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Extremal Theory for Spectrum of Random Discrete Schr\"odinger Operator. III. Localization properties

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Abstract:

Consider the Anderson Hamiltonian $\cH\iz{V}= \rrk \Delta \iz{V}+\xi(\cdot)$ on the multidimensional lattice torus V increasing to the whole of lattice, where $\xi(\cdot)$ is an i.i.d. potential with distribution function F. For $K=1,2,\ldots$, let $\psi (\cdot; \rrl\iz{K, V})$ be the eigenfunction of $\cH\iz{V}$ associated with the Kth largest eigenvalue $\rrl\iz{K,V}$, and let $z\iz{K,V} \in V$ be the coordinate of the Kth larger value $\xi\iz{K,V}$ of $\xi(\cdot)$ in V. It is well-known that if F satisfies the condition $\log\big(-\log(1-F(t))\big)=\mro(t)$ and some additional conditions on regular variation and continuity at infinity, then $\psi (\cdot; \rrl\iz{K, V})$ is (asymptotically) completely localized at the site $z\iz{\tau(K),V}$, as a localization centre for the eigenfunction for some (random) $\tau(K)=\tau\iz{V}(K) \dly 1$. In this paper, we study the asymptotic behavior in probability of the indices $\tau\iz{V}(K)$ as V increases and $K\dly 1$ is fixed. In particular, we show that if F satisfies the condition $-\log(1-F(t))=\rO(t^{3})$ (resp., $-t^{-3}\log(1-F(t))\to \infty$) and additional regularity conditions at infinity, then $\tau\iz{V}(K)=\rO(1)$ (resp., $\tau\iz{V}(K) \to \infty$) with high probability. For Weibull's and double exponential types distributions, we obtain the first order expansion formulas for $\log \tau\iz{V}(K)$.

Extremal Theory for Spectrum of Random Discrete Schrödinger Operator. III. Localization properties

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Consider the Anderson Hamiltonian $\mathcal{H}_V = \kappa\Delta_V + \xi(\cdot)$ on the multidimensional lattice torus V increasing to the whole of lattice, where $\xi(\cdot)$ is an i.i.d. potential with distribution function F. For $K = 1, 2, \ldots$, let $\psi(\cdot\,; \lambda_{K,V})$ be the eigenfunction of $\mathcal{H}_V$ associated with the Kth largest eigenvalue $\lambda_{K,V}$, and let $z_{K,V} \in V$ be the coordinate of the Kth larger value $\xi_{K,V}$ of $\xi(\cdot)$ in V. It is well-known that if F satisfies the condition $\log\big(-\log(1-F(t))\big) = \mathrm{o}(t)$ and some additional conditions on regular variation and continuity at infinity, then $\psi(\cdot\,; \lambda_{K,V})$ is (asymptotically) completely localized at the site $z_{\tau(K),V}$, as a localization centre for the eigenfunction for some (random) $\tau(K) = \tau_V(K) \geqslant 1$. In this paper, we study the asymptotic behavior in probability of the indices $\tau_V(K)$ as V increases and $K \geqslant 1$ is fixed. In particular, we show that if F satisfies the condition $-\log(1-F(t)) = \mathrm{O}(t^3)$ (resp., $-t^{-3}\log(1-F(t)) \to \infty$) and additional regularity conditions at infinity, then $\tau_V(K) = \mathrm{O}(1)$ (resp., $\tau_V(K) \to \infty$) with high probability. For Weibull's and double exponential types distributions, we obtain the first order expansion formulas for $\log \tau_V(K)$.

KEY WORDS: Anderson Hamiltonian; random potential; localization; largest eigenvalues and eigenfunctions; localization centres; convergence in probability.

1. INTRODUCTION

This paper is a continuation of our previous works [4, 5] on extreme value theory for spectrum of a finite-volume Anderson Hamiltonian

$$\mathcal{H}_V = \kappa\Delta_V + \xi_V \quad \text{on} \;\; l^2(V);$$

here V is the ν-dimensional torus obtained by identifying opposite faces of the cube $[-n; n]^\nu$ in the ν-dimensional integer lattice $\mathbb{Z}^\nu$; Δ_V is the lattice Laplacian on $l^2(V)$ with periodic boundary data (i.e., a restriction of the operator $\Delta\psi(x) := \sum_{|y-x|=1} \psi(y)$ to torus V where $|\cdot|$ is the periodic norm $|x| := \min_{y\in 2n\mathbb{Z}^\nu} \big(|x^1 - y^1| + \cdots + |x^\nu - y^\nu|\big)$ for $x = (x^1, \ldots, x^\nu) \in$

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$\mathbb{Z}^\nu$); $\kappa > 0$ stands for a diffusion constant; $\xi(x)$, $x \in \mathbb{Z}^\nu$, are independent identically distributed random variables (i.i.d. random potential) with a common distribution function F. Throughout we assume that $F(t) < 1$ for all real $t \in \mathbb{R}$, i.e., $\xi(\cdot)$ is unbounded from above potential. Note that the spectrum of $\mathcal{H}_V$ is a finite set, say, $\operatorname{Spect}(\mathcal{H}_V) = \{\lambda_{k,V} \colon\ 1 \leqslant k \leqslant |V|\}$, where

$$\lambda_{1,V} \geqslant \lambda_{2,V} \geqslant \ldots \geqslant \lambda_{|V|,V};$$

here $|V|$ stands for the volume of V. For $\kappa = 0$, this variational series becomes

$$\xi_{1,V} := \xi(z_{1,V}) \geqslant \xi_{2,V} := \xi(z_{2,V}) \geqslant \ldots \geqslant \xi_{|V|,V} := \xi(z_{|V|,V}), \tag{1.1}$$

i.e., $z_{K,V} \in V$ are the coordinates of the Kth larger values of the sample $\xi_V = \{\xi(x) \colon\ x \in V\}$.

Let $\psi(\cdot\,;\lambda) = \{\psi(x;\lambda) \colon x \in V\}$ be an eigenfunction of $\mathcal{H}_V$ associated with $\lambda \in \operatorname{Spect}(\mathcal{H}_V)$ and normalized by the condition $\sum_{x\in V} \psi(x;\lambda)^2 = 1$. Given $K = 1, 2, \ldots$, let $z_{\tau(K),V} \in V$ denote the localization centre of the Kth eigenfunction $\psi(\cdot\,;\lambda_{K,V})$ defined by

$$\psi(z_{\tau(K),V};\lambda_{K,V}) := \max_{1\leqslant l\leqslant |V|} \psi(z_{l,V};\lambda_{K,V}) \quad \text{for some} \ \ \tau(K) = \tau_V(K). \tag{1.2}$$

In the present paper, we study the asymptotic properties of the localization centres $z_{\tau(K),V}$ with high probability, as $V \uparrow \mathbb{Z}^\nu$ and $K = 1, 2, \ldots$ is fixed; see Theorems 2.1, 2.3 and 2.4 below. Let us define the (generalized) inverse function of $-\log(1-F)$ by

$$f(s) := \inf\left\{t \colon\ 1 - F(t) \leqslant \mathrm{e}^{-s}\right\} \quad (0 < s < \infty);$$

cf. [4, 5]. (Notice that f is left-continuous nondecreasing function and $f(s) \to \infty$ as $s \to \infty$). Throughout the paper, we assume that the distribution function F satisfies the following conditions:

$$\begin{gathered} \lim_{s\to\infty} f(s) - f(s\delta) = -\rho \log \delta \quad \text{for any} \quad 0 < \delta < 1 \\ \text{for some} \ \ 0 < \rho \ \leqslant \infty \ \ \text{such that} \ \ \rho/\kappa \ \ \text{is large enough,} \end{gathered} \tag{1.3}$$

and

$$\begin{gathered} (F(t+s) - F(t-s))\,|\log s|^\mu = \mathrm{O}(1) \\ \text{as} \ \ t \to \infty \ \text{and} \ \ s \downarrow 0 \ \ \text{simultaneously} \end{gathered} \tag{1.4}$$

for some $\mu > \nu$, i.e., F is log-Hölder continuous of order μ at infinity. For finite ρ, condition (1.3) is fulfilled if and only if the function $g := -\log\circ(1-F)\circ\log := -\log\bigl(1 - F(\log(\cdot))\bigr)$ is regularly varying at infinity with index $1/\rho$, i.e., $g(ct)/g(t) \to c^{1/\rho}$ as $t \to \infty$, for any $c > 0$. In the case $\rho = \infty$, (1.3) is equivalent to the slow variation of $g := -\log\circ(1 - F)\circ\log$, therefore, $\log(-\log(1-F(t))) = \mathrm{o}(t)$ as $t \to \infty$. Condition (1.4) is fulfilled, if for instance, the distribution function F has a density $p(t) := \frac{\mathrm{d}F(t)}{\mathrm{d}t}$ $(t \geqslant t_0)$. In this case, the main examples of (1.3) with $\rho = \infty$ are the following distributions:

$$p(t) = \exp\{-At^\alpha + \mathrm{O}(\log t)\} \quad \text{as} \quad t \to \infty, \tag{1.5}$$

for arbitrary $\alpha > 0$ and $A > 0$ (i.e., Weibull's type tails) as well as

$$p(t) = \exp\Big\{-\mathrm{e}^{Bt^\gamma} + \mathrm{O}(t^{\mathrm{const}})\Big\} \quad \text{as} \quad t \to \infty, \tag{1.6}$$

for $0 < \gamma < 1$ and $B > 0$ (i.e., the fractional-double exponential tails). Distributions (1.3) with $0 < \rho < \infty$ are represented by (1.6) with $\gamma = 1$ and $B = 1/\rho$ (i.e., the double-exponential tails).

In [4], we show that, under conditions (1.3) and (1.4), the eigenfunctions $\psi(\cdot\,; \lambda_{K,\,V})$ $(K \geqslant 1$ fixed) are exponentially well localized, i.e., with probability 1 there exist constants $0 < M_V \to \infty$ and (random) indices $1 \leqslant \tau(K) = |V|^{\mathrm{o}(1)}$ such that

$$|\psi(x; \lambda_{K,\,V})| \leqslant \exp\{-M_V|x - z_{\tau(K),\,V}|\} \quad (x \in V) \tag{1.7}$$

for all (large) $V \supset V_0$. Moreover, the Kth largest eigenvalue $\lambda_{K,\,V}$ is approximated by the principal (i.e., the first largest) eigenvalue of the "single-peak" Hamiltonian $\kappa\Delta_V + \sum_{|y - z_{\tau(K),\,V}| \leqslant J} \xi(y)\delta_y$ for some $J \geqslant 0$; see also Theorems 3.1 and 3.4 below. This refers to the correspondence $\lambda_{K,\,V} \leftrightarrow z_{\tau(K),\,V}$, so that the eigenvalue $\lambda_{K,\,V}$ is associated with an isolated high ξ_V-peak. The asymptotic behavior of indices $\tau_V(K)$ depends strongly on the asymptotic geometric structure of ξ_V-peaks, which in turn is determined by regularity and tail decay conditions on potential distribution [3–5].

In [5], we show that if the distribution function F satisfies condition

$$-\log(1 - F(t)) = \mathrm{o}(t^3) \tag{1.8}$$

(heavy tails) and some additional conditions on regular variation at infinity, then with probability $1 + \mathrm{o}(1)$

$$\tau_V(K) \to K \quad \text{as} \quad |V| \to \infty,$$

i.e., $\lambda_{K,\,V}$ is associated with $z_{K,\,V} \in V$, the coordinate of the Kth largest ξ_V-value. Notice that, under these conditions on F (1.8), the potential ξ_V possesses extremely sharp peaks; see Section 3 in [5].

In the present paper, we extend the results of [4, 5]. We prove that for the lighter tails than those in (1.8), limit for $\tau_V(K)$ differs from K. In particular, if F satisfies condition $-t^{-3}\log(1-F(t)) \to \infty$ and additional regularity and continuity conditions at infinity, then $\tau_V(K) \to \infty$ with probability $1 + \mathrm{o}(1)$ (cf. Theorems 2.1(ii)–(iii), 2.3 and 2.4 below). Notice that, in contrary to the case (1.8), the landscape of ξ_V now becomes "smoother " as $|V| \to \infty$, therefore, the eigenvalue $\lambda_{K,V}$ is associated with a lower and "slightly supported " ξ_V-peak [3, 4].

For the infinite Anderson model $\mathcal{H} = \kappa\Delta + \xi(\cdot)$ on $l^2(\mathbb{Z}^\nu)$, it is well known (see, e.g., recent surveys [17], [21]) that almost sure $\mathcal{H}$ has pure point spectrum at the edge of Spect$\,(\mathcal{H})$ and the corresponding eigenfunctions decay exponentially, provided F is Hölder continuous and $\xi(0)$ has some finite statistical moments. Notice also that the regions $I \subset \mathrm{Spect}_{\,pp}(\mathcal{H})$ of pure point spectrum are distinguished by Poissonian asymptotic behavior of the eigenvalues $\lambda_{l,\,V} \in I$ and their localization centres of the finite-volume model $\mathcal{H}_V$ as $|V| \to \infty$ [19, 16, 14].

For the relationship between extreme value theory for the spectrum Spect$\,(\mathcal{H}_V)$ and the long-time intermittency for the Anderson parabolic problems $\partial u/\partial s = \mathcal{H}u$, we refer to the recent surveys [12, 15, 18], where one can find a comprehensive list of references on the subject.

The organization of the paper is as follows:

In Section 2, the main results of the paper are formulated. First, we provide asymptotics for $\tau_V(K)$ under general conditions like (1.3) and (1.4) (see Theorem 2.1 in Sect. 2.1). Further on, we give the first order asymptotic expansion formulas for $\log \tau_V(K)$ in the case of Weibull's type distributions (1.5) and fractional-double exponential distributions (1.6) (see, respectively, Theorems 2.3 and 2.4 in Sect. 2.2).

Sections 3 and 4 provide the proof of the main results. In Section 3, we announce the almost sure asymptotic expansion formulas for the eigenvalues $\lambda_{K,\,V}$ and the corresponding eigenfunctions. In Sections 4.2, 4.3 and 4.4, we complete the proof of Theorems 2.1, 2.3 and 2.4, respectively, by combining the results of Section 3 and the asymptotic properties of the extreme values $\xi_{K,\,V}$ given in Section 4.1.

2. MAIN RESULTS

2.1. Limits for Localization Centres

Throughout we use the following notation and definitions. For real functions $g(\cdot)$ and $h(\cdot) > 0$, we will write $g(t) = \mathrm{O}(h(t))$ (resp., $g(t) = \mathrm{o}(h(t))$) as $t \to \infty$, if $\limsup_t |g(t)|/h(t) < \infty$ (resp., $\lim_t g(t)/h(t) = 0$). For $g(\cdot) > 0$ and $h(\cdot) > 0$, limit $g(t) \asymp h(t)$ means that $g(t) = \mathrm{O}(h(t))$ and $h(t) = \mathrm{O}(g(t))$ as $t \to \infty$. By t_0, $|V_0|$, etc. we denote various large numbers, values of which may change from one appearance to the next. Similarly, const, const′, etc. stand for various positive constants.

We suppose that all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $z_{K,V} \in V$ be the coordinates of the Kth largest ξ_V-values (1.1), and let $z_{\tau(K),V}$ be the localization centre of the Kth eigenfunction $\psi(\cdot;\lambda_{K,V})$ defined by (1.2) or (1.7).

Theorem 2.1. Assume that the distribution function F is log-Hölder continuous of order $\mu > \nu$ at infinity, i.e., (1.4) holds true. For fixed $K = 1, 2, \ldots$, we have the following limits in probability for the indices $\tau(K) = \tau_V(K)$.

(i) If

$$f(s)^2(f(s+c) - f(s)) \asymp 1 \quad \text{as} \quad s \to \infty, \quad \text{for any} \quad c > 0, \tag{2.1}$$

then

$$\limsup_V \tau_V(K) < \infty.$$

(ii) If

$$\lim_{s\to\infty} f(s)^2(f(s+c) - f(s)) = 0 \quad \text{for any} \quad c > 0 \tag{2.2}$$

and, additionally, f satisfies (1.3) with $\rho = \infty$, then

$$\lim_V \tau_V(K) = \infty \quad \text{and} \quad \lim_V \frac{\log \tau_V(K)}{\log |V|} = 0. \tag{2.3}$$

(iii) If f satisfies (1.3) with (finite) sufficiently large ρ/κ, then there exist nonrandom constants $\varepsilon_2(\rho/\kappa) \geqslant \varepsilon_1(\rho/\kappa) > 0$ such that

$$\varepsilon_1(\rho/\kappa) \leqslant \liminf_V \frac{\log \tau_V(K)}{\log |V|} \leqslant \limsup_V \frac{\log \tau_V(K)}{\log |V|} \leqslant \varepsilon_2(\rho/\kappa),$$

where $\varepsilon_i(\rho) = (\rho \log \rho)^{-2}(\nu/2 + \mathrm{o}(1))$ as $\rho \to \infty$, for $i = 1, 2$ (cf. also Theorem 3.4 below).

The *proof* of parts (i) and (ii) of Theorem 2.1 is given in Section 4.2. Part (iii) can be shown by using the same arguments as in the proof of Theorem 4.4 and Corollary 4.5 in [4] combined with Theorem 2.16 by Gärtner and Molchanov [13]; therefore, the proof of (iii) is omitted.

We now characterize the classes of distributions (2.1) and (2.2) in terms of F.

Remark 2.2. (Regularity and decay conditions for the tails $1 - F$ at infinity; see [3] for a detailed discussion and proofs).

(i) Condition (2.1) implies that

$$\frac{\log\big(1-F(t+ct^{-2})\big)}{\log\big(1-F(t)\big)} \asymp 1 \quad \text{as} \quad t\to\infty, \quad \text{for any} \quad c>0, \tag{2.4}$$

which in turn yields that $-\log(1-F(t)) \asymp t^3$ as $t\to\infty$. Moreover, there is an example of F for which (2.4) holds true, however, (2.1) fails.

(ii) If condition (2.1) is fulfilled and if a function $a(s)>0$ $(s>0)$ is chosen to satisfy $\liminf_{s\to\infty} a(s) \geqslant c_1 > 0$ and $\liminf_{s\to\infty}(s-a(s)) \geqslant c_2 > 0$, then

$$\begin{aligned}&\mathrm{const}\,(s-a(s))s^{-2/3}\\ &\leqslant f(s)-f(a(s)) \leqslant \mathrm{const}'\left(s^{1/3}-a(s)^{1/3}+a(s)^{-2/3}\right)\end{aligned}$$

for any $s\geqslant s_0$ and for some $\mathrm{const}' \geqslant \mathrm{const} > 0$.

(iii) f satisfies (2.2) if and only if

$$\lim_{t\to\infty}\frac{\log\big(1-F(t+ct^{-2})\big)}{\log\big(1-F(t)\big)} = \infty \quad \text{for any} \quad c>0. \tag{2.5}$$

In this case, $-t^{-3}\log(1-F(t))\to\infty \quad \text{as} \quad t\to\infty$.

(iv) For $0<\rho\leqslant\infty$, condition (1.3) is fulfilled if and only if

$$\lim_{t\to\infty}\frac{\log(1-F(t+c))}{\log(1-F(t))} = \mathrm{e}^{c/\rho} \quad \text{for any} \;\; c>0, \tag{2.6}$$

or equivalently, the function $-\log\circ(1-F)\circ\log$ is regularly varying at infinity with index $1/\rho$ (see, e.g., Theorems 1.5.12, 2.4.7 and Proposition 2.4.4(iv) in [8]). In the case of $\rho=\infty$, either of conditions (1.3) and (2.5) implies that $\log(-\log(1-F(t))) = \mathrm{o}(t)$ as $t\to\infty$ (see, e.g., Proposition 1.3.6(i) in [8]).

2.2. Examples. The First Order Asymptotic Expansion Formulas

We now give the first order asymptotic expansion formulas for $\log\tau_V(K)$, provided the distribution function F has a density $p(t) := \frac{\mathrm{d}F(t)}{\mathrm{d}t}$ $(t\geqslant t_0)$ satisfying the conditions of Theorem 2.1(ii). We restrict ourselves to the cases of Weibull's type density (1.5) with $\alpha>3$ and fractional-double exponential density (1.6) with $0<\gamma<1$.

Consider first the case (1.5). Clearly (1.5) implies that $1-F(t) = \exp\{-At^{\alpha}+\mathrm{O}(\log t)\}$ as $t\to\infty$, which in turn yields that

$$f(s) = (A^{-1}s)^{1/\alpha} + \mathrm{O}\left(s^{(1-\alpha)/\alpha}\log s\right) \quad \text{as} \quad s\to\infty. \tag{2.7}$$

Thus for $\alpha > 3$, F satisfies (2.5) and (2.6) with $\rho = \infty$. Abbreviate

$$l_V = \left(A^{-1} \log |V|\right)^{1/\alpha}. \tag{2.8}$$

Theorem 2.3. Let the distribution density p satisfy condition (1.5) with $\alpha > 3$. Then

$$\lim_V \frac{\log \tau_V(K)}{l_V^{\alpha(\alpha-3)/(\alpha-1)}} = 2\nu A \kappa^{2\alpha/(\alpha-1)}.$$

in probability.

Theorem 2.3 is *proved* in Sect. 4.3 by combining Theorem 3.2 below and asymptotic properties of ξ_V-extremes studied in Sect. 4.1. Theorem 2.3 is announced in [6].

Consider now the case of p (1.6). Condition (1.6) implies that $1-F(t) = \exp\left\{-\mathrm{e}^{Bt^\gamma} + \mathrm{O}(t^{\mathrm{const}_1})\right\}$ as $t \to \infty$, which in turn yields that

$$f(s) = (B^{-1} \log s)^{1/\gamma} + \mathrm{O}\left(s^{-1}(\log s)^{\mathrm{const}_2}\right) \quad \text{as} \quad s \to \infty. \tag{2.9}$$

Thus for $0 < \gamma < 1$, F satisfies (2.5) and (2.6) with $\rho = \infty$. Abbreviate

$$d_V = \left(B^{-1} \log\log |V|\right)^{1/\gamma}. \tag{2.10}$$

Theorem 2.4. Let p satisfy condition (1.6) with $0 < \gamma < 1$. Then

$$\lim_V \frac{\log \tau_V(K)}{d_V^{2\gamma-2}(\log d_V)^{-2} \log |V|} = \frac{\nu}{2}\Big(\frac{B\kappa\gamma}{1-\gamma}\Big)^2. \tag{2.11}$$

in probability.

A *sketch of the proof* is given in Sect. 4.4. The arguments repeat those of Theorem 2.3, where one exploits Theorem 3.3 instead of Theorem 3.2.

3. ASYMPTOTIC EXPANSION FORMULAS FOR THE LARGEST EIGENVALUES

In this section we announce some results of [4] on the almost sure asymptotic structure of the first K largest eigenvalues $\lambda_{k,V}$ and the corresponding (normalized) eigenfunctions $\psi(\cdot; \lambda_{k,V})$ of the Anderson Hamiltonian $\mathcal{H}_V = \kappa\Delta_V + \xi(\cdot)$ when $V \uparrow \mathbb{Z}^\nu$ and K is fixed. We again assume that the distribution function F of $\xi(0)$ satisfies conditions (1.3) and (1.4).

Let us introduce additional notation we will use throughout Sections 3 and 4. For $0 < \theta < 1/2$, write $L_{V,\theta} := f((1-\theta)\log |V|)$. Let $\widetilde{\xi}(x)\colon = \xi(x)$

if $\xi(x) < L_{V,\theta}$, and $\widetilde{\xi}(x) := 0$ otherwise, i.e., $\widetilde{\xi}(\cdot)$ is the "noise" potential. Abbreviate also $J := J_V := [|V|^{(1+\theta)/\mu}]$. Given $z \in V$, consider now the principal eigenvalue $\lambda^{(J)}(z)$ of the "single peak" Hamiltonian

$$\kappa\Delta_V + \sum_{y\colon 1\leqslant|y-z|\leqslant J} \widetilde{\xi}(y)\delta_y + \xi(z)\delta_z \quad \text{on} \quad l^2(V),$$

where δ_z stands for the Kronecker symbol. Let

$$\lambda^{(J)}_{1,V} := \lambda^{(J)}(z_{\tau(1),V}) \geqslant \lambda^{(J)}_{2,V} := \lambda^{(J)}(z_{\tau(2),V}) \geqslant \ldots \geqslant \lambda^{(J)}_{|V|,V} := \lambda^{(J)}(z_{\tau(|V|),V})$$

be the variational series of the sample $\{\lambda^{(J)}(x)\colon x \in V\}$. (Recall that the sites $z_{l,V} \in V$ $(1 \leqslant l \leqslant |V|)$ are associated with the variational series (1.1) based on ξ_V).

Theorem 3.1. (see Theorem 4.1 in [4]). Assume that F satisfies conditions (1.4) and (1.3) with $\rho = \infty$, and pick (small) $\theta > 0$ such that $\mu > (1+\theta)\nu/(1-2\theta)$. Fix $K = 1, 2, \ldots$. Then the following almost sure limits hold true:

$$\limsup_V \frac{\log\left|\lambda_{K,V} - \lambda^{(J)}_{K,V}\right|}{J_V M_V(K)} \leqslant -2,$$

$$\liminf_V \frac{\log(\lambda_{K,V} - \lambda_{K+1,V})}{J_V} \geqslant -1$$

and

$$\limsup_V \max_{x \neq z_{\tau(K),V}} \frac{\log\left|\psi(x;\lambda_{K,V})\right|}{M_V(K)|x - z_{\tau(K),V}|} \leqslant -1,$$

where $M_V(K) := \log\left(\lambda^{(J)}_{K,V} - L_{V,\theta}\right) \geqslant \log(L_{V,\varepsilon} - L_{V,\theta}) \to \infty$ as $|V| \to \infty$, for each $\varepsilon \in (0,\theta)$.

For $z := z_{\tau(K),V}$, the variable $\lambda^{(J)}(z) := \lambda^{(J)}_{K,V}$ is expanded in the series over $\widetilde{\xi}(x)/\xi(z)$ $(x \in V)$ [4]. In particular, with probability 1

$$\begin{aligned}\lambda^{(J)}(z) = \xi(z) + \kappa^2 \sum_{|x-z|=1} \frac{1}{\xi(z) - \widetilde{\xi}(x)} &+ \\ +\mathrm{O}\Bigg(\sum_{\substack{|x-z|=1\\|y-z|=1\\|u-z|=2}} \frac{1}{(\xi(z)-\widetilde{\xi}(x))(\xi(z)-\widetilde{\xi}(y))}\Bigg(\frac{1}{\xi(z)-\widetilde{\xi}(u)} + \frac{1}{\xi(z)}\Bigg)\Bigg)& \qquad (3.1)\end{aligned}$$

as $|V| \to \infty$. One can apply (3.1) to derive the asymptotic expansion formulas for $\lambda^{(J)}_{K,V}$ in the cases of Weibull's type distributions (1.5) and fractional-double exponential distributions (1.6) satisfying the conditions of Theorem 3.1.

Theorem 3.2. (see Theorem 6.3 and Corollary 6.4 in [4], or [6]). Assume that the distribution density p of $\xi(0)$ satisfies condition (1.5) with $\alpha > 3$. Fix $K = 1, 2, \ldots$, and let l_V be defined by (2.8). Then with probability $1 + \mathrm{o}(1)$

$$\lambda_{K,V}^{(J)} = l_V + c^0 l_V^{-1} + c^1 l_V^{-\frac{\alpha+1}{\alpha-1}} \big(1 + \mathrm{o}(1)\big) \quad \text{as } |V| \to \infty,$$

where $c^0 := 2\nu\kappa^2$ and $c^1 := (\alpha - 1)\alpha^{-1} 2\nu\kappa^{2\alpha/(\alpha-1)}$.

Theorem 3.3. Assume that p satisfies condition (1.6) with $0 < \gamma < 1$. Fix $K = 1, 2, \ldots$, and let d_V be defined by (2.10). Then with probability $1 + \mathrm{o}(1)$

$$\lambda_{K,V}^{(J)} = d_V + b^0 \frac{d_V^{\gamma-1}}{\log d_V} + b^1 \frac{d_V^{\gamma-1} \log\log d_V}{(\log d_V)^2} + b \frac{d_V^{\gamma-1}}{(\log d_V)^2} \big(1 + \mathrm{o}(1)\big) \quad (3.2)$$

as $|V| \to \infty$, where $b^0 := \nu B \kappa^2 \gamma (1 - \gamma)^{-1}$, $b^1 := b^0 (\gamma - 1)^{-1}$ and $b := b^1 \log \frac{2(1-\gamma)\sqrt{\mathrm{e}}}{B\kappa\gamma}$.

A *sketch of the proof* is given in Sect. 4.4. The arguments repeat those of Theorem 6.4 and Corollary 6.4 in [4] and are based on the Laplace's method for the corresponding integrals.

We now extend the results of Theorem 3.1 to the case (1.3) with finite ρ (i.e., the double exponential case), provided the constant ρ/κ is large enough.

Theorem 3.4. (see Theorem 4.4 and Corollary 4.5 in [4], and Theorem 2.16 in [13]). Assume that F satisfies conditions (1.3) and (1.4) for some constants $\mu > (1 + \theta)\nu/(1 - 2\theta)$, $0 < \theta < \frac{1}{2}$ and $0 < \rho < \infty$ such that the constant

$$\underline{M}(\rho, \kappa, \theta) := \log \left(\frac{1}{2\nu} \frac{\rho}{\kappa} \log \frac{1}{1 - \theta} \right) > 0 \quad \text{is large enough}$$

(say, $\underline{M}(\rho, \kappa, \theta) \geqslant \log(36\nu)$). Then we have the almost sure assertions of Theorem 3.1 with $M_V(K) := \log(\widetilde{\lambda}_{K,V} - L_{V,\theta}) - \log(2\nu\kappa) \geqslant \underline{M}(\rho, \kappa, \theta)$ and fixed $K \geqslant 1$.

Moreover,

$$\lim_V \big(\lambda_{K,\,V} - f\,(\log |V|)\,\big) = 2\nu\kappa q(\rho/\kappa)$$

for some nonrandom constant $q(\rho) > 0$ such that $q(\rho) = (2\rho \log \rho)^{-1} \,(1 + \mathrm{o}(1))$ as $\rho \to \infty$.

From Theorems 3.1 and 3.4, we obtain the following *Poisson limit theorem* (as $|V| \to \infty$) for the largest eigenvalues $\lambda_{K,\,V}$ and the corresponding localization centres $z_{\tau(K),\,V}$ [2, 4]:

Assume that there exist the normalizing constants $A_V > 0$ and B_V such that $\lim_V |V|\mathbb{P}(\lambda^{(J)}(0) > B_V + A_V^{-1}t) = \mathrm{e}^{-t}$ for any $t \in \mathbb{R}$, and define the point process $\mathcal{N}_V^\lambda$ on $[-1;1]^\nu \times \mathbb{R}$ by

$$\mathcal{N}_V^\lambda := \sum_{k=1}^{|V|} \delta_{X_{k,\,V}} \quad \text{where} \quad X_{k,\,V} := \left(\frac{z_{\tau(k),\,V}}{|V|^{1/\nu}}, \big(\lambda_{k,\,V} - B_V\big)A_V \right).$$

Then $\mathcal{N}_V^\lambda$ converges weakly to Poisson process on $[-1;1]^\nu \times \mathbb{R}$ with intensity measure $\mathrm{d}\,x \times \mathrm{e}^{-t}\mathrm{d}\,t$.

The proof of Theorems 3.1 and 3.4 rely on the fact that, under conditions (1.3) and (1.4), the ξ_V-peaks possess a *strongly pronounced geometric structure* which can be described as follows [1, 3]:

For arbitrary sufficiently small $0 \leqslant \varepsilon < \theta$, there exist constants $c_1 > c_2 > 0$ and (large) $C > 0$ such that almost sure

$$\min_{1 \leqslant k < n \leqslant |V|^\theta} \left| z_{k,\,V} - z_{n,\,V} \right| \geqslant |V|^{c_1},$$

$$\min_{1 \leqslant k < n \leqslant |V|^\theta} \big(\xi_{k,\,V} - \xi_{n,\,V} \big) \geqslant \mathrm{e}^{-|V|^{c_2}}$$

and, finally,

$$\xi_{[|V|^\varepsilon],\,V} - \xi_{[|V|^\theta],\,V} \geqslant C$$

for each $V \supset V_0$. In this case, the largest eigenvalue $\lambda_{K,\,V}$ is associated with an isolated high ξ_V-peak; so that the asymptotic support of the corresponding eigenfunction $\psi(\cdot;\lambda_{K,\,V})$ consists of a single site $z_{\tau(K),\,V} \in V$, i.e., $\psi(\cdot;\lambda_{K,\,V})$ is a delta-like function.

For the lighter tails than those in (1.3) (including potentials with fractional double-exponential tails (1.6) with arbitrary $\gamma > 1$ as well as bounded from above potentials), the ξ_V-peaks possess a *weakly pronounced geometric structure*; in particular, almost sure $\xi_{[|V|^\varepsilon],\,V} - \xi_{[|V|^\theta],\,V} \to 0$ as $|V| \to \infty$, for all $0 \leqslant \varepsilon < \theta < 1$. In this case, the eigenvalue $\lambda_{K,\,V}$ does not longer correspond to an isolated potential peak, but to an extremely large "island" of high ξ_V-values of comparable amplitude [13, 10, 15, 4, 9]. See also [11] for rigorous results on Poisson limit theorems and localization properties for the largest eigenvalues in the case of double exponential distributions (1.6) with $\gamma = 1$ and arbitrary $B > 0$.

4. PROOF OF THEOREMS 2.1, 2.3 and 2.4

4.1. Preliminaries: Asymptotic Properties of Extreme Values of Potential

Throughout this section, we essentially use the following representation of the i.i.d. potential. As above, given a (right-continuous) distribution function such that $F(t) < 1$ ($t \in \mathbb{R}$), let f be the (left-continuous) inverse function of $-\log(1-F)$. Let $\eta(x) = \eta^{(\omega)}(x)$ ($x \in \mathbb{Z}^\nu$; $\omega \in \Omega$) be independent exponentially distributed random variables with mean 1. Clearly the random variables $\xi(x) := f(\eta(x))$ ($x \in \mathbb{Z}^\nu$) are independent and have a (common) distribution function F. Given a sample $\eta(\cdot\,)$ in V, we associate the sites $z_{l,\,V} \in V$ $(1 \leqslant l \leqslant |V|)$ with the variational series $\eta(z_{1,\,V}) > \eta(z_{2,\,V}) > \ldots > \eta(z_{|V|,\,V})$. Therefore, the variables

$$\xi_{l,\,V} := f(\eta_{l,\,V}) := f(\eta(z_{l,\,V})) \quad (1 \leqslant l \leqslant |V|) \tag{4.1}$$

form the variational series (1.1) based on the sample ξ_V.

In this section, we briefly study the asymptotic properties of extreme values $\eta_{l,\,V}$ as $V \uparrow \mathbb{Z}^\nu$, which are transferred directly to $\xi_{l,\,V}$ under appropriate conditions on f. We first formulate the following well-known properties of exponential order statistics.

Lemma 4.1. (see, e.g., [3]). (i) For fixed $K = 1, 2, \ldots$, $\eta_{K,\,V} - \eta_{K+1,\,V} \asymp 1$ as $|V| \to \infty$ in probability.

(ii) For an arbitrary sequence $\{K_V\}$ such that $1 \leqslant K_V \leqslant |V|$,

$$\limsup_V \sqrt{K_V} \max_{K_V \leqslant l \leqslant |V|} \left| \eta_{l,\,V} - \log \frac{|V|}{l} \right| < \infty$$

in probability.

Lemma 4.2. Assume that condition (2.1) is fulfilled. Then for arbitrarily fixed $K = 1, 2, \ldots$, any $0 < \varepsilon < 1$ and any sequence of integers $n_V = \mathrm{O}(|V|^\varepsilon)$, we have the following limits in probability:

(i) $$\xi_{n_V,\,V} \asymp (\log |V|)^{1/3} \quad \text{as} \quad |V| \to \infty,$$

and

(ii) $$0 < \liminf_V \min_{K+1 \leqslant l \leqslant |V|^\varepsilon} \xi^2_{l,V}(\xi_{K,\,V} - \xi_{l,\,V}) \frac{1}{\log l} \leqslant \limsup_V \max_{K+1 \leqslant l \leqslant |V|^\varepsilon} \xi^2_{l,V}(\xi_{K,\,V} - \xi_{l,\,V}) \frac{1}{\log l} < \infty.$$

Proof. For this, apply formula (4.1), Lemma 4.1 and the assertions of Remark 2.2(ii).

Lemma 4.3. Assume that condition (2.2) holds true. Then for fixed integers $K > l \geqslant 1$, we have the following limit in probability:

$$\xi_{K,V}^2(\xi_{l,\,V} - \xi_{K,\,V}) \to 0 \text{ as } |V| \to \infty.$$

Proof. This follows from formula (4.1) and Lemma 4.1(i).

The following lemma describes asymptotic properties of η_V-values neighboring to η_V-peaks and is frequently used in this section .

Lemma 4.4. (see [3]). Fix a finite subset $U \subset \mathbb{Z}^\nu \backslash \{0\}$, $U \neq \varnothing$, and a sequence of nonrandom real functions $\{D_l(t_U)\colon\ t_U \in \mathbb{R}^{|U|}\}$, $l = 1, 2, \ldots$. Further, abbreviate $\eta(z; l) := D_l(\{\eta(z + x)\colon\ x \in U\})$ for each $z \in \mathbb{Z}^\nu$, and fix a sequence of integers $K_V = \mathrm{O}(|V|^\varepsilon)$ for some $0 < \varepsilon < \frac{1}{2}$. Then, for any V and any $t \in \mathbb{R}$,

$$\left|\mathbb{P}\left(\max_{1 \leqslant l \leqslant K_V} \eta(z_{l,\,V}; l) \leqslant t\right) - \prod_{l=1}^{K_V} \mathbb{P}(\eta(0; l) \leqslant t)\right| \leqslant c_1 |V|^{-c_2},$$

where $c_i > 0$ do not depend on V and t.

Lemma 4.5. Fix $y \in \mathbb{Z}^\nu \backslash \{0\}$ and a sequence of integers $K_V \to \infty$ such that $K_V = \mathrm{O}(|V|^\varepsilon)$ for some $0 < \varepsilon < \frac{1}{2}$. Then

$$\limsup_{V} \left|\max_{1 \leqslant l \leqslant K_V} \eta(z_{l,\,V} + y) - \log K_V\right| < \infty$$

in probability.

Proof. This assertion follows from Lemma 4.4 with $D_l(t_U) \equiv t_y$ ($l = 1, 2, \ldots$) and Lemma 4.1(ii) with K_V instead of $|V|$.

Lemma 4.6. (see, e.g., [1]). Assume that condition (1.4) holds true, and fix $0 < \varepsilon < 1$. Then with probability 1 the cardinality of the subset $\{x \in V\colon\ \xi(x) \geqslant L_{V,\,\varepsilon}\}$ equals $|V|^\varepsilon(1 + \mathrm{o}(1))$ as $|V| \to \infty$.

4.2. Proof of Theorem 2.1(i),(ii)

Clearly the conditions of Theorem 2.1(i) imply (1.4) and (1.3) with $\rho = \infty$, i.e., the conditions of Theorem 3.1 where $\theta > 0$ is chosen small enough. We now show that the conditions of Theorem 3.1 yield that almost sure $\log \tau_V(K) / \log |V| \to 0$, i.e., the second assertion in (2.3). Indeed, fix arbitrary (small) constants $0 < \varepsilon < \varepsilon' < \theta$. By expanding $\lambda^{(J)}(z_{l,\,V})$ over $\widetilde{\xi}(x)/\xi_{l,\,V}$ (see (3.1)), we find that $\lambda^{(J)}(z_{l,\,V}) - \xi_{l,\,V} = \mathrm{o}(1)$ uniformly in

$1 \leqslant l \leqslant |V|^{\varepsilon'}$, as $|V| \to \infty$. Therefore, by Lemma 4.6, with probability 1 the cardinality of the subset $\{z_{l,V} \in V\colon\ \lambda^{(J)}(z_{l,V}) \geqslant L_{V,\varepsilon}\}$ tends to infinity as $|V| \to \infty$. Using these limits and applying again Lemma 4.6, we find that with probability 1 the site $z_{\tau(K),V}$ belongs to the subset $\{z_{l,V}\colon\ \lambda^{(J)}(z_{l,V}) \geqslant L_{V,\varepsilon}\}$ which is contained in $\{z_{l,V}\colon\ 1 \leqslant l \leqslant |V|^{\varepsilon'}\}$ for each $V \supset V_0$. Since $\varepsilon' > \varepsilon > 0$ are arbitrarily small, this implies the second limit in (2.3), as claimed.

Throughout this section, we essentially exploit the following auxiliary random variables (see notation in the beginning of Sections 3 and 4.1):

$$\Lambda_V(k) := \xi_{1,V}^2 \big(\lambda^{(J)}(z_{k,V}) - \xi_{1,V} - 2\nu\kappa^2 \xi_{1,V}^{-1}\big). \tag{4.2}$$

Clearly $\Lambda_V(\tau(k)) \equiv \Lambda_{k,V}$ $(1 \leqslant k \leqslant |V|)$. For real a and b, we will write $a \vee b := \max(a,b)$, $a \wedge b := \min(a,b)$ and $a_+ := a \vee 0$.

With these remarks and abbreviations, we now are in a position to prove the assertion of part (i) and the first assertion of part (ii) of Theorem 2.1.

(i) We first show that the assertion of (i) is a consequence of the following limits in probability:

$$\limsup_{M\to\infty} \limsup_{V} \frac{1}{\log M} \max_{M \leqslant l \leqslant |V|^{\varepsilon}} \Lambda_V(l) < 0 \tag{4.3}$$

and

$$\liminf_{M\to\infty} \liminf_{V} \frac{1}{\log M} \Lambda_V(M) > -\infty \tag{4.4}$$

for some $\varepsilon \in (0,\theta)$. This is done by induction in $K \geqslant 1$:

Write $\tau_V(0) := 0$ by convention, and assume that

$$p_M := \sum_{l=0}^{K-1} \limsup_{V} \mathbb{P}(\tau_V(l) \geqslant M) \to 0 \quad \text{as} \quad M \to \infty. \tag{4.5}$$

We further abbreviate $e_M := \exp\big\{(\log M)^2\big\}$ $(M = 1, 2, \ldots)$ and observe that the inequalities $e_M \leqslant \tau_V(K) \leqslant |V|^{\varepsilon}$ and $\max_{1 \leqslant l \leqslant K-1} \tau_V(l) < M$ imply that $\Lambda_V(l) \geqslant \Lambda_V(M)$ for all $e_M \leqslant l \leqslant |V|^{\varepsilon}$. Using this implication and recalling that almost sure $\tau_V(k)|V|^{-\varepsilon} = \mathrm{o}(1)$ for fixed k, we obtain that

$$\begin{aligned} &\limsup_{V} \mathbb{P}(\tau_V(K) \geqslant e_M) \\ &\qquad \leqslant \limsup_{V} \mathbb{P}\left(\max_{e_M \leqslant l \leqslant |V|^{\varepsilon}} \Lambda_V(l) \geqslant \Lambda_V(M)\right) + p_M \to 0 \end{aligned}$$

as $M \to \infty$, by (4.3)–(4.5). I.e., the assertion of (i) is proved.

The remainder is devoted to the proof of (4.3) and (4.4). We start with (4.3). For $\delta > 0$, let $\Omega^{(1)}_{V,\delta} \in \mathcal{F}$ stand for the (measurable) subset of configurations $\xi_V = \xi_V^{(\omega)}$ satisfying the following three inequalities:

$$\xi_{1,V}^2 \max_{2 \leqslant l \leqslant |V|^\varepsilon} \big(\xi_{l,V} - \xi_{1,V}\big) \frac{1}{\log l} \leqslant -3\delta,$$
$$\xi_{1,V} \leqslant \delta^{-1} (\log |V|)^{1/3} \quad \text{and} \quad \xi_{[|V|^\varepsilon],V} - L_{V,\theta} \geqslant \delta (\log |V|)^{1/3}.$$

Now, expanding $\lambda^{(J)}(z_{l,V})$ in powers of $\widetilde{\xi}(z_{l,V} + x)/\xi_{l,V}$ with $|x| = 1$ and $l \leqslant |V|^\varepsilon$ (see (3.1)), we obtain that, for any $0 < \delta < 1$, any $M \geqslant M_0(\delta)$, any $V \supset V_0(M)$ and any $\xi_V^{(\omega)}$ ($\omega \in \Omega^{(1)}_{V,\delta}$), the following inequality holds true for all $M \leqslant l \leqslant |V|^\varepsilon$:

$$\Lambda_V(l) \leqslant -2\delta \log l + \text{const} \sum_{|x|=1} \xi_+(z_{l,V} + x) + \frac{\text{const}'}{(\log |V|)^{1/3}}.$$

From this and Lemma 4.4 with $\eta(z;l) := -2\delta \log l + \text{const} \sum_{|x|=1} f_+(\eta(z+x))$ ($M \leqslant l \leqslant |V|^\varepsilon$), we obtain that

$$\begin{aligned} &\limsup_V \mathbb{P}\Big(\Big\{ \max_{M \leqslant l \leqslant |V|^\varepsilon} \Lambda_V(l) \geqslant -\delta \log M \Big\} \bigcap \Omega^{(1)}_{V,\delta}\Big) \\ &\qquad \leqslant 1 - \prod_{l=M}^{\infty} \mathbb{P}\bigg(-2\delta \log l + \text{const} \sum_{|x|=1} \xi_+(x) < -\delta \log M \bigg). \end{aligned} \tag{4.6}$$

According to Remark 2.2(i), the right-hand side of (4.6) does not exceed

$$\sum_{l=M}^{\infty} \exp\bigg\{ -\text{const}' \bigg(\log l - \frac{1}{2} \log M \bigg)^3 \bigg\} \to 0 \quad \text{as} \quad M \to \infty.$$

This yields that

$$\limsup_{M\to\infty} \limsup_V \mathbb{P}\Big(\max_{M \leqslant l \leqslant |V|^\varepsilon} \Lambda_V(l) \geqslant -\delta \log M \Big) \leqslant \limsup_V \mathbb{P}\big(\Omega \backslash \Omega^{(1)}_{V,\delta}\big) \to 0$$

as $\delta \downarrow 0$, by Lemma 4.2. Thus, (4.3) is proved.

To show (4.4), we apply the same arguments. For $0 < \delta < 1$ and $M = 1, 2, \ldots$, let $\Omega^{(2)}_{V,M,\delta} \in \mathcal{F}$ denote the (measurable) subset of configurations $\xi_V = \xi_V^{(\omega)}$ satisfying the following three inequalities:

$$\xi_{1,V}/\xi_{M,V} < \delta^{-1/2}, \quad \xi_{M,V}^2 \big(\xi_{M,V} - \xi_{1,V}\big) > -\delta^{-1} \log M$$

and

$$\delta (\log |V|)^{1/3} < \xi_{M,V} - L_{V,\theta} < \delta^{-1} (\log |V|)^{1/3}.$$

By Lemma 4.2,

$$\sup_{M\geqslant 1}\limsup_{V}\mathbb{P}\big(\Omega\backslash\Omega^{(2)}_{V,M,\delta}\big)\to 0\quad\text{as}\quad \delta\downarrow 0. \tag{4.7}$$

Now, for any small $\delta>0$, any $M\geqslant M_0$, any $V\supset V_0(M,\delta)$ and any $\xi_V^{(\omega)}$ ($\omega\in\Omega^{(2)}_{V,M,\delta}$), we have that

$$\begin{aligned}\Lambda_V(M)&\geqslant\left(\frac{\xi_{1,V}}{\xi_{M,V}}\right)^2\Bigg[\xi^2_{M,V}\big(\xi_{M,V}-\xi_{1,V}\big)+\kappa^2\bigg(\sum_{|x|=1}\xi(z_{M,V}+x)\bigg)\bigwedge 0-\text{const}\Bigg]\\&\geqslant\delta^{-1}\Bigg[-\delta^{-1}\log M+\kappa^2\bigg(\sum_{|x|=1}\xi(z_{M,V}+x)\bigg)\bigwedge 0-\text{const}\Bigg]\Bigg\}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\limsup_V\mathbb{P}\big(\{\Lambda_V(M)<-2\delta^{-2}\log M\}\cap\Omega^{(2)}_{V,M,\delta}\big)\\&\leqslant\limsup_V\mathbb{P}\bigg(\sum_{|x|=1}\xi(z_{M,V}+x)<-\text{const}'\log M\bigg)\to 0\quad\text{as}\quad M\to\infty\end{aligned}$$

by Lemma 4.4 with $\eta(z;k)\equiv\sum_{|x|=1}f(\eta(z+x))$. This and (4.7) yield (4.4). Part (i) is proved.

(ii) We need to show the first assertion in (2.3). For a sequence $s_V=\log|V|+\mathrm{O}(1)$, we define $\widetilde{m}_V>0$ by $f(s_V-\widetilde{m}_V)\leqslant f(s_V)-f(s_V)^{-2}\leqslant f(s_V-\widetilde{m}_V+0)$, so that $\widetilde{m}_V\to\infty$ according to the assumption of (ii). Obviously, there is a sequence of integers $m_V\to\infty$ such that

$$\frac{f(2\log m_V)}{f\big(\frac12\log|V|\big)}\to 0\quad\text{and}\quad\frac{\log m_V}{\widetilde{m}_V}\to 0. \tag{4.8}$$

Let $\Lambda_V(l)$ be given by (4.2). We now show that the first limit in (2.3) is a consequence of the following two limits in probability:

$$\lim_V\max_{km_V\leqslant l<(k+1)m_V}\Lambda_V(l)=\infty\quad\text{for fixed}\quad k=1,2,\dots \tag{4.9}$$

and

$$\lim_{M\to\infty}\lim_V\frac{1}{M}\max_{1\leqslant l\leqslant M}\Lambda_V(l)=0. \tag{4.10}$$

Indeed, for all $M\geqslant 1$ and $V\supset V_0(M)$, the inequality $\min_{k\leqslant K}\tau_V(k)\leqslant M$ implies that

$$\min_{k\leqslant K}\max_{km_V\leqslant l<(k+1)m_V}\Lambda_V(l)\leqslant\max_{1\leqslant l\leqslant M}\Lambda_V(l).$$

Using this implication, we have that

$$\begin{aligned}
&\limsup_V \mathbb{P}\Big(\min_{k\leqslant K}\tau_V(k)\leqslant M\Big)\\
&\leqslant \limsup_V \mathbb{P}\Big(\min_{k\leqslant K}\max_{km_V\leqslant l<(k+1)m_V}\Lambda_V(l)\leqslant M\Big)\\
&\quad+\limsup_V \mathbb{P}\Big(\max_{1\leqslant l\leqslant M}\Lambda_V(l)>M\Big)\to 0 \quad\text{as}\quad M\to\infty,
\end{aligned}$$

by (4.9) and (4.10).

We only need to check (4.9) and (4.10), say, for $k=1$. Fix sufficiently small $\varepsilon>0$. Let $\Omega_V^{(3)}$ denote the (measurable) subset of configurations $\xi_V=\xi_V^{(\omega)}$ satisfying

$$\xi_{1,V}\geqslant L_{V,\varepsilon}\quad\text{and}\quad \xi_{2m_V,V}\geqslant \xi_{1,V}-2\xi_{1,V}^{-2},$$

and, for $M>0$, let $\Omega_{V,M}^{(4)}$ stand for the (measurable) subset of $\xi_V=\xi_V^{(\omega)}$ satisfying

$$\xi_{1,V}\geqslant L_{V,\varepsilon}\quad\text{and}\quad \max_{1\leqslant l\leqslant 2m_V}\sum_{1\leqslant|x|\leqslant 2}\xi_+(z_{l,V}+x)/\xi_{1,V}<\frac{1}{M}.$$

We propose that

$$\lim_V \mathbb{P}\big(\Omega\backslash\Omega_V^{(3)}\big)=0\quad\text{and}\quad \lim_V \mathbb{P}\big(\Omega\backslash\Omega_{V,M}^{(4)}\big)=0\quad\text{for each}\quad M>0. \tag{4.11}$$

Indeed, the first limit follows directly from formula (4.1), Lemma 4.1(ii) the second property (4.8) of m_V (cf. also Lemma 4.3). The second limit in (4.11) can be easily proved by combining formula (4.1), Lemmas 4.5 and 4.1 and the first property (4.8) of m_V. We now abbreviate $\zeta_V(l):=\sum_{|x|=1}\xi(z_{l,V}+x)+\big(\sum_{|x|=1}\xi(z_{l,V}+x)\big)\bigwedge 0$. For any $M\geqslant M_0$, any $V\supset V_0(M)$ and any $\xi_V^{(\omega)}$ ($\omega\in\Omega_V^{(3)}\bigcap\Omega_{V,M}^{(4)}$), the following inequalities hold true:

$$\begin{aligned}
\Lambda_V(l)\ &\geqslant\ \big(\xi_{1,V}\big)^2\bigg(\xi_{l,V}-\xi_{1,V}+\Big(\frac{\kappa}{\xi_{l,V}}\Big)^2\sum_{|x|=1}\xi(z_{l,V}+x)\bigg)-\text{const}\\
&\geqslant\ \kappa^2\zeta_V(l)-\text{const}'\quad\text{for all}\quad m_V\leqslant l<2m_V.
\end{aligned}$$

and

$$\Lambda_V(l)\leqslant \text{const}\sum_{|x|=1}\xi_+(z_{l,V}+x)+\frac{\text{const}\,'}{L_{V,\varepsilon}}\quad\text{for all}\quad 1\leqslant l\leqslant M.$$

Summarizing these estimates, we obtain the following limits for any $M \geqslant M_0$:

$$\begin{aligned}
&\limsup_V \mathbb{P}\Big(\Big\{\max_{m_V \leqslant l < 2m_V} \Lambda_V(l) < M\Big\} \bigcap \Omega_V^{(3)} \bigcap \Omega_{V,M}^{(4)}\Big) \\
&\quad \leqslant \limsup_V \mathbb{P}\Big(\kappa^2 \max_{m_V \leqslant l < 2m_V} \zeta_V(l) < 2M\Big) = 0
\end{aligned} \tag{4.12}$$

(where the last limit is a consequence of formula (4.1) and Lemma 4.5) and, moreover,

$$\begin{aligned}
&\limsup_V \mathbb{P}\Big(\Big\{\max_{1 \leqslant l \leqslant M} \Lambda_V(l) > \log M\Big\} \bigcap \Omega_V^{(3)} \bigcap \Omega_{V,M}^{(4)}\Big) \\
&\quad \leqslant \limsup_V \mathbb{P}\Big(\max_{1 \leqslant l \leqslant M} \sum_{|x|=1} \xi_+(z_{l,\,V} + x) \geqslant \text{const} \log M\Big) \\
&\to 0 \quad \text{as} \quad M \to \infty,
\end{aligned} \tag{4.13}$$

where the last limit follows from formula (4.1), Lemma 4.5 and Remark 2.2(iii). Now, because of (4.11), limits (4.12) and (4.13) imply, respectively, (4.9) and (4.10) for $k = 1$, as claimed.

4.3. Proof of Theorem 2.3

With constants c^0, c^1 as in Theorem 3.2 and l_V given by (2.8), we abbreviate

$$n_V := \big[\exp\{a_0 l_V^{\alpha-2}\}\big], \quad C_{V,\mu} := l_V + c^0 l_V^{-1} + (c^1 + \mu) l_V^{-\frac{\alpha+1}{\alpha-1}}$$

and

$$\sigma_V(k) := l_V - \frac{\log k}{\alpha A l_V^{\alpha-1}} + \frac{c^0}{l_V} + \left(\frac{\kappa}{l_V}\right)^2 \sum_{|x|=1} \xi_+(z_{k,V} + x) \quad (1 \leqslant k \leqslant |V|).$$

For sufficiently large $a_0 > 0$ and for each $\mu < 0$, we now prove the following auxilary limits:

$$\lim_V \mathbb{P}(\tau(K) > n_V) = 0 \quad \text{and} \quad \lim_V \mathbb{P}\big(\sigma_V(\tau(K)) < C_{V,\mu}\big) = 0. \tag{4.14}$$

To show the first limit in (4.14), we note that the random variables $\Lambda_V(k)$ (4.2) satisfy (4.9), provided a sequence m_V tends to infinity sufficiently slowly. Therefore, the first assertion in (4.14) follows from the limit

$$\lim_V \max_{n_V \leqslant k \leqslant |V|^\varepsilon} \Lambda_V(k) = -\infty \quad \text{in probability} \tag{4.15}$$

for some $0 < \varepsilon < \theta < 1/2$. To show (4.15), we apply Lemma 4.1(ii), formulas (4.1) and (2.7) to estimate $\xi_{k,V}$. Thus, the expression under the limit in (4.15) does not exceed

$$\xi_{1,V}^2(\xi_{n_V,V} - \xi_{1,V}) + \mathrm{const}\, l_V = -\frac{\log n_V}{\alpha A l_V^{\alpha-3}}(1 + \mathrm{o}(1)) + \mathrm{const}\, l_V$$

in probability. Since the latter does not exceed $-\mathrm{const}' l_V$ where $\mathrm{const}' > 0$, we arrive at (4.15).

Let us show the second assertion in (4.14). By combining (2.7), formula (4.1) and Lemmas 4.1(ii) and 4.5, we obtain that, for each $\delta > 0$,

$$\limsup_V \mathbb{P}\Big(\big(\lambda^{(J)}(z_{\tau(K),V}) - \sigma_V(\tau(K))\big) l_V^{\frac{\alpha+1}{\alpha-1}} > \delta, M \leqslant \tau(K) \leqslant n_V\Big) \to 0$$

as $M \to \infty$. Using this together with Theorems 3.2 and 2.1(ii) and the first limit in (4.14), we arrive at the second limit in (4.14).

With abbreviation $t_V := \exp\big\{2\nu A \kappa^{2\alpha/(\alpha-1)} l_V^{\alpha(\alpha-3)/(\alpha-1)}\big\}$, we now are in a position to prove that, for each $\delta > 0$,

$$\mathbb{P}(\tau(K) > t_V^{1+\delta}) \to 0 \quad \text{and} \quad \mathbb{P}(\tau(K) < t_V^{1-\delta}) \to 0 \tag{4.16}$$

as $|V| \to \infty$. To show the first limit in (4.16), we observe that

$$\mathbb{P}\big(t_V^{1+\delta} \leqslant \tau(K) \leqslant n_V, \sigma_V(\tau(K)) \geqslant C_{V,\mu}\big) \leqslant \mathbb{P}\Big(\max_{t_V^{1+\delta} \leqslant k \leqslant n_V} \sigma_V(k) \geqslant C_{V,\mu}\Big) \tag{4.17}$$

By Lemma 4.4 with

$$\eta(z;k) := -\frac{\log k}{\alpha A l_V^{\alpha-1}} + \left(\frac{\kappa}{l_V}\right)^2 \sum_{|x|=1} f_+(\eta(z+x)),$$

we obtain that the right-hand side of (4.17) does not exceed

$$\sum_{t_V^{1+\delta} \leqslant k \leqslant n_V} \mathbb{P}\Bigg(\sum_{|x|=1} \xi_+(x) \geqslant \left(\frac{l_V}{\kappa}\right)^2 \left[\frac{\log k}{\alpha A l_V^{\alpha-1}} + (c^1 + \mu) l_V^{-\frac{\alpha+1}{\alpha-1}}\right]\Bigg) + |V|^{-\mathrm{const}} \tag{4.18}$$

For any small $\delta > 0$, we now pick $\mu = \mu(\delta) < 0$ to satisfy

$$\rho' := \frac{\delta}{\alpha} + \frac{\mu}{2\nu}\kappa^{-2\alpha/(\alpha-1)} > 0 \quad \text{and} \quad \delta' := (1+\rho')^\alpha - 1 - \delta > 0.$$

Using the fact that

$$-\log \mathbb{P}\Bigg(\sum_{|x|=1} \xi_+(x) > t\Bigg) = A(2\nu)^{1-\alpha} t^\alpha + \mathrm{O}(\log t) \quad \text{as} \quad t \to \infty$$

(see, pp. 627 in [20], or [7]) and applying then the inequality $(a+b)^\alpha \geqslant b^\alpha + \alpha a b^{\alpha-1}$ for positive a and b, we obtain that the sum (4.18) does not exceed

$$(\log t_V)^{\text{const}} \sum_{t_V^{1+\delta} \leqslant k \leqslant n_V} \exp\left\{ -\left(\frac{1}{\alpha \log t_V} \log\left(\frac{k}{t_V^{1+\delta}}\right) + 1 + \rho'\right)^\alpha \log t_V \right\}$$

$$\leqslant (\log t_V)^{\text{const}} \exp\left\{ -(1+\rho')^\alpha \log t_V \right\} \sum_{k \geqslant t_V^{1+\delta}} \exp\left\{ -(1+\rho')^{\alpha-1} \log \frac{k}{t_V^{1+\delta}} \right\}$$

$$\leqslant t_V^{-\delta'} (\log t_V)^{\text{const}} \to 0.$$

Consequently, the right-hand side of (4.17) tends to zero. Together with (4.14), this yields the first limit in (4.16).

We show the second limit in (4.16) by the same arguments. For any small $\delta > 0$, we pick $\mu = \mu(\delta) < 0$ such that

$$\rho'' := \frac{\delta}{\alpha} - \frac{\mu}{2\nu} \kappa^{-2\alpha/(\alpha-1)} < 1 \quad \text{and} \quad \delta'' := (1-\rho'')^\alpha + \delta - 1 > 0,$$

and notice that

$$\mathbb{P}\Big(\tau(K) \leqslant t_V^{1-\delta},\ \sigma_V(\tau(K)) \geqslant C_{V,\mu} \Big)$$

$$\leqslant \sum_{k \leqslant t_V^{1-\delta}} \mathbb{P}\left(\sum_{|x|=1} \xi_+(x) \geqslant \left(\frac{l_V}{\kappa}\right)^2 \left[\frac{\log k}{\alpha A l_V^{\alpha-1}} + (c^1 + \mu) l_V^{-\frac{\alpha+1}{\alpha-1}} \right] \right) + |V|^{-\text{const}}$$

$$\leqslant (\log t_V)^{\text{const}} \exp\left\{ -(1-\rho'')^\alpha \log t_V \right\}$$

$$\times \sum_{k \leqslant t_V^{1-\delta}} \exp\left\{ -(1-\rho'')^{\alpha-1} \log \frac{k}{t_V^{1-\delta}} \right\} + |V|^{-\text{const}}$$

$$\leqslant t_V^{-\delta''} (\log t_V)^{\text{const}\,'} \to 0.$$

This and the second limit (4.14) conclude the proof of (4.16).

4.4. Sketch of the Proof of Theorems 3.3 and 2.4

Combining Theorem 2.1(ii), Lemmas 4.1(ii), 4.5, formulas (4.1) and (2.9) similarly as in the proof of the first assertion of (4.14), we obtain with probability $1 + \mathrm{o}(1)$ that $\tau_V(K) \to \infty$ and $\tau_V(K) \leqslant N_V := \exp\left\{ d_V^{2\gamma-2} \log|V| \right\}$. Therefore, taking into account (3.1) and again applying Lemmas 4.1(ii), 4.5 and formulas (4.1) and (2.9), we see that $\lambda_{K,V}^{(J)}$ is approximated by the Kth largest values of the following random variables

$$\Xi_V(x) := \xi(x) + \sum_{|y-x|=1} \frac{\kappa^2}{d_V - \widetilde{\xi}(y)} \qquad (x \in V)$$

and

$$\chi_V(k) := d_V - \frac{\log k}{\gamma B d_V^{\gamma-1}\log|V|} + \sum_{|y-z_{k,V}|=1} \frac{\kappa^2}{d_V - \widetilde{\xi}(y)} \quad (1 \leqslant k \leqslant N_V). \quad (4.19)$$

Namely, with probability $1+\mathrm{o}(1)$

$$\lambda_{K,V}^{(J)} = \Xi_{K,V} + \mathrm{O}\big(d_V^{3\gamma-3}\big) = \chi_{K,V} + \mathrm{O}\big(d_V^{3\gamma-3}\big). \quad (4.20)$$

Write B_V for the right-hand side of (3.2), and let $B_{V,\mu} := B_V + \mu d_V^{\gamma-1}(\log d_V)^{-2}$. By applying Laplace's method for certain integrals similarly as in the proof of Lemma 6.7 in [4], we obtain that for each $\mu > 0$, $|V|\mathbb{P}\big(\Xi_V(0) > B_{V,\mu}\big) \to 0$ and $|V|\mathbb{P}\big(\Xi_V(0) > B_{V,-\mu}\big) \to \infty$ as $|V| \to \infty$. This and the first assertion of (4.20) conclude the proof of Theorem 3.3.

To prove Theorem 2.4, we exploit the random variables $\chi_V(\cdot)$ (4.19). Write $T_V := \exp\Big\{C d_V^{2\gamma-2}(\log d_V)^{-2}\log|V|\Big\}$, where constant C stands for the right-hand side of (2.11). Similarly as in the proof (4.16), we obtain that for each small $\delta > 0$ and some $\mu = \mu(\delta) < 0$,

$$\begin{aligned}\mathbb{P}\big(\tau(K) \geqslant T_V^{1+\delta}\big) &\leqslant \mathbb{P}\Big(\max_{k \geqslant T_V^{1+\delta}} \chi_V(k) \geqslant B_{V,\mu}\Big) + \mathrm{o}(1) \\ &\leqslant \sum_{k \geqslant T_V^{1+\delta}} \mathbb{P}\bigg(\sum_{|x|=1} \frac{\kappa^2}{d_V - \widetilde{\xi}(x)} \geqslant B_{V,\mu} - d_V + \frac{\log k}{\gamma B d_V^{\gamma-1}\log|V|}\bigg) + \mathrm{o}(1). \quad (4.21)\end{aligned}$$

To estimate the summands in (4.21) for $k < T_V^{\mathrm{O}(1)}$, we apply the following asymptotic tail bound as $|V| \to \infty$:

$$\mathbb{P}\bigg(\sum_{|x|=1} \frac{\kappa^2}{d_V - \widetilde{\xi}(x)} \geqslant B_{V,t} - d_V\bigg) \leqslant \exp\Big\{-\big(\log T_V\big)\big(\mathrm{e}^{t\gamma B/C - 1} + \mathrm{o}(1)\big)\Big\}$$

uniformly for t in finite intervals of $\mathbb{R}$, which is derived again by Laplace's method for integrals. For k larger than $T_V^{\mathrm{O}(1)}$ in (4.21), we replace the random variables $\widetilde{\xi}(x)$ ($|x| = 1$) by their maximum $\xi^{\max} := \max_{|x|=1} \widetilde{\xi}(x)$, where $\mathbb{P}\big(\xi^{\max} > t\big) \leqslant 2\nu(1 - F(t))$ for all t. Together with these estimates, direct calculations show that the right-hand side of (4.21) tends to 0 as $|V| \to \infty$. Similarly, $\mathbb{P}\big(\tau(K) < T_V^{1-\delta}\big) = \mathrm{o}(1)$. These limits conclude the proof of Theorem 2.4.

4.5. Conclusions and Remarks

(i) Under the conditions of Theorem 2.1(i), we see from the proof in Sect. 4.2 that $\liminf_V \mathbb{P}(\tau_V(K) \geqslant m) > 0$ for each $m \geqslant 1$; meanwhile, $\limsup_V \mathbb{P}(\tau_V(K) \geqslant M) \to 0$ as $M \to \infty$.

(ii) For Weibull's type distributions (1.5), the derivation of the first order formula for $\log \tau_V(K)$ claims the third order expansion formula for the eigenvalue $\lambda_{K,V}$; cf. Theorems 3.1, 3.2 and 2.3. Meanwhile, for the fractional-double exponential distributions (1.6), we need the forth order expansion formula for $\lambda_{K,V}$ to derive limit (2.11) for $\log \tau_V(K)$; cf. Theorems 3.3 and 2.4.

In both cases, the main contribution to $\log \tau_V(K)$ is determined by an isolated ξ_V-peak and its nearest neighbor values; see the proof of Theorems 2.3 and 2.4. Notice also that the O-terms in (1.5) and (1.6) do not reflect on the first order asymptotics for $\log \tau_V(K)$.

(iii) In Theorem 2.1(iii) treating the double exponential case, we obtain O-type asymptotic bounds for $\log \tau_V(K)$. In order to prove more explicit limits (for instance, $\varepsilon_1(\cdot) \equiv \varepsilon_2(\cdot)$ in Theorem 2.1(iii)), we need to apply more refined variational arguments. We notice that, in this case, the main contribution to $\log \tau_V(K)$ is determined by a lower isolated ξ_V-peak and a huge flat "island" of its neighbor values; cf. Theorem 3.4.

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