/\|q\|^{2\deg(q)/\deg(x_i)}, \overline{q}/\|q\|^2)$$

is algebraic. Indeed, consider the fiber product $C \times_D C$, where the first morphism $C \to D$ is π, and the second is $i \circ \pi$. Let C_0 denote the connected component in $(C \times_D C)(\mathbb{C})$, given as the set of pairs $((x_i, q), j(x_i, q))$. It is algebraic, and the morphism $C_0 \to C$ induced by the first projection is finite of degree one, thus an isomorphism. So the theorem is proved. ■

We call this involution i of $QH^*(X, \mathbb{C})_{loc}$ the *complex involution*.

3 Grassmannians

Let $G(d, n)$ denote the Grassmannian of d-dimensional subspaces of an n-dimensional vector space. Its quantum cohomology ring can be described as

$$QA^*(G(d, n)) = \mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{S}_d \times \mathcal{S}_{n-d}}[q]/(e_1, \ldots, e_{n-1}, e_n - (-1)^{n-d}q),$$

where $e_1, \ldots, e_n$ are the elementary symmetric functions in the n indeterminates $x_1, \ldots, x_n$ [ST]. Here the symmetric groups $\mathcal{S}_d$ and $\mathcal{S}_{n-d}$ act by permutation of the first d and last $n - d$ variables, so we only consider symmetric functions in these two sets of variables. Usually, the relations $e_1, \ldots, e_{n-1}, e_n$ are used to eliminate one of these two sets of variables, but we will not do that. As an algebra, $QH^*(G(d, n))$ is generated by q and the special Schubert classes $\sigma(k)$, which are represented by the k-th elementary symmetric functions $e_k(x_1, \ldots, x_d)$ in the first d variables.

Proof of Proposition 1.1 The equations defining $Z(\mathfrak{sl}_n)$ in $\mathbb{C}^n$ are $e_1 = \cdots = e_{n-1} = e_n - (-1)^{n-d} = 0$. Thus $Z(\mathfrak{sl}_n)$ is the set of n-tuples $(\zeta_1, \ldots, \zeta_n)$ of distinct n-th roots of $(-1)^{d-1}$. This is certainly a free orbit of the symmetric group $\mathcal{S}_n$. ■

We can therefore interpret the quantum cohomology ring of $G(d, n)$ at $q = 1$, as

$$QH^*(G(d, n))_{q=1} = \mathbb{Q}[Z(G(d, n))],$$

where $Z(G(d, n))$ denotes the set of (unordered) d-tuples of distinct n-th roots of $(-1)^{d-1}$.

Proof of Theorem 1.3 We need to prove that the complex involution maps a special Schubert class $\sigma(k)$ to the class $\sigma(n - d, 1^{d-k})$. To check this we simply compute the complex conjugate of $\sigma(k)$ as follows:

$$\begin{aligned} e_k(\bar{\zeta}_1, \ldots, \bar{\zeta}_d) &= e_k(\zeta_1^{-1}, \ldots, \zeta_d^{-1}) \\ &= (\zeta_1 \cdots \zeta_d)^{-1} e_{d-k}(\zeta_1, \ldots, \zeta_d) \\ &= (-1)^{n-d}(\zeta_{d+1} \cdots \zeta_n) e_{d-k}(\zeta_1, \ldots, \zeta_d). \end{aligned}$$

Observe that $(-1)^{n-d}\zeta_{d+1}\cdots\zeta_n = (-1)^{n-d}e_{n-d}(\zeta_{d+1},\ldots,\zeta_n) = h_{n-d}(\zeta_1,\ldots,\zeta_d)$. Therefore $\bar{e}_k = h_{n-d}e_{d-k} = s_{n-d,1^{d-k}}$. ■

For completeness we deduce the complex conjugate of any Schubert class and recover the formulas given by Postnikov [P].

Proposition 3.1 *The complex conjugate of the Schubert class $\sigma(\lambda)$, is the Schubert class $\sigma(\iota(\lambda))$.*

Here $\iota(\lambda)$ denotes the partition deduced from λ by a simple combinatorial process (see [P,CMP2]). Recall that λ is a partition whose diagram is contained in a $d\times(n-d)$ rectangle. Let c be the size of the Durfee square of λ, that is, the largest integer such that $\lambda_c \geq c$. Write λ as $(c+\mu,\nu)$, where now μ is contained in a $c\times(n-d-c)$ rectangle and μ in a $(d-c)\times c$ rectangle. Denote by $p(\mu)$ and $p(\nu)$ the complementary partitions in these respective rectangles. Then $\iota(\lambda) = (c+p(\mu), p(\nu))$.

Proof We use the fact that the Giambelli formulas hold in the quantum cohomology ring, as proved by Bertram in [B]. Thus, for any partition λ,

$$\begin{aligned}\bar{\sigma}(\lambda) &= (h_{n-d})^{n-d}\det(\sigma(d-\lambda_i^*+i-j))_{1\leq i,j\leq n-d}\\ &= (h_{n-d})^{n-d}\det(\sigma(d-\lambda^*_{n-d+1-i}-i+j))_{1\leq i,j\leq n-d}\\ &= (h_{n-d})^{n-d}\sigma(p(\lambda)),\end{aligned}$$

where $\sigma(p(\lambda))$ is the Schubert class Poincaré dual to $\sigma(\lambda)$. But h_{n-d} is invertible in the quantum cohomology ring, with inverse $\sigma(d)$, and $\sigma(d)^{n-d}$ is the punctual class $\sigma(pt)$. Since the multiplication by the punctual class is given (for $q=1$) by the formula $\sigma(pt)*\sigma(\mu) = \sigma(p\iota(\mu))$ (see [CMP2, Theorem 3.3]), we finally get

$$\bar{\sigma}(\lambda) = \sigma(pt)^{-1}*\sigma(p(\lambda)) = \sigma(\iota(\lambda)).$$

Note the interesting fact that we obtain directly the relation that holds for any (co)minuscule homogeneous space, between the complex conjugation, Poincaré duality, and the quantum product by the punctual class. ■

4 Orthogonal Grassmannians and Quadrics

4.1 Orthogonal Grassmannians

Let $G_Q(n+1,2n+2)$ denote the orthogonal Grassmannian, that is, one of the two families of maximal isotropic subspaces in some vector space of dimension $2n+2$ endowed with a nondegenerate quadratic form. Its quantum cohomology ring can be described as

$$QA^*(G_Q(n+1,2n+2)) = \mathbb{Z}[x_1,\ldots,x_{n+1}]^{\mathfrak{S}_{n+1}}[q]/(E_1,\ldots,E_{n-1},E_n-4q,e_{n+1}),$$

where $E_1,\ldots,E_n,E_{n+1}=e_{n+1}^2$ are now the elementary symmetric functions in the squares of the $n+1$ indeterminates $x_1,\ldots,x_{n+1}$. Moreover, the quantum cohomology ring is generated by q and the special Schubert classes $\sigma(k)$, for $1\leq k\leq n$. The class $\sigma(k)$ is represented by $e_k/2$, where e_k is the k-th elementary symmetric function in $x_1,\ldots,x_n$ (see [KT, Theorem 1]).

Proof of Proposition 1.1 The equations defining $Z(\mathfrak{so}_{2n+2})$ are $E_1 = \cdots = E_{n-1} = E_n - 4 = e_{n+1} = 0$. The set of solutions to this equation is

$$Z(\mathfrak{so}_{2n+2}) = \{(\zeta_1, \dots, \zeta_k, 0, \zeta_{k+1}, \dots, \zeta_n)\},$$

where the squares of $\zeta_1, \dots, \zeta_n$ are the n distinct n-th roots of $4.(-1)^{n-1}$. This is certainly a free orbit of the Weyl group $W(D_{n+1}) = \mathcal{S}_{n+1} \times \mathbb{Z}_2^n$. ■

We can therefore interpret the quantum cohomology ring of $G_Q(n+1, 2n+2)$ at $q = 1$, as

$$QH^*(G_Q(n+1, 2n+2))_{q=1} = \mathbb{Q}[Z(G_Q(n+1, 2n+2))],$$

where $Z(G_Q(n+1, 2n+2))$ is identified with the set of (unordered) n-tuples of square roots of the n distinct n-th roots of $(-1)^{n-1}.4$. Note that $\#Z(G_Q(n+1, 2n+2)) = 2^n$, as expected.

Proof of Theorem 1.3 We need to prove that the complex involution maps a special Schubert class $\sigma(k)$ to a suitable multiple of the class $\sigma(n, n-k)$. We compute the complex conjugate of $\sigma(k)$ as follows:

$$\begin{aligned} e_k(\bar{\zeta}_1, \dots, \bar{\zeta}_n) &= 4^{k/n} e_k(\zeta_1^{-1}, \dots, \zeta_n^{-1}) \\ &= 4^{k/n} (\zeta_1 \dots \zeta_n)^{-1} e_{n-k}(\zeta_1, \dots, \zeta_n) \\ &= 4^{(k/n)-1} (\zeta_{d+1} \dots \zeta_n) e_{n-k}(\zeta_1, \dots, \zeta_n), \end{aligned}$$

where we have used the fact that $(\zeta_{d+1} \dots \zeta_n)^2 = e_n(\zeta_1, \dots, \zeta_n)^2 = 4$. Otherwise stated,

$$\bar{\sigma}(k) = 2^{(2k/n)-1} \sigma(n) \sigma(n-k) = 2^{(2k/n)-1} \sigma(n, n-k),$$

which is in complete agreement with Proposition 4.7 in [CMP2]. Finally, note that since $e_n^2 = 4$, e_n is real, hence $\sigma(n)$ is fixed by the complex involution, and we are done. ■

We deduce that the complex conjugate of any Schubert class $\sigma(\lambda)$ is given by a suitable multiple of $\sigma(\iota(\lambda))$, where $\iota(\lambda)$ is defined as follows. Write $\lambda = (\lambda_1 > \cdots > \lambda_{2\delta(\lambda)})$, ending with a zero part if necessary. Then $\iota(\lambda) = (n - \lambda_{2\delta(\lambda)} > \cdots > n - \lambda_1)$. Let $z(\lambda) = \frac{2|\lambda|}{n} - (\ell(\lambda) + \delta_{\lambda_1, n})$.

Proposition 4.1 *The complex involution maps the Schubert class $\sigma(\lambda)$ to $2^{z(\lambda)} \sigma(\iota(\lambda))$.*

Proof We first consider classes $\sigma(i, j)$, with $i > j > 0$. Suppose for example that $i + j > n$; then

$$\sigma(i, j) = \sigma(i)\sigma(j) + 2 \sum_{k=1}^{n-i-1} (-1)^k \sigma(i+k)\sigma(j-k) + (-1)^{n-i} \sigma(n)\sigma(i+j-n).$$

Using the fact that $\sigma(n)^2 = 1$, we deduce that

$$\bar{\sigma}(i,j) = 2^{\frac{2i+2j}{n}-2}\{\sigma(n-j)\sigma(n-i) + 2\sum_{k=1}^{n-i-1}(-1)^k\sigma(n-j+k)\sigma(n-i-k) + (-1)^{n-i}\sigma(2n-i-j)\},$$

that is, $\bar{\sigma}(i,j) = 2^{\frac{2i+2j}{n}-2}\sigma(n-j,n-i)$. A similar computation shows that this formula also holds for $i+j \le n$. But then the Giambelli type formula (see [BKT1, Theorem 7])

$$\sigma(\lambda) = \mathrm{Pfaff}(\sigma(\lambda_i,\lambda_j))_{1\le i<j\le 2\delta(\lambda)}$$

immediately implies that $\bar{\sigma}(\lambda) = 2^{z(\lambda)}\sigma(\iota(\lambda))$. ■

4.2 Quadrics

We will only treat the case of quadrics of even dimension (which are minuscule). The case of quadrics of odd dimensions (which are cominuscule) is very similar.

So let $\mathbb{Q}^{2n}$ be a quadric of even dimension $2n$. There are two Schubert classes σ_+ and σ_- in middle dimension n, and a single one σ_k in every other dimension $k \neq n$. In terms of the hyperplane class $H = \sigma_1$, one has $\sigma_k = H^k$ for $k \le n-1$, $\sigma_k = H^k/2$ for $k \ge n+1$, and $H^n = \sigma_+ + \sigma_-$ (in the classical Chow ring). See [CMP1, Section 4.1] for more details.

The quantum cohomology ring can be described as

$$QH^*(\mathbb{Q}^{2n}) = \mathbb{Z}[x_0, x_1, \ldots, x_n]^{\mathcal{D}_n}[q]/(E_1, \ldots, E_{n-1}, E_n - (-1)^{n-1}4q, e_{n+1}),$$

where $\mathcal{D}_n$ denotes the Weyl group of type D_n. Therefore $Z(\mathfrak{so}_{2n+2})$ is defined by the equations $E_1 = \cdots = E_{n-1} = E_n - (-1)^{n-1}4 = e_{n+1} = 0$, and thus it is the set of $(n+1)$-tuples $(\zeta_1, \ldots, \zeta_k, 0, \zeta_{k+1}, \ldots, \zeta_n)$, where the squares of $\zeta_1, \ldots, \zeta_n$ are the n distinct n-th roots of 4. The Weyl group W_P of the parabolic P defining $\mathbb{Q}^{2n}$ is the fixator of the first coordinate. It has $2n+2$ orbits in $Z(\mathfrak{so}_{2n+2})$. First, there are $2n$ orbits $O(\zeta)$ defined by their non zero first coordinate ζ, which can be any $2n$-th root of 4. Second, there are two orbits $O(+)$ and $O(-)$ with zero first coordinate and defined by the fact that the product of the non zero coordinates is ± 2. Since $2n+2$ is also the dimension of $H^*(\mathbb{Q}^{2n})$, this confirms that $Z(\mathbb{Q}^{2n})$ is reduced and $QA^*(\mathbb{Q}^{2n})_{q=1}$ is semisimple.

The W_P-invariant polynomials are generated by t_0, the first coordinate, and the product $P = t_1 \cdots t_n$ of the other coordinates. The algebra $QA^*(\mathbb{Q}^{2n})_{q=1}$, considered as an algebra of functions on the set $Z(\mathbb{Q}^{2n})$, is determined by the following table.

	H	H^n	P
$O(\zeta)$	ζ	ζ^n	0
$O(+)$	0	0	2
$O(-)$	0	0	-2

Observe that the two degree n classes H^n and P are real. Moreover we easily get that $\overline{H} = 4^{\frac{1-n}{n}} H^{2n-1}$. There remains to express σ_+ and σ_- in terms of H^n and P. Using the formulas of [CMP2], we see that this expression depends on the parity of n; we get

$$\sigma_{\pm} = \frac{1}{2}(H^n \pm iP) \quad \text{for } n \text{ even} \quad \text{and} \quad \sigma_{\pm} = \frac{1}{2}(H^n \pm P) \quad \text{for } n \text{ odd.}$$

Thus $\overline{\sigma}_{\pm} = \sigma_{\mp}$ for n even, but $\overline{\sigma}_{\pm} = \sigma_{\pm}$ for n odd. Comparing with [CMP2], we see that strange duality coincides with the complex involution.

5 Lagrangian Grassmannians

Let $G_\omega(n, 2n)$ denote the Lagrangian Grassmannian parametrizing maximal isotropic subspaces in some symplectic vector space of dimension $2n$. This is a cominuscule, but not minuscule, homogeneous space. Its quantum cohomology ring can be described as

$$QA^*(G_\omega(n, 2n)) = \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}[q]/(R_1, \dots, R_n),$$

where the relations are $R_k = E_k - (-1)^{2k-n-1} q e_{2k-n-1}$ for $1 \leq k \leq n$ [T]. Recall that e_k (resp. E_k) is the k-th elementary symmetric functions of the variables $x_1, \dots, x_n$ (resp. of their squares). Moreover, we used the convention that $e_k = 0$ for negative k, so that the relations have no quantum correction in degree less than or equal to $n + 1$. All the higher degree relations involve a quantum term, in contrast with the minuscule case for which only the relation of highest degree receives a quantum correction. This means that the quantum relations are no longer W-invariants, so there is no point in proving Proposition 1.1. Instead, we directly prove that $Z(G_\omega(n, 2n))$ is reduced, hence the semi-simplicity of the quantum cohomology ring at $q = 1$.

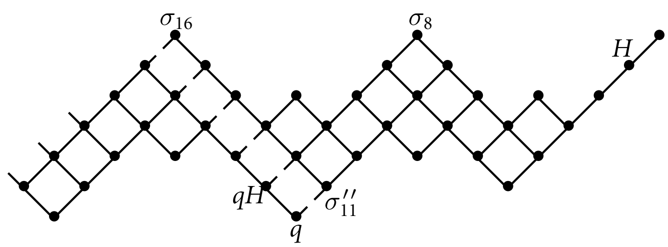
Proof of Corollary 1.2 Observe that the relations take a simpler form if we introduce formally an auxiliary variable x_0 and let $q = e_{n+1}$. The point is that the R_k's are then formally the same as the k-th elementary symmetric functions of the squares of $x_0, x_1, \dots, x_n$. This implies that the spectrum of the quantum cohomology ring at $q = 1$ is supported on the set of (unordered) $(n + 1)$-tuples $(\zeta_0, \zeta_1, \dots, \zeta_n)$ such that $\zeta_0\zeta_1 \cdots \zeta_n = 1$ and $\zeta_0^2, \zeta_1^2, \dots, \zeta_n^2$ are the $(n+1)$-th different roots of $(-1)^n$. There are exactly 2^n such unordered $(n + 1)$-tuples, as was already observed in [Ch]. Since this coincides with the dimension of $QA^*(G_\omega(n, 2n))_{q=1}$ as a vector space, this algebra is semisimple. ■

Proof of Theorem 1.3 We leave this to the reader, since this case is even easier than the previous ones. Indeed, the complex conjugation sends a special Schubert class $\sigma(k)$ to a suitable multiple of another special Schubert class $\sigma(n + 1 - k)$. A consequence is that the fact that this defines an involution of the quantum cohomology can be directly checked on the relations: R_k is simply changed into a multiple of R_{n+1-k}, and the claim follows. ■

6 The Exceptional Cases

6.1 The Cayley Plane

We first recall some facts about the quantum cohomology ring of the Cayley plane $\mathbb{OP}^2 = E_6/P_1$. This is a sixteen dimensional variety of index 12. The Schubert classes are organized in the following Hasse diagram.



This diagram could be extended indefinitely on the left, with a period increasing degrees by 8. Then it would exactly encode the quantum Chevalley formula in the sense that the product of any Schubert class by the hyperplane class H would be the sum of the Schubert classes (possibly with some q-coefficient) connected to it on its immediate left. Dashed edges represent quantum corrections, so that the rightmost component of the diagram obtained by disconnecting the dashed edges in just the classical Hasse diagram encoding the classical Chevalley formula. In particular the diagram above, the *quantum Hasse diagram*, makes it easy to compute powers of H in the quantum cohomology ring. This will be useful a little later on.

Over $\mathbb{Q}[q]$, it was proved in [CMP2] that the quantum cohomology ring is generated by the three classes H, σ_8, and σ''_{11}. The complex involution is thus determined by the images of these classes, which are sent by strange duality to

$$\iota(H) = 12^{1/4}q^{-1}\sigma''_{11}, \quad \iota(\sigma_8) = q^{-2}\sigma_{16}, \quad \iota(\sigma''_{11}) = 12^{1/4}q^{-1}H.$$

Proof of Proposition 1.1 The first difficulty is to understand the $W(E_6)$-invariant polynomials on a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{e}_6$. For this, we consider the realisation of $\mathfrak{t}$ and $W(E_6)$ studied in [M]. This is based on the observation that $\mathfrak{e}_6$ contains a full rank subalgebra isomorphic with $\mathfrak{sl}_6 \times \mathfrak{sl}_2$. As modules over this subalgebra, $\mathfrak{e}_6$ and its minimal representation J, of dimension 27, can be decomposed as

$$\mathfrak{e}_6 = \mathfrak{sl}_6 \times \mathfrak{sl}_2 \oplus \wedge^3\mathbb{C}^6 \otimes \mathbb{C}^2, \quad J = (\mathbb{C}^6)^* \otimes \mathbb{C}^2 \oplus \wedge^2\mathbb{C}^6.$$

We choose for our Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{e}_6$ the product of Cartan subalgebras in $\mathfrak{sl}_6$ and $\mathfrak{sl}_2$. We thus have generators of $\mathfrak{t}^*$ given by $(x_i)_{1\le i\le 6}$ with $\sum x_i = 0$ (for the factor in $\mathfrak{sl}_6$), and y (for the factor in $\mathfrak{sl}_2$). Moreover, the $W(E_6)$-invariants are generated by the obvious invariants given by the sums of powers of weights in J, that is,

$$I_k(x, y) = \sum_{1\le i,j\le 6}(x_i + x_j)^k + \sum_i[(-x_i + y)^k + (-x_i - y)^k],$$

for $k \in \{2, 5, 6, 8, 9, 12\}$. So, the equations we have to solve are $I_k(x, y) = 0$ for $k \in \{2, 5, 6, 8, 9\}$ and $I_{12}(x, y) = c$, where c is any nonzero constant.

We claim that all the solutions of these equations, for a certain choice of c, are given by the $W(E_6)$-orbit of the solution

$$\begin{cases} x_2 = -x_1 = 1 \\ x_4 = -x_3 = i \\ x_6 = -x_5 = e^{i\pi/8} \\ y = e^{5i\pi/8}, \end{cases}$$

and that this $W(E_6)$-orbit is free. In fact, let us first check that this is indeed a solution. Since $x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = 0$, all I_k's with odd k vanish. For even k, I_k reduces to a symmetric polynomial in x_1^2, x_3^2, x_5^2 and y^2. Therefore, to prove that (x, y) annihilates I_2, I_6 and I_8, it is enough to show that

$$\sum_{i\in\{1,3,5,7\}} x_i^2 = 0, \qquad \sum_{i\in\{1,3,5,7\}} x_i^6 = 0, \qquad \sum_{i\in\{1,3,5,7\}} x_i^8 = 0,$$

where we have set $x_7 := y$. The first two equations hold because they are of odd degree in the x_i^2, and the values of x_i^2 are two opposite pairs. To check that the last one also holds is a straighforward computation.

Let us now prove that the $W(E_6)$-orbit of our solution is free. To this end, let $w \in W(E_6)$ such that $w(x, y) = (x, y)$. We know that w can be represented by a 7×7 matrix $(a_{i,j})$ with rational coefficients. Since x_1, x_3, x_5, y are independant over $\mathbb{Q}$, the diagonal coefficient $a_{7,7}$ of this matrix must be 1. Using the fact that w preserves I_2, we deduce that w stabilises the vector $(0, 0, 0, 0, 0, 0, 1)$. This means that w belongs to the symmetric group $\mathfrak{S}_6 \subset W(E_6)$, acting by permutations of $x_1, \dots, x_6$. But then it follows easily that w is the identity. ∎

Proof of Theorem 1.3 Let us first show that the complex conjugate of σ_8 is σ_{16}, as a warm-up. For this we recall that σ_8 is an idempotent of the quantum cohomology ring. More precisely, we have the following statement, which is a special case of [CMP2, Corollary 5.2].

Lemma 6.1 *The multiplication by σ_8 in $QH^*(E_6/P_1)$ sends any Schubert class to its translate by eight steps on the left in the quantum Hasse diagram. In particular, $\sigma_8^2 = \sigma_{16}$ and $\sigma_8^3 = q^2$.*

In particular, we deduce that σ_8, considered as a function on $Z(E_6/P_1)$, takes its values among the third roots of unity. But for such a root ζ, we have $\overline{\zeta} = \zeta^2$, and therefore

$$\overline{\sigma_8} = \sigma_8^2 = \sigma_{16}.$$

Let us now show that the complex conjugate of H is $12^{1/4}\sigma''_{11}$. This is much more tricky, but the idea is the same as above. Suppose we can find a non zero polynomial P such that $P(H) = 0$; suppose that we can also find a polynomial Q such that $Q(\zeta) = \overline{\zeta}$ for any ζ root of P. Then we can conclude that $\overline{H} = Q(H)$. (That's exactly what we have done above with $P(\zeta) = \zeta^3 - 1$ and $Q(\zeta) = \zeta^2$.)

We first find an equation for H. For this we will find a dependance relation between H^{25}, H^{13}, and H (recall that the index of the Cayley plane is twelve). We compute in the specialisation $QH^*(E_6/P_1)_{q=1}$. With the help of the quantum Chevalley formula, or equivalently, of the quantum Hasse diagram above, we get

$$\begin{cases} H^{13} = 78\sigma_{13} + 57H \\ H^{25} = 21060\sigma_{13} + 15417H, \end{cases}$$

hence the equation $P(H) = H^{25} - 270H^{13} - 27H = 0$. Note that the roots of P are given by $\zeta = 0$ or $\zeta^{12} = 135 \pm 78\sqrt{3} = (3 \pm 2\sqrt{3})^3$.

The quantum Hasse diagram also gives us

$$\begin{cases} H^{11} = 33\sigma'_{11} + 12\sigma''_{11} \\ H^{23} = 8901\sigma'_{11} + 3258\sigma''_{11}. \end{cases}$$

Therefore, $234\sigma''_{11} = H^{11}(11H^{12} - 2967)$.

To check that $\overline{H} = 12^{1/4}\sigma''_{11}$, we therefore just need to verify that for every root ζ of the polynomial $P(z) = z^{25} - 270z^{13} - 27z$, one has $234.12^{1/4}\overline{\zeta} = \zeta^{11}(11\zeta^{12} - 2967)$, or, equivalently,

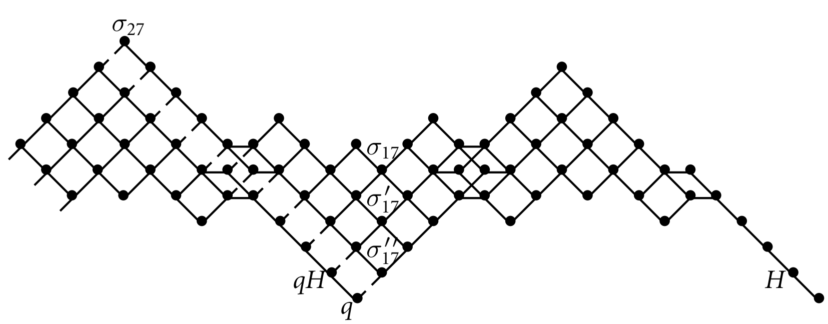
$$234\zeta\overline{\zeta} = \zeta^{12}(11\zeta^{12} - 2967).$$

This is clear if $\zeta = 0$. If $\zeta^{12} = 135+78\sqrt{3}$, an explicit computation yields $\zeta^{12}(11\zeta^{12} - 2967) = 702 + 234\sqrt{3}$, whereas $12^{1/4}\zeta\overline{\zeta} = \sqrt{(3+2\sqrt{3}).2\sqrt{3}} = 3 + \sqrt{3}$. The computation is similar when $\zeta^{12} = 135 - 78\sqrt{3}$, and we can therefore conclude that $\overline{H} = 12^{1/4}\sigma''_{11}$, as claimed.

Of course we can deduce that, conversely, the complex conjugate of σ''_{11} is $12^{-1/4}H$. Since $QH^*(E_6/P_1)$ is generated by H, σ''_{11}, and σ_8, this completes the proof. ∎

6.2 The Freudenthal Variety

We recall some facts on the quantum cohomology ring of the Freudenthal variety E_7/P_7. This is a 27 dimensional variety, of index 18. The quantum Hasse diagram is as follows.



Over $\mathbb{Q}[q]$, the quantum cohomology ring of the Freudenthal variety is generated by the hyperplane class H and the Schubert classes σ_{17} and σ_{27}, the punctual class. The strange duality ι maps these classes to

$$\iota(H) = 3456^{1/9}q^{-1}\sigma_{17}, \quad \iota(\sigma_{17}) = 3456^{-1/9}q^{-1}H, \quad \iota(\sigma_{27}) = q^{-3}\sigma_{27}.$$

Proof of Proposition 1.1 In this case, we were not able to give explicitely any point in $Z(E_7)$, so we give a more abstract argument. First recall that the $W(E_7)$-invariants polynomials on a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{e}_7$ are generated by homogeneous polynomials J_l of degrees $l = 2, 6, 8, 10, 12, 18$. To compute $Z(\mathfrak{e}_7)$, we will, as in the preceeding cases, give a (nonexplicit) example of an element in this scheme, and show that it belongs to a free $W(E_7)$-orbit, so that $Z(\mathfrak{e}_7)$ is exactly this orbit and is reduced.

The orthogonal of ω_7 in $\mathfrak{t}$ is a Cartan algebra for $\mathfrak{e}_6$. Let $I_k, k \in \{2, 5, 6, 8, 9, 12\}$ denote $W(E_6)$-invariants of degree k in $\omega_7^\perp$ that generate the algebra of invariants. Let u in $\mathfrak{t}$ be a point in $\omega_7^\perp$ such that $I_2(u) = I_5(u) = I_6(u) = I_8(u) = 0$ and $I_9(u) = 1$.

First we claim that $u \in Z(\mathfrak{e}_7)$. In fact, the $J_l, l \in \{2, 6, 8, 10, 12\}$ restrict on $\omega_7^\perp$ to $W(E_6)$-invariants, therefore to polynomials in the $I_k, k \in \{2, 5, 6, 8, 9, 12\}$, and since $I_2(u) = I_5(u) = I_6(u) = I_8(u) = I_{12}(u) = 0$, we must have $J_l(u) = 0$.

Let $w \in W(E_7)$ such that $w.u = u$. Let J denote the first fundamental representation of $\mathfrak{e}_6$ and consider the characteristic polynomial of the action of u on J. Since it is a $W(E_6)$-invariant of degree 27 and $I_2(u) = I_5(u) = I_6(u) = I_8(u) = I_{12}(u) = 0$, it must be of the form

$$Q(u, T) = \prod_\eta (T - \eta(u)) = T^{27} + q_9 T^{18} + q_{18} T^9 + q_{27},$$

where the product is taken over the 27 weights η of J.

Let η_1 be a weight of J such that $x := \eta_1(u) \neq 0$. Let ζ be a primitive 9-th root of unity. Since the set $\{\eta(u)\}$ is the set of roots of $Q(u, T)$, which is a polynomial in T^9, there must exist weights $\eta_2, \ldots, \eta_6$ such that

$$(\eta_1(u), \ldots, \eta_6(u)) = (x, x\zeta, x\zeta^2, \ldots, x\zeta^5).$$

Recall that the product $\prod_\theta (T - \theta)$ over the 6 primitive 9-th roots of unity, a cyclotomic polynomial, is irreducible. Thus $T^6 + T^3 + 1$ is the minimal polynomial of ζ and the family $(1, \zeta, \ldots, \zeta^5)$ is free over $\mathbb{Q}$. Therefore, $(x, x\zeta, \ldots, x\zeta^5)$ is also free over $\mathbb{Q}$, and a fortiori $(\eta_1, \ldots, \eta_6)$ is free over $\mathbb{Q}$, and thus (η_i, ω_7) form a base over $\mathbb{Q}$ of the weight vector space.

Thus, setting

$$w.\omega_7 = a\omega_7 + \sum_{1 \le i \le 6} a_i \eta_i,$$

we get $0 = \omega_7(w^{-1}.u) = \sum a_i \eta_i(u)$, so $\forall i \in \{1, \ldots, 6\}, a_i = 0$, and $w.\omega_7 = a\omega_7$. By the same argument, we also have $w.\eta_i = \eta_i$ modulo ω_7, so w preserves $\omega_7^\perp$ and is trivial on it. Recall that in any Weyl group the only nontrivial elements acting trivially on some hyperplane are the reflections s_α, where α is some root (see [Bou, Chap. V, §3.2, Théorème 1(iv)]). But ω_7 is not a root, so the line it generates is not preserved by any reflection. Hence we must have $w = id$. ■

Proof of Theorem 1.3 We know from [CMP2, Theorem 3.3] that $\sigma_{27}^2 = q^3$. When we specialize at $q = 1$, we deduce that σ_{27} is real, hence equal to its complex conjugate.

Let us now show that the complex conjugate of H is $3456^{1/9}\sigma_{17}$. For this, we first express σ_{17} as a polynomial in H. Using the quantum Hasse diagram, we compute

$$H^{17} = 78\sigma_{17} + 442\sigma'_{17} + 748\sigma''_{17},$$
$$H^{35} = 2252088\sigma_{17} + 12969160\sigma'_{17} + 22121896\sigma''_{17},$$
$$H^{53} = 66396246672\sigma_{17} + 382360744192\sigma'_{17} + 652206892048\sigma''_{17}.$$

So we get the relation $\sigma_{17} = H^{17}Q(H^{18})$, where we have set

$$Q(T) = \frac{4237743313}{721278} - \frac{33629825}{77976}T + \frac{84371}{5770224}T^2.$$

In fact, these formulas can be proved simply using the periodicity of the quantum Hasse diagram of E_7/P_7, and the fact that the restriction of the multiplication by H^{18} to the degree 8 part of $QH^*(E_7/P_7)_{q=1}$ has matrix

$$\begin{pmatrix} 598 & 1710 & 1938 \\ 3420 & 9832 & 11172 \\ 5814 & 16758 & 19066 \end{pmatrix}$$

in the base $\sigma_{26}, \sigma_8, \sigma'_8$. Moreover, this shows that if

$$P(T) = 64 - 401808T + 29496T^2 - T^3,$$

the characteristic polynomial of this matrix, then $H^8P(H^{18}) = 0$. Now, it is easy to check that P has three real roots, all positive. Moreover, our key point is the following observation.

Fact $(TQ(T))^{18} = (3456T)^2$ modulo P.

In fact, somewhat painful computations lead to

$$(TQ(T))^2 \equiv -1/40071\, T^2 + 799/1083\, T + 11696/13357,$$
$$(TQ(T))^6 \equiv -40/13357\, T^2 + 34544/361\, T + 8768/13357,$$

and finally

$$(TQ(T))^{18} = (3456T)^2 \quad \text{mod } P.$$

Therefore, if ζ is such that $P(\zeta^{18}) = 0$, then $(\zeta^{18}Q(\zeta^{18}))^{18} = (3456\zeta^{18})^2$, and since ζ^{18} is real, this is equal to $3456^2(\zeta\overline{\zeta})^{18}$. We have noticed that ζ^{18}, being a root of P, is positive. One also checks that $Q(\zeta^{18}) \geq 0$. So we can extract 18-th roots, and deduce that

$$\overline{\zeta} = 3456^{1/9}\zeta^{17}Q(\zeta^{18}).$$

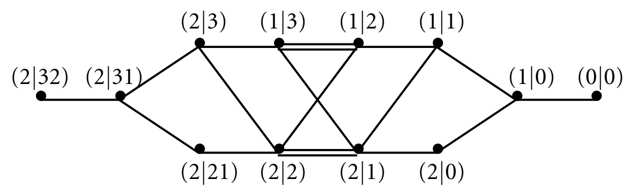
Note that this also holds for $\zeta = 0$, so this is exactly what we need to conclude that the complex conjugate of H is

$$\overline{H} = 3456^{1/9} H^{17} Q(H^{18}) = 3456^{1/9} \sigma_{17}.$$

Of course, we can deduce that the complex conjugate of σ_{17} is the expected multiple of H, and this concludes the proof. ■

7 A Nonreduced Example

In this section we consider the symplectic Grassmannian $G_\omega(2,6)$. Its dimension is 7 and its index is 5. Its Schubert classes are indexed by pairs $(a|\alpha)$, where $0 \leq a \leq 2$, and α is a strict partition whose parts are not bigger than 3, and whose length is at most a. The cohomological degree of the corresponding class is a plus the sum of the parts of α. The Hasse diagram is the following.



Tamvakis proved in [T] that the quantum cohomology ring of $G_\omega(2,6)$ is generated by the special Schubert classes $\sigma_k = \sigma_{(1|k-1)}$, for $1 \leq k \leq 4$, with the following relations:

$$\sigma_2^2 - 2\sigma_1\sigma_3 + 2\sigma_4 = \sigma_3^2 - 2\sigma_2\sigma_4 + q\sigma_1 = 0,$$

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ 1 & \sigma_1 & \sigma_2 \\ 0 & 1 & \sigma_1 \end{pmatrix} = 0, \qquad \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ 1 & \sigma_1 & \sigma_2 & \sigma_3 \\ 0 & 1 & \sigma_1 & \sigma_2 \\ 0 & 0 & 1 & \sigma_1 \end{pmatrix} = 0.$$

The last two relations allow us to express σ_3 and σ_4 as polynomials in σ_1, σ_2. Plugging these expressions in the first two relations, and letting $q = 1$, we get

$$\sigma_2(3\sigma_2 - 2\sigma_1^2) = 2\sigma_2^2\sigma_1^2 - 2\sigma_2\sigma_1^4 + \sigma_1^6 - 2\sigma_2^3 + \sigma_1 = 0.$$

These two equations define the finite scheme $Z(G_\omega(2,6))$. This scheme has length 12 and is the union of a subscheme Z_1, the union of the 5 simple points given by $\sigma_2 = 0, \sigma_1^5 = -1$, a subscheme Z_2, the union of the other 5 simple points given by $3\sigma_2 = 2\sigma_1^2, \sigma_1^5 = 27$, and a length two scheme Z_0 supported at the origin, with Zariski tangent space $\sigma_1 = 0$. The quantum cohomology ring of $G_\omega(2,6)$ at $q = 1$ identifies with the algebra of functions on that scheme Z. It is convenient to identify each Schubert class with such a function given by a triple of functions on the subschemes

Z_0, Z_1, Z_2. For this we just need the quantum Chevalley formula, and the result is as follows.

	$\sigma_{(1\|0)}$	$\sigma_{(1\|1)}$	$\sigma_{(0\|2)}$	$\sigma_{(2\|1)}$	$\sigma_{(1\|2)}$	$\sigma_{(2\|2)}$	$\sigma_{(1\|3)}$	$\sigma_{(2\|3)}$	$\sigma_{(2\|21)}$	$\sigma_{(2\|31)}$	$\sigma_{(2\|32)}$
Z_0	0	ϵ	$-\epsilon$	0	0	0	0	-1	1	0	$-\epsilon$
Z_1	s	0	s^2	s^3	$-s^3$	s^4	$-s^4$	0	-1	$-s$	$-s^2$
Z_2	s	$\frac{2}{3}s^2$	$\frac{1}{3}s^2$	$\frac{1}{3}s^3$	$\frac{1}{3}s^3$	$\frac{1}{9}s^4$	$\frac{1}{9}s^4$	2	1	s	$\frac{1}{3}s^2$

Note in particular that $\sigma_{(2|32)} - \sigma_{(1|1)} + \sigma_{(0|2)}$ is nilpotent: it generates the radical of the quantum cohomology ring at $q = 1$. In fact, even before specializing q, we have

$$(\sigma_{(2|32)} - q\sigma_{(1|1)} + q\sigma_{(0|2)})^2 = 0 \qquad \text{in} \quad QA^*(G_\omega(2,6)).$$

It would now be completely straightforward to write down the multiplication table of Schubert classes in the quantum cohomology ring. But the point we want to stress is the following: if we consider a degree two class, $\sigma_{(1|1)}$ or $\sigma_{(0|2)}$, its complex conjugate is nonzero on Z_0, and therefore it cannot be expressed as a linear combination of the degree three classes $\sigma_{(2|1)}$ and $\sigma_{(1|2)}$, which vanish on Z_0. It is therefore impossible to lift the complex conjugation to the localized quantum cohomology ring of $G_\omega(2,6)$, in such a way that a degree k class is mapped to a class of degree $-k$. The existence of a nonreduced point, although as simple as possible, definitely prevents us from doing that.

The fact that the first non(co)minuscule example leads to a nonsemisimple quantum cohomology algebra suggests the following

Conjecture *Consider a rational homogeneous variety G/P with Picard number one. Then $QA^*(G/P)_{q=1}$, its quantum cohomology algebra specialized at $q = 1$, is semisimple, if and only if G/P is minuscule or cominuscule.*

8 Vafa-Intrilagator Type Formulas

8.1 The Quantum Euler Class

Abrams introduced in [A], for any projective variety X, with Picard group of arbitrary rank, and in fact, in an even more general setting, a *quantum Euler class* $e(X)$. Let us denote by pt the element of W^P that defines the punctual class $\sigma(pt)$. Let $\varphi\colon QH^*(X,\mathbb{C}) \simeq H^*(X,\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[q] \to \mathbb{C}[q]$ be defined by $\varphi(\sum P_\lambda(q)\sigma(\lambda)) = P_{pt}$. Then the bilinear map $(x,y) \mapsto \varphi(x * y)$ defines a Frobenius algebra structure on $QH^*(X,\mathbb{C})$ over $\mathbb{C}[q]$. Let (e_i) be a base of $QH^*(X,\mathbb{C})$ over $\mathbb{C}[q]$ and (e_i^*) the dual base. Our bilinear form identifies $QH^*(X,\mathbb{C})$ and its dual; let $e_i^\sharp \in QH^*(X,\mathbb{C})$ correspond to e_i^*. The Euler class is defined by $e(X) := \sum e_i.e_i^\sharp$. By [A, Theorem 3.4], in a given specialization of the quantum parameters, the invertibility of $e(X)$ is equivalent to the semisimplicity of that algebra. We will identify that class explicitly when $X = G/P$ is any homogeneous variety of Picard number one.

Recall that the invariant algebras

$$\mathbb{Q}[\mathfrak{t}]^{W_P} = \mathbb{Q}[X_1,\ldots,X_r] \quad \text{and} \quad \mathbb{Q}[\mathfrak{t}]^W = \mathbb{Q}[Y_1,\ldots,Y_r]$$

are polynomial algebras over certain homogeneous invariants by the famous Chevalley theorem. The classical and quantum cohomology rings of G/P are

$$H^*(G/P)_{\mathbb{Q}} = \mathbb{Q}[\mathfrak{t}]^{W_P}/(Y_1, \dots, Y_r) \quad \text{and}$$
$$QA^*(G/P)_{\mathbb{Q}} = \mathbb{Q}[\mathfrak{t}]^{W_P}[q]/(Y_1(q), \dots, Y_r(q)),$$

where $Y_1(q), \dots, Y_r(q)$ are certain homogeneous deformations of the classical relations $Y_1, \dots, Y_r$.

Denote by $\Phi(G/P) = \Phi^+ - \Phi_P^+$ the set of positive roots of $\mathfrak{g}$ that are not roots of $\mathfrak{p} = \mathrm{Lie}(P)$. This is also the set of weights in $\mathfrak{g}/\mathfrak{p}$, the P-module whose associated vector bundle on G/P is the tangent bundle. Each root in $\Phi(G/P)$ defines a linear functional on $\mathfrak{t}$, and since $\Phi(G/P)$ is preserved by W_P, the product of the roots contained in it defines a W_P-invariant polynomial function on $\mathfrak{t}$, hence a class in the cohomology ring $QA^*(G/P)_{q=1}$.

Proposition 8.1 *Let G/P be any rational homogeneous space of Picard number one. The quantum Euler class of G/P is*

$$e(G/P) = \prod_{\alpha \in \Phi(G/P)} \alpha.$$

Proof Choose a basis $t_1, \dots, t_r$ of $\mathfrak{t}^*$. There is a constant $c \neq 0$ such that

$$\det\Big(\frac{\partial Y_p}{\partial t_q}\Big)_{1 \le p,q \le r} = c \prod_{\alpha \in \Phi^+} \alpha.$$

Indeed both terms are proportional to the minimal degree anti-invariant of the Weyl group (see [Bou, Chap. V, § 5.4, Proposition 5]). Similarly, for W_P, we get

$$\det\Big(\frac{\partial X_p}{\partial t_q}\Big)_{1 \le p,q \le r} = c_P \prod_{\alpha \in \Phi_P^+} \alpha,$$

and both terms are proportional to the minimal degree anti-invariant of W_P. We deduce that

$$J := \det\Big(\frac{\partial Y_p}{\partial X_q}\Big)_{1 \le p,q \le r} = d_P \prod_{\alpha \in \Phi(G/P)} \alpha$$

for some nonzero constant d_P.

It was proved in [A, Proposition 6.3] that in our situation, J and the quantum Euler class $e(G/P)$ coincide up to some invertible class in $QH^*(G/P)$. Moreover, since J is homogeneous of degree $\#\Phi(G/P) = \dim(G/P)$, this invertible class is in fact a constant (see the proof of [A, Proposition 6.4]). So $e(G/P)$ must be a constant multiple of $\prod_{\alpha \in \Phi(G/P)} \alpha$.

We claim that this constant is one. Indeed, the quantum Euler class is a deformation of the classical Euler class (see the claim after [A, Proposition 5.1]). But the latter coincides with $c_{top}(T_{G/P})$, the top Chern class of the tangent space, which in the classical setting is precisely given by the product of the roots in $\Phi(G/P)$. This implies our claim, and the proof is complete. ■

8.2 The Gromov–Witten Invariants

From now on G/P is again a (co)minuscule, rational, homogeneous space. Its Picard number is one so Proposition 8.1 applies. Moreover the quantum Euler class is invertible.

For a class σ in $QH^*(G/P)_{q=1}$, denote by trace(σ) the trace of the multiplication operator by σ. In terms of our finite set $Z(G/P)$, we have

$$\mathrm{trace}(\sigma) = \sum_{z \in Z(G/P)} \sigma(z).$$

Recall that we denote by pt the element of W^P that defines the punctual class $\sigma(pt)$. Recall also that the Schubert class that is Poincaré dual to $\sigma(\mu)$ is denoted by $\sigma(p(\mu))$.

Lemma 8.2 *For any Schubert classes $\sigma(\lambda)$ and $\sigma(\mu)$ on G/P, we have*

$$\mathrm{trace}\Big(\frac{\sigma(\lambda)}{e(G/P)}\Big) = \delta_{\lambda,pt}, \quad \mathrm{trace}\Big(\frac{\sigma(\lambda)\sigma(\mu)}{e(G/P)}\Big) = \delta_{\lambda,p(\mu)}.$$

Proof The first equality is a general fact in Frobenius algebras. Recall that for $x \in QH^*(X,\mathbb{C})$ we denoted by $\varphi(x)$ the coefficient of x in σ_{pt}. We want to show that

$$\mathrm{trace}(x) = \varphi(e(G/P)x) \text{ for } x \in QH^*(G/P,\mathbb{C}).$$

But we have

$$\varphi(e(G/P)x) = \sum_i \varphi(e_i^\sharp e_i x) = \sum e_i^*(e_i x) = \mathrm{trace}(x)$$

(the second equality follows from the definition of $e_i^\sharp$).

We deduce that trace($\sigma(\lambda)\sigma(\mu)/e(G/P)$) is the coefficient of $\sigma(pt)$ in the quantum product $\sigma(\lambda) * \sigma(\mu)$, that is, the Gromov–Witten invariant $I(\sigma(\lambda),\sigma(\mu),1)$. By the associativity of the quantum product, this is also the coefficient of the Poincaré dual class $\sigma(p(\mu))$ inside $\sigma(\lambda) * 1 = \sigma(\lambda)$. ■

Our Vafa–Intriligator type formula follows immediately (up to the identification of $J = e(G/P)$, this formula appears in [ST, Section 4]).

Proposition 8.3 *Let G/P be a (co)minuscule, rational, homogeneous space. The three-point, genus zero, Gromov–Witten invariants of G/P can be computed as*

$$I(\sigma(\lambda),\sigma(\mu),\sigma(\nu)) = \mathrm{trace}\Big(\frac{\sigma(\lambda)\sigma(\mu)\sigma(\nu)}{e(G/P)}\Big) = \sum_{z\in Z(G/P)} \frac{\sigma(\lambda)(z)\sigma(\mu)(z)\sigma(\nu)(z)}{\prod_{\alpha\in\Phi(G/P)}\alpha(z)}.$$

Note that we have not indicated the degree d of this invariant. It is determined by the Schubert classes involved through the relation $\deg\sigma(\lambda) + \deg\sigma(\mu) + \deg\sigma(\nu) = \dim(G/P) + \mathrm{ind}(G/P)d$.

In fact, following [ST] we have a general formula for the genus g Gromov–Witten invariants:

$$I^g(\sigma(\lambda), \sigma(\mu), \sigma(\nu)) = \mathrm{trace}\Big(e(G/P)^{g-1}\sigma(\lambda)\sigma(\mu)\sigma(\nu)\Big).$$

It would be interesting to understand better the quantum Euler class of a (co)minuscule G/P. We conjecture that, as functions on $Z(G/P)$, we should have

$$e(G/P) = |e(G/P)|\sigma(pt).$$

Recall that the punctual class $\sigma(pt)$ is always an idempotent, so it takes its values among the roots of unity. So the punctual class would give the argument of the quantum Euler class. We have checked this on Grassmannians and quadrics, but we have no explanation of this intriguing relation.

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