International Journal of Algebra, Vol. 7, 2013, no. 13, 597 - 606
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ija.2013.3775

Nontransferable Individual Payoffs in Cooperative Stochastic Dynamic Games

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Abstract

In cooperative dynamic games with non-transferable payoffs, the players' agreed-upon cooperative actions would determine the resulting payoff that each player receives. This article develops a mechanism for the derivation of individual player's payoff functions in cooperative stochastic dynamic games with nontransferable payoffs. This is the first time that individual player's payoff functions are characterized in an analytically derivable form in such a framework. An illustrative example is provided.

Mathematics Subject Classifications: 91A12, 91A25

Keywords: Cooperative stochastic dynamic games, nontransferable payoffs, stochastic difference equations

1 Introduction

In cooperative games, players negotiate to establish an agreement on how to act and assign their payoffs. A necessary condition is that the agreement must satisfy individual rationality. To verify individual rationality, individual players' payoff functions have to be derived. In the case when the game is dynamic and stochastic while payoffs are nontransferable, analytically tractable individual payoff functions, though difficult to be obtained, are needed for this verification process. Yeung (2004) provided a formulation to characterize the players' individual payoffs in continuous-time cooperative stochastic differential with non-transferable payoffs. In this article, we develop a mechanism for the derivation of individual player's payoff functions for discrete-time cooperative stochastic dynamic games. An illustrative example is given.

2 Game Framework

Consider the general $T-$ stage $n-$ person nonzero-sum discrete-time stochastic dynamic game with initial state x_1^0. The state space of the game is $X \in R^m$ and the state dynamics of the game is characterized by the stochastic difference equation:

$$x_{k+1} = f_k\,(x_k, u_k^1, u_k^2, \cdots, u_k^n) + \theta_k\,, \tag{2.1}$$

for $k \in \{1,2,\cdots,T\} \equiv \kappa$ and $x_1 = x_1^0$,

where $u_k^i \in R^{m_i}$ is the control vector of player i at stage k, $x_k \in X$ is the state, and θ_k is a set of statistically independent random variables.

The objective of player i is

$$E_{\theta_1,\theta_2,\cdots,\theta_T}\left\{ \sum_{k=1}^{T} g_k^i(x_k, u_k^1, u_k^2, \cdots, u_k^n) + q^i(x_{T+1}) \right\}, \tag{2.2}$$

for $i \in \{1,2,\cdots,n\} \equiv N$,

where $E_{\theta_1,\theta_2,\cdots,\theta_T}$ is the expectation operator with respect to the statistics of $\theta_1, \theta_2, \cdots, \theta_T$, and $q^i(x_{T+1})$ is the terminal payoff that player i will received in stage $T+1$.

The payoffs of the players are not transferable. Let $\{\phi_k^i(x)$, for $k \in \kappa$ and $i \in N\}$ denote a set of strategies that provides a feedback Nash equilibrium solution (if it exists) to the game (2.1)-(2.2), and $V^i(k,x)$, for $k \in K$, denote the value functions indicating the expected payoff to player i over the stages from k to T. The theorems characterizing a Nash of the game (2.1)-(2.2) can be found in standard textbooks (for instance see Theorem 13.1 in Yeung and Petrosyan (2012)).

For the sake of exposition, we sidestep the issue of multiple equilibria and focus on solvable games in which a particular noncooperative Nash equilibrium is chosen by the players in the entire game.

3 Cooperation Scheme and Individual Payoffs

Now consider the case when the players agree to cooperate and enhance their payoffs according to an agreed-upon cooperative scheme. In the scheme, the players would adopt the agreed-upon cooperative strategies which would directly determine the payoffs of the players.

3.1. Optimal Cooperative Strategies

To obtain a group optimal outcome one has to consider the derivation of a set of cooperative strategies using payoff weights in which the players agree to adopt a vector of constant payoff weights $\alpha = (\alpha^1, \alpha^2, \cdots, \alpha^n)$ for $\sum_{j=1}^{n} \alpha^j = 1$. Dockner and Jorgensen (1984), Hamalainen et al (1986), Leitmann (1974), and Yeung and Petrosyan (2005) provided analysis along this line. Conditional upon an agreed-upon vector of weights α, the agents' optimal cooperative strategies can be generated by solving the following stochastic control problem of maximizing their joint weighted expected payoff:

$$E_{\theta_1,\theta_2,\cdots,\theta_T}\left\{ \sum_{j=1}^{n} \left[\sum_{k=1}^{T} \alpha^j g_k^j(x_k, u_k^1, u_k^2, \cdots, u_k^n) + \alpha^j q^j(x_{T+1}) \right] \right\} \tag{3.1}$$

subject to (2.1).

An optimal solution to the problem (3.1)-(2.1) can be characterized by the following Theorem.

Theorem 3.1. A set of strategies $\{\psi_k^{(\alpha)i}(x)$, for $k \in \kappa$ and $i \in N\}$ provides an optimal solution to the problem (3.1)-(2.1) if there exist functions $W^{(\alpha)}(k, x)$, for $k \in K$, such that the following recursive relations are satisfied:

$$W^{(\alpha)}(T+1, x) = \sum_{j=1}^{n} \alpha^j q^j(x_{T+1}), \tag{3.2}$$

$$\begin{aligned} W^{(\alpha)}(k, x) = \max_{u_k^1, u_k^2, \cdots, u_k^n} E_{\theta_k} \Bigg\{ & \sum_{j=1}^{n} \alpha^j g_k^j(x_k, u_k^1, u_k^2, \cdots, u_k^n) \\ & + W^{(\alpha)}[k+1, f_k(x_k, u_k^1, u_k^2, \cdots, u_k^n) + \theta_k] \Bigg\} \\ = E_{\theta_k} \Bigg\{ & \sum_{j=1}^{n} \alpha^j g_k^j[x, \psi_k^{(\alpha)1}(x), \psi_k^{(\alpha)2}(x), \cdots, \psi_k^{(\alpha)n}(x)] \\ & + W^{(\alpha)}[k+1, f_k(x, \psi_k^{(\alpha)1}(x), \psi_k^{(\alpha)2}(x), \cdots, \psi_k^{(\alpha)n}(x)) + \theta_k] \Bigg\}, \end{aligned} \tag{3.3}$$

Proof. The results in (3.2)-(3.3) comes directly from the stochastic optimal control techniques (See Basar and Olsder (1999) and Yeung and Petrosyan (2012)). ■

Substituting the optimal control $\{\psi_k^{(\alpha)i}(x)$, for $k \in \kappa$ and $i \in N\}$ into the state dynamics (2.1), one can obtain the dynamics of the cooperative trajectory as:

$$x_{k+1} = f_k\,(x_k, \psi_k^{(\alpha)1}(x_k), \psi_k^{(\alpha)2}(x_k), \cdots, \psi_k^{(\alpha)n}(x_k)) + \theta_k\,, \tag{3.4}$$

We use $x_k^{(\alpha)} \in X_k^{(\alpha)}$ to denote the value of the state at stage k generated by (3.4), where $X_k^{(\alpha)}$ is the set of realizable values of $x_k^{(\alpha)}$. The term $W^{(\alpha)}(k,x)$ gives the expected weighted cooperative payoff over the stages from k to T if $x_k^{(\alpha)} = x \in X_k^{(\alpha)}$ is realized at stage $k \in \kappa$.

3.2. Individual Payoff under Cooperation

Given that all players are adopting the cooperative strategies in Section 3.1 the expected payoff of player i under cooperation can be obtained as:

$$W^{(\alpha)i}(t,x) = E_{\theta_t,\theta_{t+1},\cdots,\theta_T}\left\{ \sum_{k=t}^{T} g_k^i[x_k^{(\alpha)}, \psi_k^{(\alpha)1}(x_k^{(\alpha)}), \psi_k^{(\alpha)2}(x_k^{(\alpha)}), \cdots, \psi_k^{(\alpha)n}(x_k^{(\alpha)})] \right.$$

$$\left. + q^i(x_{T+1}^{(\alpha)}) \,\middle|\, x_t = x \right\}, \qquad \text{for } i \in N \text{ and } t \in \kappa. \tag{3.5}$$

To allow the derivation of the functions $W^{(\alpha)i}(t,K)$ in a more direct way we establish a discrete-time analog of Yeung's (2004) characterization of the players' individual expected payoffs under cooperation in the following Theorem.

Theorem 3.2.
The expected payoff of player i can be characterized as

$$W^{(\alpha)i}(T+1,x) = q^i(x_{T+1}),$$

$$W^{(\alpha)i}(t,x) = E_{\theta_t}\left\{ g_t^i[x, \psi_k^{(\alpha)1}(x), \psi_t^{(\alpha)2}(x), \cdots, \psi_t^{(\alpha)n}(x)] \right.$$

$$\left. + W^{(\alpha)i}[t+1, f_t\,(x, \psi_t^{(\alpha)1}(x), \psi_t^{(\alpha)2}(x), \cdots, \psi_t^{(\alpha)n}(x)) + \theta_t] \right\},$$

for $i \in N$ and $t \in \kappa$. (3.6)

Proof. Note that $W^{(\alpha)i}(t,x)$ in (3.5) can be expressed as:

$$W^{(\alpha)i}(t,x) = E_{\theta_t,\theta_{t+1},\cdots,\theta_T}\left\{ g_t^i[x, \psi_t^{(\alpha)1}(x), \psi_t^{(\alpha)2}(x), \cdots, \psi_t^{(\alpha)n}(x)] \right.$$

$$\left. + \sum_{k=t+1}^{T} g_k^i[x_k^{(\alpha)}, \psi_k^{(\alpha)1}(x_k^{(\alpha)}), \psi_k^{(\alpha)2}(x_k^{(\alpha)}), \cdots, \psi_k^{(\alpha)n}(x_k^{(\alpha)})] + q^i(x_{T+1}^{(\alpha)}) \,\middle|\, x_t^{(\alpha)} = x \right\}. \tag{3.7}$$

Invoking (3.5), we have:

$$W^{(\alpha)i}(t+1, x_{t+1}^{(\alpha)}) =$$

$$E_{\theta_{t+1},\theta_{t+2},\cdots,\theta_T}\left\{ \sum_{k=t+1}^{T} g_k^i[x_k^{(\alpha)},\psi_k^{(\alpha)1}(x_k^{(\alpha)}),\psi_k^{(\alpha)2}(x_k^{(\alpha)}),\cdots,\psi_k^{(\alpha)n}(x_k^{(\alpha)})] + q^i(x_{T+1}^{(\alpha)}) \right\}, \tag{3.8}$$

Using (3.7) and (3.8), we have

$$W^{(\alpha)i}(t,x) = E_{\theta_t}\left\{ g_t^i[x,\psi_k^{(\alpha)1}(x),\psi_t^{(\alpha)2}(x),\cdots,\psi_t^{(\alpha)n}(x)] \right.$$
$$\left. + W^{(\alpha)i}[t+1, f_t\ (x,\psi_t^{(\alpha)1}(x),\psi_t^{(\alpha)2}(x),\cdots,\psi_t^{(\alpha)n}(x)) + \theta_t] \right\},$$

for $i \in N$ and $t \in \kappa$. (3.9)

Hence Theorem 3.2 follows. ■

For individual rationality to be maintained at the outset, it is required that:

$$W^{(\alpha)i}(1,x_1^0) \geq V^i(1,x_1^0), \text{ for } i \in N. \tag{3.10}$$

For individual rationality to be maintained throughout all the stages $k \in \kappa$, it is required that:

$$W^{(\alpha)i}(k,x_k^{(\alpha)}) \geq V^i(k,x_k^{(\alpha)}), \text{ for } i \in N \text{ and } k \in \kappa. \tag{3.11}$$

If there exists an agreed-upon set of solution weights α that satisfies (3.11) the cooperative solution satisfies both individual rationality and group optimality throughout the cooperative duration.

4. An Illustration

We consider a non-transferable payoff version of the game in Yeung and Petrosyan (2010). Consider two economies which can extract a renewable resource. The planning horizon for resource extraction begins at stage 1 and ends at stage 3 for these two extractors. Let u_k^i denote the quantity of resource extracted by extractor i at stage k, for $i \in \{1,2\}$. Let U^i be the set of admissible extraction, and $x_k \in X \subset R^+$ the size of the resource stock at stage k. In particular, we have $U^i \in \mathrm{R}^+$ and $u_k^1 + u_k^2 \leq x_k$. The social plus private extraction cost for extractor i at stage k is $c_i\,(u_k^i)^2 / x_k$. The benefit of a unit of resource to economy i is P_i.

The growth dynamics of the resource is governed by the stochastic difference equation:

$$x_{k+1} = x_k + a - bx_k - \sum_{j=1}^{2} u_k^j + \theta_k, \qquad \text{for } k \in \{1,2,3\} \text{ and } x_1 = x^0, \tag{4.1}$$

where θ_k is a random variable with non-negative range $\{\theta_k^1,\theta_k^2,\theta_k^3\}$ and corresponding probabilities $\{\lambda_k^1,\lambda_k^2,\lambda_k^3\}$.

The objective of extractor $i \in \{1,2\}$ is to maximize the present value of the expected payoff:

$$E_{\theta_1\theta_2\theta_3}\left\{ \sum_{k=1}^{3} \left[P_i\, u_k^i - \frac{c_i}{x_k}(u_k^i)^2 \right] \left(\frac{1}{1+r} \right)^{k-1} \right\}, \quad \text{for } i \in \{1,2\}, \tag{4.2}$$

subject to (4.1).

The payoffs are nontransferable. Now consider the case when the extractors agree to use the weight $\alpha = (\alpha^1, \alpha^2)$ to maximize their expected weighted payoff

$$E_{\theta_1\theta_2\theta_3}\left\{ \sum_{j=1}^{2} \alpha^j \sum_{k=1}^{3} \left[P_j\, u_k^j - \frac{c_j}{x_k}(u_k^j)^2 \right] \left(\frac{1}{1+r} \right)^{k-1} \right\} \tag{4.3}$$

subject to (4.1).

Invoking Theorem 3.1, one can characterize the optimal controls in the stochastic dynamic programming problem (4.1) and (4.3). In particular, a set of control strategies $\{\psi_k^{(\alpha)i}(x)$, for $k \in \{1,2,3\}$ and $i \in \{1,2\}\}$ provides an optimal solution to the problem (4.1) and (4.3) if there exist functions $W^{(\alpha)}(k,x): R \to R$, for $k \in \{1,2,3\}$, such that the following recursive relations are satisfied:

$$W^{(\alpha)}(k,x) = \max_{u_k^1, u_k^2} \left\{ \sum_{j=1}^{2} \alpha^j \left[P_j\, u_k^j - \frac{c_j}{x}(u_k^j)^2 \right] \left(\frac{1}{1+r} \right)^{k-1} \right.$$

$$\left. + \sum_{y=1}^{3} \lambda_k^y\, W^{(\alpha)}[k+1,\ x + a - bx - \sum_{j=1}^{2} u_k^j + \theta_k^y] \right\}, \quad \text{for } k \in \{1,2,3\}.$$

$$W^{(\alpha)}(T+1, x) = 0. \tag{4.4}$$

Performing the indicated maximization in (4.4) yields the optimal cooperative strategies:

$$\psi_k^{(\alpha)i}(x) =$$

$$\left(P_i - \frac{1}{\alpha^i} \sum_{y=1}^{3} \lambda_k^y\, W_{x_{k+1}}^{(\alpha)}[k+1,\ x + a - bx - \sum_{j=1}^{2} \psi_k^{(\alpha)j}(x) + \theta_k^y]\, (1+r)^{k-1} \right) \frac{x}{2c_i},$$

for $i \in \{1,2\}$ and $k \in \{1,2,3\}$. (4.5)

Proposition 4.1. The value function

$$W^{(\alpha)}(k,x) = [A_k^{(\alpha)} x + C_k^{(\alpha)}], \text{ for } k \in \{1,2,3\}, \tag{4.6}$$

where $A_k^{(\alpha)}$ and $C_k^{(\alpha)}$, for $k \in \{1,2,3\}$, are constants given in (A.4), (A.9) and (A.12) in Appendix A.

Proof. See Appendix A. ■

Using (4.5) and Proposition 4.1, the optimal cooperative strategies of the agents can be expressed as:

$$\psi_k^{(\alpha)i}(x) = [P_i - \frac{A_{k+1}^{(\alpha)}(1+r)^{k-1}}{\alpha^i}] \frac{x}{2c_i}, \quad \text{for } i \in \{1,2\} \text{ and } k \in \{1,2,3\}. \tag{4.7}$$

Invoking Theorem 3.2 we can characterize the value function $W^{(\alpha)i}(k,x)$ which indicates the expected payoff to player i under cooperation by:

$$W^{(\alpha)i}(k,x)=\left[\; P_i\;[P_i\;-\frac{A_{k+1}^{(\alpha)}(1+r)^{k-1}}{\alpha^i}]\frac{x}{2c_i}-[P_i\;-\frac{A_{k+1}^{(\alpha)}(1+r)^{k-1}}{\alpha^i}]^2\frac{x}{4c_i}\;\right]\left(\frac{1}{1+r}\right)^{k-1}$$

$$+\sum_{y=1}^{3}\lambda_k^y\,W^{(\alpha)i}\left(\;k+1,\;x+a-bx-\sum_{j=1}^{2}\;[P_j\;-\frac{A_{k+1}^{(\alpha)}(1+r)^{k-1}}{\alpha^j}]\frac{x}{2c_j}+\theta_k^y\;\right),$$

for $k\in\{1,2,3\}$,

$$W^{(\alpha)i}(4,x)=0. \tag{4.8}$$

Proposition 4.2. The value function

$$W^{(\alpha)i}(k,x)=[A_k^{(\alpha)i}x+C_k^{(\alpha)i}],\text{ for } k\in\{1,2,3\}\text{ and } i\in[1,2], \tag{4.9}$$

where $A_k^{(\alpha)i}$ and $C_k^{(\alpha)i}$, for $k\in\{1,2,3\}$ and $i\in[1,2]$, are constants given in (B.2), (B.3) and (B.4) of Appendix B.

Proof. See Appendix B. ■

5. Concluding Remarks

This article develops a mechanism for the derivation of individual player's payoff functions in cooperative stochastic dynamic games with nontransferable payoffs. The analysis can be readily applied to cooperative dynamic games by removing the stochastic elements. Further applications of the results in discrete-time dynamic games are expected.

Appendix A: Proof of Proposition 4.1.

Consider first the last stage, that is stage 3. Invoking that $W^{(\alpha)}(3,x)=\;[A_3^{(\alpha)}x+C_3^{(\alpha)}]$ from Proposition 4.1 and $W^{(\alpha)}(4,x)=0$, the conditions in equation (4.4) become

$$W^{(\alpha)}(3,x)=[A_3^{(\alpha)}x+C_3^{(\alpha)}]=\max_{u_3^1,u_3^2}\left\{\;\sum_{j=1}^{2}\alpha^j\left[P_j\,u_3^j-\frac{c_j}{x}(u_3^j)^2\right]\left(\frac{1}{1+r}\right)^2\;\right\}. \tag{A.1}$$

Performing the indicated maximization in (A.1) yields the optimal cooperative strategies in stage 3 as:

$$\psi_3^{(\alpha)i}(x)=\frac{P_i\,x}{2c_i},\qquad\text{for } i\in\{1,2\}. \tag{A.2}$$

Substituting (A.2) into (A.1) yields:

$$[A_3^{(\alpha)}x+C_3^{(\alpha)}]=\left(\frac{1}{1+r}\right)^2\sum_{j=1}^{2}\alpha^j\frac{(P_j)^2}{4c_j}x,\qquad\text{for } i\in\{1,2\}, \tag{A.3}$$

where $A_3^{(\alpha)} = \left(\frac{1}{1+r}\right)^2 \sum_{j=1}^{2} \alpha^j \frac{(P_j)^2}{4c_j}$ and $C_3^{(\alpha)} = 0$. (A.4)

Now we proceed to stage 2, the conditions in equation (4.4) become

$$W^{(\alpha)}(2,x) = [A_2^{(\alpha)} x + C_2^{(\alpha)}] = \max_{u_2^1, u_2^2} \left\{ \sum_{j=1}^{2} \alpha^j \left[P_j u_2^j - \frac{c_j}{x} (u_2^j)^2 \right] \left(\frac{1}{1+r}\right) + \sum_{y=1}^{3} \lambda_2^y \, W^{(\alpha)}[3, x + a - bx - \sum_{j=1}^{2} u_2^j + \theta_2^y] \right\}. \quad \text{(A.5)}$$

Invoking (A.3), the condition in (A.5) can be expressed as:

$$[A_2^{(\alpha)} x + C_2^{(\alpha)}] = \max_{u_2^1, u_2^2} \left\{ \sum_{j=1}^{2} \alpha^j \left[P_j u_2^j - \frac{c_j}{x} (u_2^j)^2 \right] \left(\frac{1}{1+r}\right) + \sum_{y=1}^{3} \lambda_2^y \, A_3^{(\alpha)}[x + a - bx - \sum_{j=1}^{2} u_2^j + \theta_2^y] \right\}. \quad \text{(A.6)}$$

Performing the indicated maximization in (A.6) yields the optimal cooperative strategies in stage 2 as:

$$\psi_2^{(\alpha)i}(x) = [P_i - \frac{(1+r)}{\alpha^i} A_3^{(\alpha)}] \frac{x}{2c_i}, \qquad \text{for } i \in \{1,2\}. \quad \text{(A.7)}$$

Substituting (A.7) into (A.6) yields:

$$[A_2^{(\alpha)} x + C_2^{(\alpha)}], \quad \text{(A.8)}$$

where

$$A_2^{(\alpha)} = \sum_{j=1}^{2} \alpha^j [P_j - \frac{(1+r)}{\alpha^j} A_3^{(\alpha)}] \frac{P_j + (1+r) A_3^{(\alpha)} / \alpha^j}{4c_j} + A_3^{(\alpha)} (1-b) - \sum_{j=1}^{2} [P_j - \frac{(1+r)}{\alpha^j} A_3^{(\alpha)}] \frac{A_3^{(\alpha)}}{2c_j}, \text{ and}$$

$$C_2^{(\alpha)} = A_3^{(\alpha)} \left(a + \sum_{y=1}^{3} \lambda_2^y \theta_2^y \right). \quad \text{(A.9)}$$

Finally, we proceed to the first stage, using (A.8) the conditions in equation (4.4) become

$$[A_1^{(\alpha)} x + C_1^{(\alpha)}] = \max_{u_1^1, u_1^2} \left\{ \sum_{j=1}^{2} \alpha^j \left[P_j u_1^j - \frac{c_j}{x} (u_1^j)^2 \right] + \sum_{y=1}^{3} \lambda_1^y \left(A_2^{(\alpha)}[x + a - bx - \sum_{j=1}^{2} u_1^j + \theta_1^y] + C_2^{(\alpha)} \right) \right\}. \quad \text{(A.10)}$$

Performing the indicated maximization in (A.10) yields the optimal cooperative strategies in stage 1 and upon substitution of the optimal strategies into (A.10) yields:

$$[A_1^{(\alpha)} x + C_1^{(\alpha)}], \quad \text{(A.11)}$$

where

$$A_1^{(\alpha)} = \sum_{j=1}^{2} \alpha^j (P_j - \frac{A_2^{(\alpha)}}{\alpha^j}) \frac{P_j + A_2^{(\alpha)}}{4c_j} + A_2^{(\alpha)} (1-b) - \sum_{j=1}^{2} (P_j - \frac{A_2^{(\alpha)}}{\alpha^j}) \frac{A_2^{(\alpha)}}{2c_j}, \text{ and}$$

$$C_1^{(\alpha)} = A_2^{(\alpha)} \left(a + \sum_{y=1}^{3} \lambda_1^y \theta_1^y \right) + C_2^{(\alpha)}. \tag{A.12}$$

Hence Proposition 4.1 follows. *Q.E.D.*

Appendix B: Proof of Proposition 4.2.

From (4.8) we have:

$$W^{(\alpha)i}(k,x) = \left[P_i \; [P_i - \frac{A_{k+1}^{(\alpha)}(1+r)^{k-1}}{\alpha^i}] \frac{x}{2c_i} - [P_i - \frac{A_{k+1}^{(\alpha)}(1+r)^{k-1}}{\alpha^i}]^2 \frac{x}{4c_i} \right] \left(\frac{1}{1+r} \right)^{k-1}$$
$$+ \sum_{y=1}^{3} \lambda_k^y \, W^{(\alpha)i} \left(k+1, \; x + a - bx - \sum_{j=1}^{2} [P_j - \frac{A_{k+1}^{(\alpha)}(1+r)^{k-1}}{\alpha^j}] \frac{x}{2c_j} + \theta_k^y \right),$$

for $k \in \{1,2,3\}$.

$$W^{(\alpha)i}(T+1,x) = 0. \tag{B.1}$$

Following the analysis in the proof of Proposition 4.1 in Appendix A one can readily obtain:

$$A_3^{(\alpha)i} = \left(\frac{1}{1+r} \right)^2 \frac{(P_i)^2}{4c_i} \quad \text{and} \quad C_3^{(\alpha)i} = 0; \tag{B.2}$$

$$A_2^{(\alpha)i} = [P_i - \frac{(1+r)}{\alpha^i} A_3^{(\alpha)}] \frac{P_i + (1+r) A_3^{(\alpha)} / \alpha^i}{4c_i}$$
$$+ A_3^{(\alpha)i} (1-b) - \sum_{j=1}^{2} [P_j - \frac{(1+r)}{\alpha^j} A_3^{(\alpha)}] \frac{A_3^{(\alpha)i}}{2c_j}, \text{ and}$$

$$C_2^{(\alpha)i} = A_3^{(\alpha)i} \left(a + \sum_{y=1}^{3} \lambda_2^y \theta_2^y \right); \tag{B.3}$$

$$A_1^{(\alpha)i} = (P_i - \frac{A_2^{(\alpha)}}{\alpha^i}) \frac{P_i + A_2^{(\alpha)}}{4c_i} + A_2^{(\alpha)i} (1-b) - \sum_{j=1}^{2} (P_j - \frac{A_2^{(\alpha)}}{\alpha^j}) \frac{A_2^{(\alpha)i}}{2c_j}, \text{ and}$$

$$C_1^{(\alpha)i} = A_2^{(\alpha)i} \left(a + \sum_{y=1}^{3} \lambda_1^y \theta_1^y \right) + C_2^{(\alpha)i}. \tag{B.4}$$

Q.E.D.

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Received: August 1, 2013