

On the Subdifferentials of Quasiinvex and Pseudoinvex Functions and Cyclic Inmonicity

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Abstract: The subdifferential characteristic of quasiinvex and pseudoinvex functions are vital role in convex optimization literature. Inmonicity and cyclic inmonicity properties are equally on the same line. In this paper, we studied the relation among subdifferential characteristics, inmonicity, and cyclic inmonicity of quasiinvex and pseudoinvex functions.

Keywords: convex functions; invex functions; subdifferentials; inmonicity; cyclic inmonicity; monotone functions

1 Introduction

Let X be a Banach space $f : X \rightarrow R \cup \{+\infty\}$ and a lower semi continuous (lsc) function. Recently Correa, Joffre and Thibault (1992), recently showed that { Correa et al. (1992) for reflexive and Correa et al. (1994) for arbitrary banach spaces.}, the function f is convex if and only if its Clarke-Rockafeller sub differential ∂f is monotone. Much work has been done to characterize the generalized convexity of lsc functions by a corresponding monotonicity of the subdifferential.

Luc (1993) and Ansel, Corvellec and Lassonde (1994), showed that f is quasiconvex if and only if ∂f is quasimonotone. Penot and Quang (1997) showed that if the function f is also radially continuous, then f is pseudo convex iff ∂f is pseudomonotone (in view of Karamardian and Schaible (1990), as generalized for multivalued operators by Yao (1994).

The concept of invexity is introduced by Hanson (1981), and showed that the Kuhn-Tucker conditions are sufficient for (global) optimality of non-linear programming problems under invexity conditions. Kaul and Kaur (1982) discussed strictly pseudo invex, quasi invex and their applications in nonlinear programming Weir and Mond (1988) introduced the concept of preinvex functions and its application and established the sufficient optimality conditions and duality in non-linear programming. Pini (1991) introduced prepsuedoinvex and prequasiinvex functions with the relationship between invexity and generalized convexity. Mohan and Neogy (1995) showed that under certain condition an invex function is preinvex and a quasiinvex function is prequasiinvex. Mukherjee and Mishra [13], Mishra and Rueda (2002) have studied the different aspects of invex functions.

It is well known that the convexity of real valued function is equivalent to the monotonicity of corresponding gradient function. So, one can say monotonicity has played a very key role for existence and solution methods of variation inequality problems. Different aspects has been dealt by various authors, for more detail readers may consult some references, Noor (2003), Zeidler, E. (1990, 1985) Zalmai, G.J. (1995) Smart and Mond (1990).

Recently, X.M. Yang, X.Q. Yang and K.L. Teo (2003) used the word inmon for invariant monotone (pointed out that originally used by Prof B.D. Craven, like invex for invariant-convex), and introduced the various properties like generalized quasi, pseudo and strict inmonicity of invex function and its differentials. Several examples have been shown and established that these generalized inmonities are proper generalization of the corresponding generalized monotonicities.

Since the Clarke-Rockafellar subdifferential of a convex function coincides with the classical Fenchel-Moreau subdifferential (Rockafellar, 1980), it is not only monotone, but also cyclically monotone (R. Phelps, 1991), cyclic (generalized) monotonicity describes the behaviour of an operator around a cycle consisting of a finite number of points. V.L. Levin

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(1995) established the relation among quasi-convex function, quasi-monotone operator and their cyclic quasi monotonicity of their Gateaux differentials . Daniilidis and Hadjisavvas (1999), reviewed the previous results and showed that in the most cases the radial continuity assumption is not necessary . They define analogues notions of cyclic quasimonotonicity and cyclic pseudomonotonicity and showed that the subdifferential of quasimonotone and pseudomonotone function have cyclical properties, respectively. Cyclic monotonicity is not only stronger property than the corresponding generalized monotonicity but it reveals specific property .In particular ,an operator can even be strongly monotone without being cyclically quasimonotone .They also considered the convex hull of such a cycle and showed successfully that the definitions of monotone and pseudomonotone operators can be equivalently stated in terms of this convex hull. It is not so far quasimonotone operator, hence introduced properly quasimonotone operator and showed that it is often easier to handle while retaining the important characteristics of quasimonotonicity (In particular, f is quasi convex if and only if ∂f is properly quasimonotone). They stated that it is closely related to the KKM property of multivalued maps and showed by an application to variational inequalities. Also stated that the quasimonotonicity and proper quasimonotonicity are identical on one-dimensional spaces ,which is probably the reason due to which it has escaped attention.

In this paper, the total attention has been paid to study the relation among the invex functions and their inmonicity and also cyclic inmonicity. In section 2, we review these relations, together with some notions and definitions between generalized invexity and generalized inmonicity .In section 3, we define analogues notions of cyclic quasimonotonicity and cyclic pseudomonotonicity and showed that the subdifferential of quasimon and pseudomon function have these properties, respectively. Cyclic generalized inmonicity is not just a stronger property than the corresponding generalized inmonicity but it expresses a behavior of a specific kind. In section 4, we showed that the definitions of inmon and pseudomon operators can be equivalently stated in terms of convex hull.

2 Relations between Generalized Invexity and Generalized Inmonicity

Denote X^* the dual of X and by $(f(x), x)$, the value of $f(x) \in X^*$ at $x \in X$. For $x, y \in X$, set $[x, y] = \{y + \lambda\eta(x, y) : 0 \leq \lambda \leq 1\}$ and define $(x, y), [x, y)$ and (x, y) analogously. Given a lsc function $f : X \rightarrow R \cup \{+\infty\}$ with domain $dom f = \{x \in X : f(x) < +\infty\} \neq \emptyset$, the Clarke-Rockafellar generated derivative of f at $x_0 \in dom(f)$ in the direction of $t \in X$ is given(see[21]).

$$f^\uparrow(x_0, t) = \sup_{\epsilon > 0} \limsup_{\substack{x \rightarrow x_0 \\ \lambda \searrow 0}} \sup_{t' \in B_\epsilon(t)} \inf_{\lambda} \frac{f(x + \lambda t') - f(x)}{\lambda} \tag{2.1}$$

where $B_\epsilon(t) = \{t' \in X : \|t' - t\| < \epsilon\}$, $\lambda \searrow 0$ indicates the fact that $\lambda > 0$ and $\lambda \rightarrow 0$ and $x \rightarrow x_0$ means that both $x \rightarrow x_0$ and $\rightarrow f(x_0)$.

The Clarke-Rockafellar subdifferential of f at x_0 is defined by

$$\partial f(x_0) = \{z \in X^* : (z, t) \leq f^\wedge(x_0, t) \forall t \in X\} \tag{2.2}$$

We recall that a function f is called pre quasiinvex, if for any $x, y \in X \subseteq R^n$ be an invex set with respect to $\eta : R^n \times R^n \rightarrow R^n$ and $f : X \rightarrow R$ and $\lambda \in [0, 1]$.

$$f(y + \lambda\eta(x, y)) \leq \max\{f(x), f(y)\} \tag{2.3}$$

A lsc function f is called pseudoinvex , if for every $x, y \in w \subseteq R^n$ (X is an invex set), the following implication holds. There exists

$$z \in \partial f(x) : (z, \eta(x, y)) \geq 0 \Rightarrow f(x) \leq f(y) \tag{2.4}$$

It is known , Penot and Quang (1997) that a lsc pseudoconvex function which is also radially continuous (i.e. its restriction to line segments is continuous) is quasiconvex. Both prequasiinvexity and pseudoinvexity of functions are often used in optimization and other areas of applied mathematics when an invexity assumption would be too restrictive. For convexity see Penot and Quang (1997) .

Let $f : X \rightarrow 2^{X^*}$ be a multivalued operator with domain $D(f)$ an invex set. $D(f) = \{x \in X : f(x) \neq \emptyset\}$, the operator f is called

(1) Inmon, if for any $x, y \in X$ (an invex set)

$$\eta(x, y)^T f(y) + \eta(y, x)^T f(x) \leq 0 \tag{2.5}$$

(2) Pseudoimmon, if for any $x, y \in X$ the following implication holds:

$$\eta(y, x)^T f(x) \geq 0 \Rightarrow \eta(x, y)^T f(y) \leq 0 \tag{2.6a}$$

$$\eta(y, x)^T f(x) > 0 \Rightarrow \eta(x, y)^T f(y) < 0 \tag{2.6b}$$

(3) Quasiimmon, if for any $x, y \in X$ the following implication holds:

$$\eta(y, x)^T f(x) > 0 \Rightarrow \eta(x, y)^T f(y) \leq 0 \tag{2.7}$$

we recall, the known results connecting generalized convexity with generalized monotonicity.

Theorem 1 Let $f : X \rightarrow R \cup \{+\infty\}$ be a lower semi-continuous function. Then

- (1) f is convex if and only if ∂f is monotone in this case of is also cyclically monotone (R.Phelps,1991).
- (2) f is quasiconvex if and only if ∂f is quasimonotone [Luc(1993),Anssel et al. (1994)].
- (3) Let f be also radically continuous, then f is pseudoconvex if and only if ∂f is pseudomonotone [Penot and Quang (1997),Anssel (1998)].

On the some how same line [Yang et al.(2003)] showed some facts for a real-valued function.

Theorem 2 Let W be an invex set with respect to $\eta : X \times X \rightarrow R^n$, be a vector-valued function .Let f and η satisfy assumption A and C, respectively and f is differentiable on W .

- (i) Then f is preinvex function with respect to the same η on W , if and only if ∇f is inmon with respect to η on W .
- (ii) Then f is prequasiinvex with respect the to same η on W if and only if ∇f is quasiimmon with respect to same η on W .
- (iii) Then f is pseudoinvex with respect to η on W if and only if ∇f is pseudoimmon with respect to η on W .

Assumption A: Let W be an invex set with respect to η and let $f : W \rightarrow R$ then

$$f(y + \lambda\eta(x, y)) \leq f(x) \text{ for any } x, y \in W$$

Assumption C : [6]. Let $\eta : X \times X \rightarrow R^n$. Then for any $x, y \in R^n$ and for any $\lambda \in [0, 1]$,

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y)$$

$$\eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y)$$

[Yang et al.(2003)] proves a lemma, by which

Lemma 3 Let f and η satisfy assumptions A and C, respectively. If the differentiable function f is pseudoinvex with respect to η on an invex set W of R^n , then f is prequasiinvex with respect to the same η on W .

We now show that pseudoinvexity of a function f implies quasiinvexity of f and pseudoimmonicity of ∂f .

Proposition 4 Let $f : X \rightarrow R \cup \{+\infty\}$ be a lsc, pseudoinvex function with invex domain. Then

- I. f is quasiinvex.
- II. ∂f is pseudoimmon.

Proof. (i) Suppose that for some $x, y \in \text{dom}(f)$ and some $z \in (x, y)$. We have $f(z) > \max\{f(x), f(y)\}$. Set $m = \max\{f(x), f(y)\}$. Since f is lower semi continuous, there exist some $\varepsilon > 0$ such that $f(z') > m$ for all $z' \in B_\varepsilon(z)$. From(2.4) it follows that the sets of local and global minimizers of the function f coincide; so the point z can not be a local minimizer, so there exist $z_1 \in B_\varepsilon(z)$ such that $f(z_1) < f(z)$. Applying Zagrodnys Mean Value Theorem to the segment $[z_1, z]$, we obtain $u \in [(z_1, z)]$, a sequence $u_n \rightarrow u$ and $u_n \in \partial f(u_n)$, such that $(u_n^*, z - u_n) > 0$. Since $z \in c_0\{x, y\}$, it follows that $(u_n^*, x_i - u_n) > 0$ for some $i \in \{1, 2\}[x_1 = x, x_2 = y]$. Using relation (2.4) we get $m \geq f(x_i) \geq f(u_n)$ and, since f is lower semi continuous, $m \geq f(u)$. This clearly contradicts the fact that $u \in B_\varepsilon(z)$.

(ii) Let $x \in \partial f(x)$ be such that $(x^*, \eta(y, x)) > 0$. By part (i), f is prequasiinvex, so applying Theorem (2)(ii). We conclude that ∂f is quasiimmon. Hence $(z^*, \eta(z, x)) \geq 0$, for all $z^* \in \partial f(z)$. Suppose to the contrary that for some $z^* \in \partial f(z)$. We have $(z^*, \eta(z, x)) = 0$, from relation (2.4), we obtain $f(x) \geq f(z)$. On the other hand, since $f^\uparrow(x; \eta(z, x)) > 0$, there exists $\varepsilon_1 > 0$, such that for some $x_n \rightarrow x, t_n \searrow 0$ and for all $z' \in B_{\varepsilon_1}(z)$. We have, quasiinvexity of f implies $f(z') > f(x_n)$, for every $z' \in B_{\varepsilon_1}(z)$. In particular $f(z') > f(x)$ (Since f is lsc), hence $f(z') > f(z)$. The latter shows that z is a local minimizer, hence a global one. This is a contradiction, since we have at least $f(z) > f(x_n)$. ■

Proposition 5 Let f be a lsc function such that ∂f is pseudoinmon . Then f has the following properties :

- (i) If $0 \in \partial f(x)$ then x is a global minimizer.
- (ii) There exist $x^* \in \partial f(x) : (x^*, \eta(y, x)) > 0 \Rightarrow f(y) > f(x)$.

Proof. (i) Suppose that $f(y) < f(x)$. Then using Zagrodny’s Mean Value Theorem, we can find $z_n \rightarrow z \in [y, x]$ and $z^* \in \partial f(z_n)$, such that $(z_n^*, \eta(x, z_n)) > 0$. By pseudoinmonicity, $(x^*, \eta(x, z_n)) > 0$ for all $x^* \in \partial f(x)$ i.e. $0 \notin \partial f(x)$.
 (ii) Let us assume that for some $x^* \in \partial f(x)$. We have $x^*, \eta(y, x) > 0$. We may choose $\varepsilon > 0$; such that $(x^*, \eta(y', x)) > 0$, for all $y' \in B_\varepsilon(y)$. Since ∂f is obviously quasiinmon , from theorem 2(ii), we conclude that f is prequasiinvox ; it then follows that $f(y) \geq f(x)$. Suppose to the contrary that $f(x) = f(y)$ then $f(y') \geq f(x) = f(y)$, so f has a local minimum at y , It follows that $0 \in \partial f(x)$. However ∂f is pseudoinmon, hence we should have $(y^*, \eta(y, x)) > 0$ for all $y^* \in \partial f(y)$ a contradiction. ■

3 Generalized cyclic inmonicity

We first introduce cyclic quasiinmonicity.

Definition 1 An operator $T : X \rightarrow 2^{X^*}$ is called cyclically quasiinmon, if for every $x_i \in X$ there exist an $i \in \{1, 2, \dots, n\}$ such that

$$T(x_i^*), \eta(x_{i+1}, x_i) \leq 0 \forall x_i^* \in T(x_i) \tag{3.1}$$

(Where $x_{n+1} = x_1$).

It is easy to see that a cyclically inmon operator is cyclically quasiinmon, while a cyclically quasiinmon operator is quasiinmon.

Theorem 6 Let $f : X \rightarrow R \cup \{+\infty\}$ be a lower semicontinuous function, then f is prequasiinvox if and only if cyclically quasiinmon.

Proof. In view of theorem (2)(ii), we have only to prove that if f is prequasiinvox then ∂f is cyclically quasiinmon. Assume to the contrary that there exist $x_i \in X \forall i \in \{1, 2, \dots, n\}$, and $x_i \in \partial f(x_i)$ such that $(x_i^*, \eta(x_{i+1}, x_i)) > 0$ for $i = 1, 2, \dots, n$ (Where $x_{n+1} = x_1$). It follows that $f^\uparrow(x_i^*, \eta(x_{i+1}, x_i)) > 0$. In particular, for every i there exists $\varepsilon_i > 0$, such that

$$\lim_{x'_i \searrow 0 \rightarrow x_i} \sup_{a \in B_{\varepsilon_i}(\eta(x_{i+1}, x_i))} \inf_{\frac{f(x'_i + \lambda a) - f(x'_i)}{\lambda} > \delta_i} > 0 \tag{3.2}$$

We set $\varepsilon = \min_{i=1,2,\dots,n} \varepsilon_i$ and $\delta = \min_{i=1,2,\dots,n} \delta_i$, for any $y \in B_{\frac{\varepsilon}{2}}(x_i)$ and $x'_{i+1} \in B_{\frac{\varepsilon}{2}}(x_{i+1})$, and we have $\eta(y, x'_{i+1}) \in B_\varepsilon(\eta(x'_{i+1}, x_i))$ hence we can choose $\bar{x}_{i+1} \in B_{\frac{\varepsilon}{2}}(x_i)$ and $\lambda \in (0, 1)$ such that

$$\inf_{x'_{i+1} \in B_{\frac{\varepsilon}{2}}(x_{i+1}, x_i)} \frac{f(\bar{x}'_i + \lambda_i \eta(x'_{i+1}, \bar{x}_i)) - f(\bar{x}_i)}{\lambda} > \delta > 0 \tag{3.3}$$

Equivalently,

$$f(\bar{x}'_i + \lambda_i \eta(x'_{i+1}, \bar{x}_i)) > f(\bar{x}_i) + \lambda_i \delta, \forall x'_{i+1} \in B_{\frac{\varepsilon}{2}}(x_{i+1}) \tag{3.4}$$

for $i = 1, 2, \dots, n$. Now for every i , we choose $x'_{i+1} = \bar{x}_{i+1}$, hence (3.4) will become,

$$(f(\bar{x}'_i + \lambda_i \eta(\bar{x}_{i+1}, \bar{x}_i)) > f(\bar{x}_i) + \lambda_i \delta \tag{3.5}$$

for $i = 1, 2, \dots, n$. Since f is prequasiinvox (3.5) implies that

$$f(\bar{x}_i) > f(\bar{x}'_i + \lambda_i \eta(x'_{i+1}, \bar{x}_i)) \tag{3.6}$$

for $i = 1, 2, \dots, n$, combining with (3.5) and adding for $i = 1, 2, \dots, n$. We get $0 > \delta(\sum_{i=1}^n \lambda_i)$ a contradiction. ■

Proposition 7 Every quasiinmon operator $T : R \rightarrow 2^R$ is cyclically quasiinmon.

Proof. We assume to the contrary that the operator T is quasiinmon and there exist $x_1, x_2, \dots, x_n \in R, x_i^* \in T(x_i)$, such that

$$(x_i^*, \eta(x_{i+1}, x_i)) \tag{3.7}$$

For $i = 1, 2, \dots, n$, (where $x_{n+1} = x_1$). Set $x_N = \max_{i=1,2,\dots,n} x_i$, then relation (3.7) implies that $x_N^* < 0$ on the other hand since $x_{N-1}^* < x_N$, we conclude from (3.7) that $x_{N-1}^* > 0$. Thus $(x_{N-1}^*, \eta(x_N, x_{N-1})) > 0$, while $(x_N^*, \eta(x_N, x_{N-1})) < 0$, which contradicts the definition of quasiinmonicity. ■

We now introduce cyclic pseudoinmonicity.

Definition 2 An operator $T : X \rightarrow 2^{R^*}$ is called cyclic pseudoinmon if for every x_1, x_2, \dots, x_n the following implication holds :

$$\exists i \in \{1, 2, \dots, n\}, \exists x_i^* \in T(x_i) : (x_i^*, \eta(x_{i+1}, x_i)) \geq 0 \Rightarrow \exists j \in \{1, 2, \dots, n\} \forall x_j^* \in T(x_j) : (x_j^*, \eta(x_{j+1}, x_j)) < 0 \tag{3.8}$$

where $x_{n+1} = x_1$.

One can check that every cyclically inmon operator is cyclically pseudoinmon, while every cyclically pseudoinmon operator is pseudoinmon and cyclically quasiinmon and every pseudoinmon is inmon and quasiinmon.

Theorem 8 Let $f : X \rightarrow R \cup \{+\infty\}$ be a lsc function. If f is pseudoinvex then ∂f cyclically pseudoinmon. Conversely, if ∂f is pseudoinmon and f is radially continuous, then f is pseudoinvex.

Proof.

Again we have only to show that f is pseudoinvex then ∂f is cyclically pseudoinmon. Assume to the contrary that there exist $x_1, x_2, \dots, x_n \in D(\partial f)$ and $x_i^* \in \partial f(x_i)$ such that $(x_i^*, \eta(x_{i+1}, x_i)) \geq 0$ for $i = 1, 2, \dots, n$, where $x_{n+1} = x_1$ while for some j and $(x_j^* \in \partial f(x_j)) > 0$ we have,

$$(x_j^*, \eta(x_{j+1}, x_j)) > 0 \tag{3.9}$$

By the definition of pseudoinvexity (relation (2.4)). We have $f(x_{i+1}) > f(x_i)$, for $i = 1, 2, \dots, n$ hence all $f(x_i)$ are equal. In particular, $f(x_{j+1}) = f(x_j)$, which contradicts (3.9) in view of proposition 5. ■

4 Proper Quasiinmonicity

The definition of inmonicity and pseudoinmonicity have an equivalent formulation , which involves a finite cycle of points and its convex null when $\eta(x, y) = x - y$.

Proposition 9 (i) An operator T is inmon if and only if for any $x_1, x_2, \dots, x_n \in X$ and every $z_k = x_{k-1} + \eta(x_k, x_{k-1})$ then $Z = \sum_{k=1}^n \lambda_k Z_k$ with $\sum_{k=1}^n \lambda_k = 1$ and $\lambda_k > 0$, one has

$$\sum_{k=1}^n \lambda_k \sup_{x_k^* \in T(x_k)} (x_k^*, \eta(z, x_k)) \leq 0 \tag{4.1}$$

(ii) an operator T with invex domain $D(T)$ is pseudoinmon if and only if for any $x_1, x_2, \dots, x_n \in X$ and every $\sum_{k=1}^n \lambda_k (x_{k-1} + \eta(x_k, x_{k-1}))$, with $\sum_{k=1}^n \lambda_k = 1$ and $\lambda_k > 0$ the following implication holds

$$\exists k \in \{1, 2, \dots, n\}, \exists x_k^* \in T(x_k) : (x_k^*, \eta(z, x_i)) > 0 \Rightarrow \exists j \in \{1, 2, \dots, n\} \forall x_j^* \in T(x_j) : (x_j^*, \eta(z, x_j)) < 0 \tag{4.2}$$

Proof.

If the operator T satisfies condition (4.1) resp. (4.2) then by choosing $z = x_2 + \frac{1}{2}\eta(x_1, x_2)$, we conclude that it is inmon (resp. pseudoinmon). Hence it remains to show the two opposite direction. Let us first suppose that T is inmon then for any $x_1, x_2, \dots, x_n \in X$, any $x_k^* \in T(x_k)$ (for $k = 1, 2, 3, \dots, n$) and any $\sum_{k=1}^n \lambda_k z_k$ with $\sum_{k=1}^n \lambda_k = 1$ and $\lambda_k > 0$, we have $\sum_{k=1}^n \lambda_k (x_k^*, \eta(z, x_k)) = \sum_{k=1}^n \sum_{j=1}^n \lambda_k \lambda_j (x_k^*, \eta(x_j, x_k)) \Rightarrow \sum_{k>1} \lambda_k \lambda_j [(x_k^*, \eta(x_j, x_k)) + (x_k^*, \eta(x_k, x_j))] \leq 0$. Where the last inequality is a consequences of the inmonicity of T . Hence T satisfied (4.1).

We now suppose that the operator T is pseudoinmon if relation (4.2) does not hold, then there exist $x_1, x_2, \dots, x_n \in X$, any $x_k^* \in T(x_k)$ (for $k = 1, 2, 3, \dots, n$) and some $z = \sum_{k=1}^n \lambda_k z_k = \sum_{k=1}^n \lambda_k (x_{k-1} + \eta(x_k, x_{k-1}))$ with $\sum_{k=1}^n \lambda_k = 1$ and $\lambda_k > 0$, such that

$$(x_k^*, \eta(z, x_k)) \geq 0 \quad (4.3)$$

While for at least one k say ($k = 1$)

$$x_1^*, \eta(z, x_1) > 0 \quad (4.4)$$

In particular, we have $x_1, x_2, \dots, x_n \in D(T)$, hence $T(z) \neq \emptyset$, choose any $z^* \in T(z)$ relation (2.6 a and 4.3) show that

$$z^*, \eta(z, x_k) \geq 0 \quad (4.5)$$

Now, for all $z^* \in T(z)$ and all k 's, since $\sum_k \lambda_k (z^*, \eta(z, x_k)) = 0$, the relation (4.5) shows that for all k 's $(z^*, \eta(z, x_k)) = 0$. On the other hand, relation (4.4) together with relation (2.6 b) imply that $(z^*, \eta(z, x_k)) > 0$, a contradiction. ■

In view of proposition (9), one could speak an equivalent formulation for the definition of quasiinmonicity which would involve again the convex hull of a finite cycle, however in contrast to inmon and pseudoinmon operator leads to a different more restrictive definition.

Definition 3 An operator $T : X \rightarrow 2^{X^*}$ is called properly quasiinmon if for every $x_1, x_2, \dots, x_n \in X$, and every $z = \sum_{k=1}^n \lambda_k z_k$, where $z_k = (x_{k-1} + \eta(x_k, x_{k-1}))$ with $\sum_{k=1}^n \lambda_k = 1$ and $\lambda_k > 0$. Then there exists j such that

$$\forall x_j^* \in T(x_j^*) : (x_j^*, \eta(z, x_j)) \leq 0 \quad (4.6)$$

Choosing $z = (x_1 + \frac{1}{2}\eta(x_2, x_1))$, we see that a properly quasiinmon operator is quasiinmon as in proposition (7), it is easy to show that the converse is true whenever $X = R$, however it is not true in general, as the following example shows

Example 1

Let $\lambda = k^2, x_1 = (0, 1), x_2 = (0, 0), x_3 = (1, 0)$, we define $T : R^2 \rightarrow R^2$ by $T(x_1) = (-1, -1), T(x_2) = (1, 0), T(x_3) = (0, 1)$ and $T(x) = 0$ properly quasiinmon. It satisfies to consider $z = \sum \frac{1}{3} z_3, z_3 = x_2 + \frac{1}{3}\eta(x_3, x_2), z_2 = x_1 + \frac{1}{3}\eta(x_2, x_1), z_1 = x_3 + \frac{1}{3}\eta(x_1, x_3)$. Not mention as $\frac{1}{3}(\frac{1}{3}x_3 + \frac{2}{3}x_2 + \frac{1}{3}x_2 + \frac{2}{3}x_1 + \frac{1}{3}x_1 + \frac{2}{3}x_3)$ as $z_n = z_1$.

The class of properly quasiinmon operators though strictly smaller than the class of quasiinmon operators is in a sense not much smaller this is shown in the next proposition.

Proposition 10 (i) Every pseudoinmon operator with invex domain is properly quasiinmon.

(ii) Every cyclically quasiinmon operator is properly quasiinmon.

Proof.

(i) This is a consequence of proposition 9(ii).

(ii) Suppose that the operator T is not properly quasiinmon then there would exist $x_1, x_2, \dots, x_n \in D(T), x_i^* \in T(x_i)$ and $z = \sum_{k=1}^n \lambda_k z_k$ with $\lambda_k > 0$ such that

$$(z^*, \eta(z, z_k)) > 0 \quad (4.7)$$

for $i = 1, 2, \dots, n$ set $z_{i(1)} = z_1$ relation (4.7) implies that $\sum_j \lambda_j (z_{i(1)}^*, \eta(z_j, z_{i(1)})) > 0$, it follows that for some $z_j \neq z_1$. We have, $(z_{i(1)}^*, \eta(z_j, z_{i(1)})) > 0$, we let $z_{i(2)} = z_j$ and apply relation (4.7) again continuing in this way we define a sequence $z_{i(1)}, z_{i(2)}, z_{i(3)}, \dots$ such that

$$(z_{i(k)}^*, \eta(z_{k+1}, z_{i(k)})) > 0 \quad (4.8)$$

for all $k \in N$. Since the set z_1, z_2, \dots, z_n is finite, so there exist $m, k \in N, m < k$ such that $z_{i(k+1)} = z_{i(m)}$ thus for the finite sequence of points $z_{i(m)}, z_{i(m+1)}, \dots, z_{i(k)}$ (4.8) holds. This means that T is not cyclic quasiinmon. ■

Combining proposition (10) and theorem (6), we get the following corollary.

Corollary 11 A lower semi continuous function f is quasiinvex if and only if ∂f is properly quasiinmon. The converse of proposition(10) does not hold for instance the operator T is defined in example(1) is properly quasiinmon (since it is inmon, hence pseudoinmon). But not cyclic quasiinmon on the other hand any differential of a continuous, quasiinvex function f is properly quasiinmon but not pseudoinmon unless f is also pseudoinvex, thus between the various generalized inmonicity properties, we consider the following strict implication hold and none other,

$$\begin{array}{c} \text{Cyclicinmon} \xrightarrow{\text{Inmon}} \\ \downarrow \text{CyclicPseudoinmon} \xrightarrow{\text{Pseudoinmon}} \\ \downarrow \text{CyclicQuasiinmon} \xrightarrow{\text{ProperlyQuasiinmon}} \\ \downarrow \text{Quasiinmon} \end{array}$$

5 Conclusion

The subdifferentials of quasiinvex and pseudoinvex functions in the view of inmonicity and cyclic inmonicity is better tool to understand the most theoretical and practical applications of convex optimization area.

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