Some Connections Between Modern and Classical Control Concepts

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This is a tutorial paper that discusses the synthesis of optimum constant-gain feedback controllers for stationary linear systems. These controllers minimize the mean value of a weighted sum of squared output error and squared input in the presence of stationary random gaussian disturbances. Symmetric root locus is shown to be a useful graphical technique for visualizing closed loop pole locations as functions of the performance index weighting parameters and the disturbance spectral densities. The main component of the optimal controller is a minimum variance observer that estimates the system state variables using a measurement of the output and a set of observer gains. These estimated states are fed back to the input with a set of optimal regulator gains. This optimal controller is interpreted here as a classical compensator. A fourth order example is used throughout the paper to help clarify the concepts.

Stationary Linear Systems With Quadratic Criteria and No Disturbances

Consider a stationary linear dynamic system with a single input $u(t)$, a single output $y(t)$, and no disturbances. Such a system may be described by giving the transfer function $Y(s)$, where

$$Y(s) \triangleq y(s)/u(s) \qquad \text{and} \qquad Y(s) = N(s)/\Delta(s). \tag{1}$$

$[y(s),\ u(s)]$ are the Laplace transforms of $[y(t),\ u(t)]$, and

$$\Delta(s) = s^n + a_1 s^{n-1} + \ldots a_n, \tag{2}$$

$$N(s) = b_1 s^{n-1} + b_2 s^{n-2} + \ldots b_n \tag{3}$$

The system may also be described in terms of arbitrary state variables

$$\dot{x} = Fx + gu; \qquad y = h^T x, \tag{4}$$

where x is a state vector with n-components $[x_1,\ \ldots,\ x_n]$, F is the dynamics matrix, g is the input distribution vector, and h is the output distribution vector. Clearly

$$Y(s) = h^T(sI - F)^{-1}g.$$

If only (1), (2), and (3) are given, a convenient state variable form is obtained by using y and its first $n - 1$ derivatives as the states, i.e.,

$$x^T = [y,\ y_1,\ \ldots,\ y_{n-1}], \tag{5}$$

where

$$y_i = d^{(i)}y/dt^{(i)}, \qquad i = 1,\ \ldots,\ n - 1. \tag{6}$$

In this case it can be shown that

$$F = \begin{bmatrix} 0 & I \\ \hline -a_n, & \ldots, \quad -a_1 \end{bmatrix}, \tag{7}$$

$$g = \begin{bmatrix} 1 & & & \bigcirc \\ a_1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{n-1} & \cdots & a_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \tag{8}$$

$$h^T = [1,\ 0,\ \ldots,\ 0]. \tag{9}$$

The form (5) is sometimes called the "observability canonical form" and the quantities g_i are called "Markov parameters."

A simple set of criteria for comparing regulator designs is as follows:

Specify the initial conditions,

$$[x_1(0),\ \ldots,\ x_n(0)] \equiv x^T(0) \tag{10}$$

Evaluate a weighted sum of the integral-square output error and the integral-square input:

$$J = {}^1\!/_2 \int_0^\infty (ay^2 + bu^2)dt \tag{11}$$

where a and b are specified constants.

The *optimal regulator* minimizes [11] subject to (1) and [10]; it involves *linear feedback* of the state vector (see e.g. [1]):

$$u = -c_1x_1 \ldots - c_nx_n \triangleq -c^Tx, \tag{12}$$

where $c^T \triangleq [c_1,\ \ldots,\ c_n]$ are the feedback gains, which are independent of the initial conditions. The closed-loop eigenvalues are determined by the gains as the roots of the characteristic equation

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$$|sI - F + gc^T| = 0, \quad (13)$$

which is obtained by substituting (12) into (2):

$$\dot{x} = (F - gc^T)x \rightarrow [sI - F + gc^T]x(s) = 0. \quad (14)$$

Symmetric Root Locus for Determining the Poles of the Optimal Regulator

It has been shown [2, 3, 4] that the closed-loop poles (or eigenvalues) of the optimal regulator are the stable (left-half plane) roots of

$$aY(s)Y(-s) + b = 0, \quad (15)$$

which we shall call the "symmetric root characteristic equation" (SRCE) because its roots are symmetric with respect to the real and imaginary axes in the s-plane. In Evans' root locus form, the SRCE is

$$-b/a = N(s)N(-s)/\Delta(s)\Delta(-s) \quad (16)$$

Thus, to plot a locus of the optimal closed-loop poles versus the parameter a/b, one "reflects" the poles and zeros of the open-loop transfer function across the imaginary axis of the s-plane and plots either the 0 deg or the 180 deg root locus, whichever one has no locus segments crossing the $j\omega$ axis (cf. [4]). We shall call this a "symmetric root locus" (SRL).

An *example* helps to clarify the concept. Consider a system consisting of the two equal masses connected by a spring (see Fig. 1), where we wish to position the right mass at $y = 0$, using a control force u, acting on the left mass. We shall take as performance index,

$$J = {}^1\!/_2 \int_0^\infty (ay^2 + bu^2)dt. \quad (17)$$

If we measure time in units of $\sqrt{m/k}$ and u in units of k, then the transfer function from u to y becomes

$$y(s)/u(s) = 1/s^2(s^2 + 2) \quad \triangleq N(s)/\Delta(s) \equiv Y(s). \quad (18)$$

The SRL equation is, therefore,

$$-b/a = 1/s^4(s^2 + 2)^2. \quad (19)$$

Fig. 2 shows the SRL versus a/b.

For $a/b = 1$, the roots of (19) are:

$$s = (\pm 0.156 \pm 1.457j), \pm(0.531 \pm 0.428j). \quad (20)$$

For $a/b = 0$, the roots of (19) are double the open-loop roots of $s = 0, 0, \pm\sqrt{2}j$ and the obvious minimum of (17) with $a = 0$, $b \neq 0$, is $u = 0$. This is a trivial result since with $a = 0$ we are not weighting the output error.

For $a/b \rightarrow \infty$, the roots of (19) are

$$s \rightarrow \pm \left(\frac{a}{b}\right)^{1/8} \left(\cos\frac{\pi}{8} \pm j\sin\frac{\pi}{8}\right),$$
$$\pm \left(\frac{a}{b}\right)^{1/8} \left(\cos\frac{3\pi}{8} \pm j\sin\frac{3\pi}{8}\right), \quad (21)$$

the so-called "Butterworth configuration" of poles. The control amplitude and bandwidth become very large; in fact, the control tends to a sum of impulses and derivatives of impulses (cf. [5]).

Determination of the Optimal Regulator Gains

For single-input, single-output systems (y and u scalars), the state feedback gains (12) may be determined by comparing coefficients of the closed-loop characteristic equation (13) with those of the desired characteristic equation, i.e., a characteristic equation whose roots are the stable (left half plane) roots of (15).

For the *example* described by equation (16),

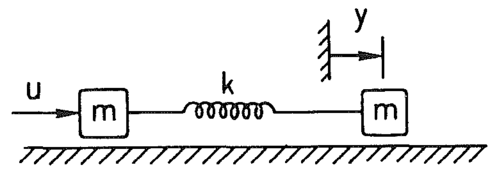

Fig. 1 Example system. Output is y, input is u.

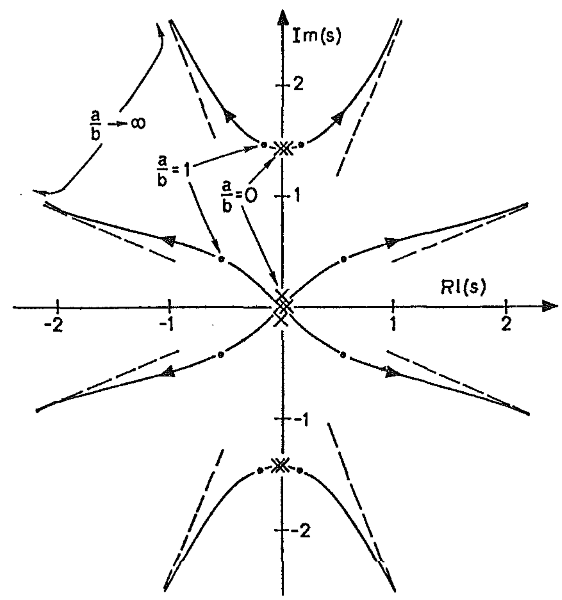

Fig. 2 Symmetric root locus of optimal regulator poles versus ratio of weighting parameters a/b

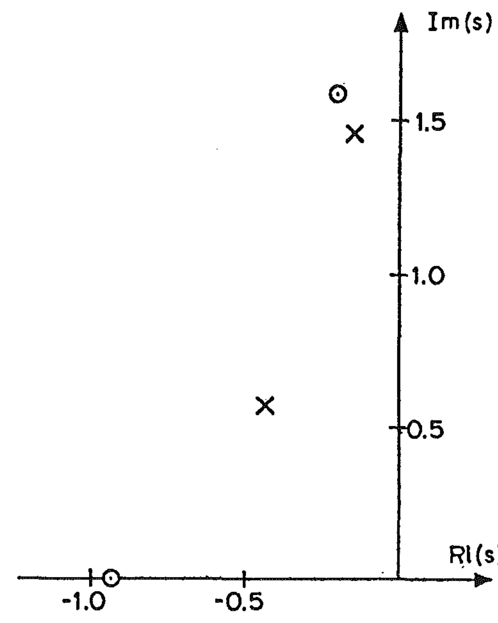

Fig. 3 Pole-zero configuration of Laplace transform of output $y(s)$ divided by Δl, where $y(0) = \Delta l$, $= \dot{y}(0) = \ddot{y}(0) = \dddot{y}(0) = 0$

$$u = -c_1y - c_2\dot{y} - c_3\ddot{y} - c_4\dddot{y}, \quad (22)$$

and the closed-loop characteristic equation (19) is given by

$$s^4 + c_4s^3 + (2 + c_3)s^2 + c_2s + c_1 = 0. \quad (23)$$

For $a/b = 1$, the desired characteristic equation is, from (20):

$$[(s + 0.156)^2 + (1.457)^2][(s + 0.531)^2 + (0.429)^2] = 0,$$

or

$$s^4 + 1.374s^3 + 2.943s^2 + 2.425s + 0.997 = 0. \quad (24)$$

Comparing coefficients of like powers of s in (23) and (24) gives:

$$c_1 = 0.997, \quad c_2 = 2.425, \quad c_3 = 0.943, \quad c_4 = 1.374. \quad (25)$$

An efficient computer program for calculating the poles and gains of the optimal regulator is described in [6].

The Laplace transform of the response of the closed-loop system to the initial conditions

$$y(0) = \Delta l,\ \dot{y}(0) = \ddot{y}(0) = \dddot{y}(0) = 0 \quad (26)$$

is shown in Fig. 3 in the form of a pole-zero plot. From the figure it is clear that the response will be primarily associated with the complex poles at $s = -0.531 \pm 0.428j$, since the other complex poles at $s = -0.156 \pm 0.457j$ are quite close to zeroes. Thus most of the output error will attenuate with a time constant of $1/0.531 \cong 1.9$ time units, but there will be a small amplitude oscillation at a frequency of 1.457 rad/time unit that will attenuate with a time constant of $1/0.156 \cong 6.4$ time units.

Minimum Variance Observers for Stationary Linear Systems With Random Disturbances

Consider the same stationary linear system (4) with stationary random disturbances added:

$$\dot{x} = Fx + gu + \gamma w \quad (27)$$

$$z = h^T x + v, \quad (28)$$

where w and v are independent zero-mean, white noise processes with constant spectral densities q and r, respectively. The initial conditions are also random with zero-mean and covariance matrix, X_0.

If the transfer function from w to y is given

$$y(s)/w(s) = N_w(s)/\Delta(s), \quad (29)$$

where

$$N_w(s) = d_1 s^{n-1} + d_2 s^{n-2} \ldots + d_n, \quad (30)$$

and y and its derivatives are used as states (cf. equation (5)), then by an analogy to (8)

$$\gamma = \begin{bmatrix} 1 & & & \\ a_1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{n-1} & \cdots & a_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} d_1 \\ \cdot \\ \cdot \\ d_n \end{bmatrix} \quad (31)$$

An *observer* or *state variable estimator* for this system has the form of (cf. [7]):

$$\dot{\hat{x}} = F\hat{x} + gu + k(z - h^T\hat{x}), \quad (32)$$

$$\hat{x}(0) = 0, \quad (33)$$

where the constant gain vector, k, is to be chosen.

The gain vector of the *minimum variance observer* is chosen so as to minimize the expected value of

$$\int_0^\infty (z - h^T\hat{x})^2 dt. \quad (34)$$

The minimum variance observer is identical to the *steady-state Kalman filter* [8].

The estimate error is defined as

$$\tilde{x} \triangleq \hat{x} - x, \quad (35)$$

By subtracting (27) from (32) and using (28) to eliminate z, we have

$$\dot{\tilde{x}} = (F - kh^T)\tilde{x} + kv - \gamma w \quad (36)$$

Thus, the eigenvalues of the estimate-error equations are determined by the gain vector k as the roots of the characteristic equation

$$|sI - F + kh^T| = 0. \quad (37)$$

Symmetric Root Locus for the Poles of the Minimum Variance Observer

The poles of the minimum variance observer (cf. [4] are the left half plane roots of

$$1/rZ(s)Z(-s) + 1/q = 0, \quad (38)$$

where

$$Z(s) \triangleq y(s)/w(s) = h^T(sI - F)^{-1}\gamma, \quad (39)$$

is the transfer function from the disturbance w to the output $y \triangleq h^T x$. Equation (38) is another symmetric root characteristic equation (SRCE) like (15). In Evans' root locus form, the SRCE is:

$$-r/q = N_w(s)N_w(-s)/\Delta(s)\ \Delta(-s), \quad (40)$$

where

$$Z(s) \triangleq N_w(s)/\Delta(s), \quad (41)$$

Obviously a *symmetric root locus* (SRL) versus the parameter q/r may be plotted using the form (40).

For the *example* introduced in (18), let use assume that a random force acts on the right mass so that (with time in units of $\sqrt{m_r/k}$, and (u, w) in units of k) the transfer function from w to y is

$$y(s)/w(s) = (s^2 + 1)/s^2(s^2 + 2) \triangleq N_w(s)/\Delta(s) \equiv Z(s). \quad (42)$$

The SRL equation is, therefore,

$$-r/q = (s^2 + 1)^2/s^4(s^2 + 2)^2 \quad (43)$$

Fig. 4 shows the SRL versus q/r. For $q/r = 1$, the roots of (43) are:

$$s = \pm(0.181 \pm 1.378j),\ \pm(0.438 \pm 0.571j). \quad (44)$$

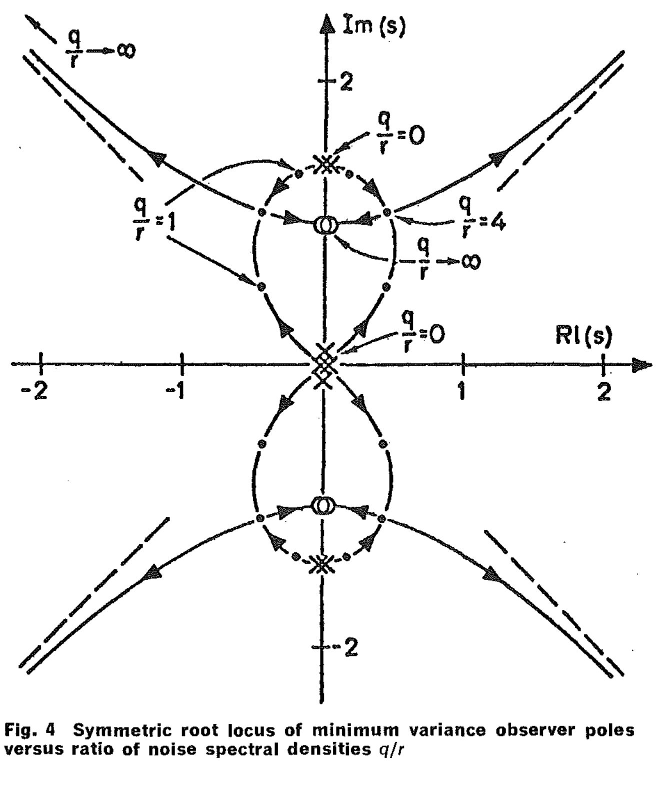

Fig. 4 Symmetric root locus of minimum variance observer poles versus ratio of noise spectral densities q/r

For $q/r \to 0$, the roots of (43) are:

$$s \to \pm(1 \pm j)^{1/2}\left(\frac{q}{r}\right)^{1/4}, \pm\left(\frac{1}{8}\sqrt{\frac{2q}{r}} \pm \sqrt{2j}\right). \quad (45)$$

Thus for $q/r = 0$, the roots are double the open-loop roots at $s = 0, 0, \pm\sqrt{2}j$ and the corresponding gain vector is $k = 0$. This is an *unsatisfactory result,* since the measurement z is not being used ($k = 0$) and the estimate-errors will not attenuate (the Kalman filter has time-varying gains that depend on X_0 and the estimate-errors do tend asymptotically to zero). In this case, we might add a *constraint* that the real parts of the estimate-error poles must be less than some negative number:

$$Rl(s_i) \leq -\sigma. \quad (46)$$

Such *constrained minimum variance observers* are discussed in [9].

For $q/r \to \infty$, the roots of (43) are

$$s \to \pm(1 \pm j)\left(\frac{q}{4r}\right)^{1/4}, \pm\left(1/2\sqrt{\frac{r}{q}} \pm j\right), \quad (47)$$

i.e., four of the poles go to the four zeros at $\pm j$, while four other poles tend to infinity in another Butterworth configuration. The observer bandwidth becomes very large; in fact, the observer tends to become partly a differentiator:

$$\left.\begin{array}{l}\hat{y} = z,\ \hat{y}_1 = \dot{z},\\ \dot{\hat{y}}_2 = \hat{y}_3,\ \dot{\hat{y}}_3 = -\hat{y}_2 - \ddot{z} + u\end{array}\right\} \text{ for } r = 0, q \neq 0. \quad (48)$$

Thus for $r/q = 0$, we again have an unsatisfactory result, since the estimate errors associated with y_2 and y_3 will not attenuate (again the Kalman filter has time-varying gains that depend on X_0 which bring all the estimate errors to zero asymptotically). In this case we might add an estimate-error eigenvalue constraint like (46), and also an upper bound on the magnitude of the eigenvalues

$$|s_i| \leq \omega. \quad (49)$$

Determination of the Minimum Variance Observer Gains

For single-disturbance single-measurement systems (w and v scalars), the observer gains may be determined by comparing coefficients of the closed-loop characteristic equation (37) with those of the desired characteristic equation, i.e., a characteristic equation whose roots are the stable (left half plane) roots of (38).

For the *example* described by equation (42), the Laplace transform of the estimate-error equation (36) is:

$$\begin{bmatrix} s+k_1 & -1 & 0 & 0 \\ k_2 & s & -1 & 0 \\ k_3 & 0 & s & -1 \\ k_4 & 0 & 2 & s \end{bmatrix}\begin{bmatrix}\tilde{y}(s)\\ \tilde{y}_1(s)\\ \tilde{y}_2(s)\\ \tilde{y}_3(s)\end{bmatrix} = \begin{bmatrix}k_1\\k_2\\k_3\\k_4\end{bmatrix} v(s) + \begin{bmatrix}0\\-1\\0\\1\end{bmatrix} w(s) \quad (50)$$

Hence the closed-loop characteristic equation (37) is:

$$s^4 + k_1 s^3 + (2 + k_2)s^2 + (2k_1 + k_3)s + 2k_2 + k_4 = 0. \quad (51)$$

For $q/r = 1$, the desired characteristic equation is, from (44):

$$[(s + 0.181)^2 + (1.378)^2][(s + 0.438)^2 + (0.571)^2] = 0,$$

or

$$s^4 + 1.238s^3 + 2.767s^2 + 1.880s + 1.000 = 0. \quad (52)$$

Comparing coefficients of like powers of s in (51) and (52) gives

$$k_1 = 0.642, \quad k_2 = 0.233, \quad k_3 = 1.238, \quad k_4 = 0.767. \quad (53)$$

An efficient computer program for calculating the poles and gains of the minimum variance observer is described in [6].

Stationary Linear Systems With Quadratic Performance Criteria and Random Disturbances

If we consider the system (27)–(28) and ask for the control history $u(t)$ that minimizes the *expected value* of

$$\int_0^\infty (ay^2 + bu^2)dt, \quad (54)$$

it has been shown (cf. [10], [11], [12], and [1]) that the minimizing solution is to *feed back the estimated state* from the Kalman filter (which, in general, has time-varying gains, $k(t)$) with the optimal regulator gains, i.e.,

$$u = -c^T\hat{x}, \quad (55)$$

$$\dot{\hat{x}} = F\hat{x} + gu + k(z - h^T\hat{x}), \quad \hat{x}(0) = 0. \quad (56)$$

If we replace the Kalman filter by the minimum variance (MV) observer (i.e., the steady-state Kalman filter), we *may* get a reasonably good sub-optimal solution. However, this is *not* the case when the MV observer has neutrally-stable eigenvalues; in this case eigenvalue constraints may be placed on the MV observer, as mentioned above equation (46); (cf. [9]).

One of the remarkable and useful facts about this "certainty-equivalence" solution is that the closed-loop system has eigenvalues that are the MV estimate-error eigenvalues plus the optimal regulator eigenvalues. This is easily seen by using x and $\tilde{x}$ as states instead of x and $\hat{x}$:

$$\begin{bmatrix}\dot{x}\\ \dot{\tilde{x}}\end{bmatrix} = \begin{bmatrix} F - gc^T & -gc^T \\ 0 & F - kh^T\end{bmatrix}\begin{bmatrix}x\\ \tilde{x}\end{bmatrix} + \begin{bmatrix}0\\k\end{bmatrix} v + \begin{bmatrix}\gamma\\ -\gamma\end{bmatrix} w, \quad (57)$$

The one-way coupling shows that the eigenvalues are those of (13) and (37).

Interpretation of Estimated-State-Feedback as an Optimal Compensation

Feeding back the estimated-state from the minimum variance observer with the optimal regulator gains gives observer equations:

$$\dot{\hat{x}} = (F - gc^T - kh^T)\hat{x} + kz. \quad (59)$$

Thus, for $v = 0$, transfer function from the output $y = z$ to the control, $u = -c^T\hat{x}$, is

$$u(s)/y(s) = -c^T(sI - F + gc^T + kh^T)^{-1}k \triangleq -T_c(s), \quad (59)$$

which may be interpreted as a classical *compensator.* Note the poles of this transfer function are neither the poles of the optimal regulator nor the poles of the minimum variance estimator.

If we multiply the right side of (59) by a scalar gain, K, then a closed-loop root locus versus K for the compensated system should have the poles of (13) and (37) when $K = 1$:

$$y(s) = Y(s)u(s), \quad (60)$$

$$u(s) = -KT_c(s)y(s), \quad (61)$$

$$\to -\frac{1}{K} = Y(s)T_c(s).$$

For the *example* with $a/b = q/r = 1$, the compensator transfer function may be determined from:

$$\begin{bmatrix} s+k_1 & -1 & 0 & 0 \\ k_2 & s & -1 & 0 \\ k_3 & 0 & s & -1 \\ k_4+c_1 & c_2 & 2+c_3 & s+c_4 \end{bmatrix} \begin{bmatrix} \hat{y}(s) \\ \hat{y}_1(s) \\ \hat{y}_2(s) \\ \hat{y}_3(s) \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} z(s), \tag{62}$$

$$u = -c_1\hat{y} - c_2\hat{y}_1, - c_3\hat{y}_2 - c_4\hat{y}_3, \tag{63}$$

which yields:

$$\frac{u(s)}{y(s)} = -\frac{1.801(s+0.233)[(s+0.010)^2+(1.545)^2]}{[(s+1.097)^2+(0.820)^2][(s+0.210)^2+(1.605)^2]} \tag{64}$$

The plant transfer function is given in equation (18). The root locus versus K for the compensated system is shown in Fig. 5. The compensator poles (x) and zeros (o) are marked with a C, the unmarked poles are those of the plant (which happens to have no zeros in this example). The black dots marked with an R are the optimal regulator poles, and the black dots marked with an E are the estimator (minimum variance observer) poles. Note the root locus versus K does pass through the black dots for $K = 1$.

This compensator has a real zero to compensate the rigid body double pole at $s = 0$ and a "notch" near the plant vibration poles at $s = \pm \sqrt{2}j$. i.e., there is both a compensator pole

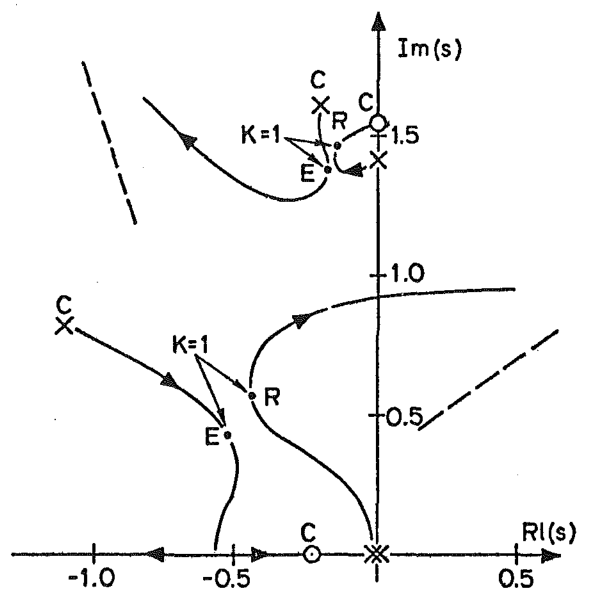

Fig. 5 Root locus versus overall gain K for compensated system, $a/b = q/r = 1$. Optimal compensation → $K = 1$.

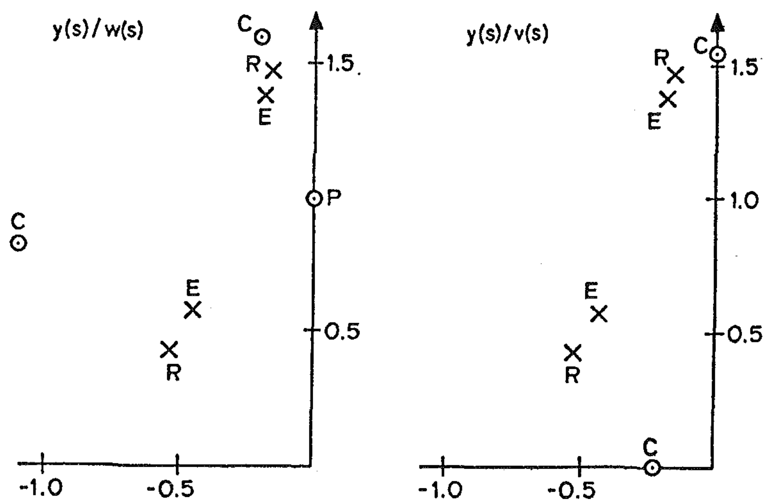

Fig. 6 Pole-zero configurations of the closed-loop transfer functions from disturbances w and v to output error y for $q/r = a/b = 1$

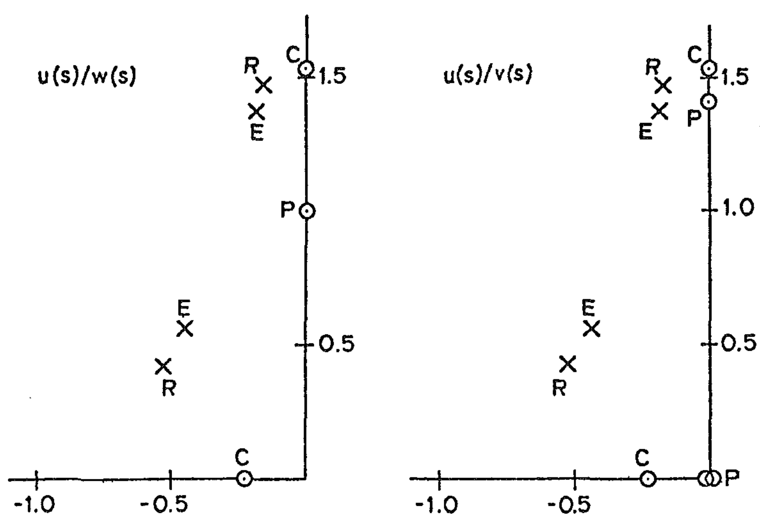

Fig. 7 Pole-zero configurations of the closed-loop transfer functions from disturbances w and v to control input u for $q/r = a/b = 1$

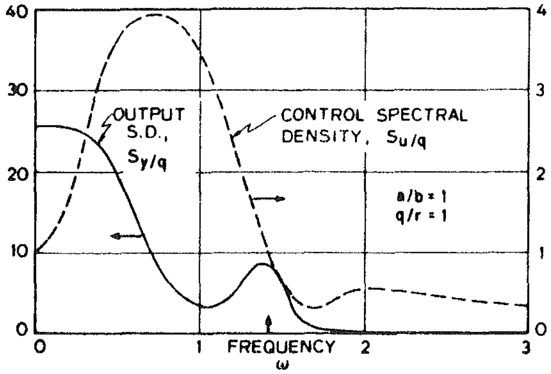

Fig. 8 Output and control spectral densities for closed-loop system with $q/r = a/b = 1$

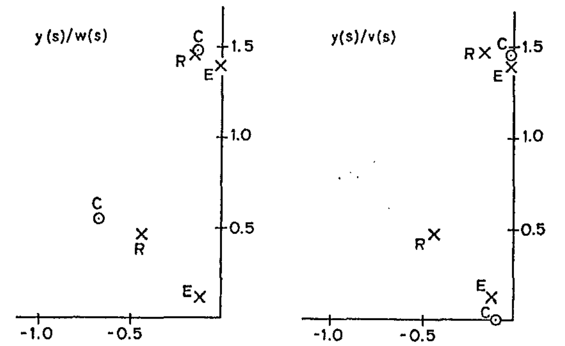

Fig. 10 Pole-zero configurations of the closed-loop transfer functions from disturbances w and v to output error y for $a/b = 1$, $q/r = 1/256$

and compensator zero quite close to the plant vibration pole. The real zero is standard in classical design. "Notch compensation" is also well known to classical designers, who know, too, that such compensation is quite sensitive to errors in estimating the plant vibration frequency!

This is one of the drawbacks of modern control synthesis, namely, it may yield sensitive compensation; small changes in the plant parameters away from their design values give degraded performance and may even produce instability (cf. e.g. [13]). Nonetheless, the pole-zero configuration of the optimal compensator may suggest unusual types of compensation; small changes in the pole-zero positions could reduce the sensitivity of the design with only small increases in the performance index (cf. e.g. [9]).

Closed Loop Response of Output and Input to Disturbances

The objective of the stochastic design problem described above equation (54) is to minimize the expected value of a weighted sum of output y squared and control u squared in the presence of random disturbances. Hence it is of interest to look at the transfer functions from the disturbances w and v to y and u for the closed-loop system.

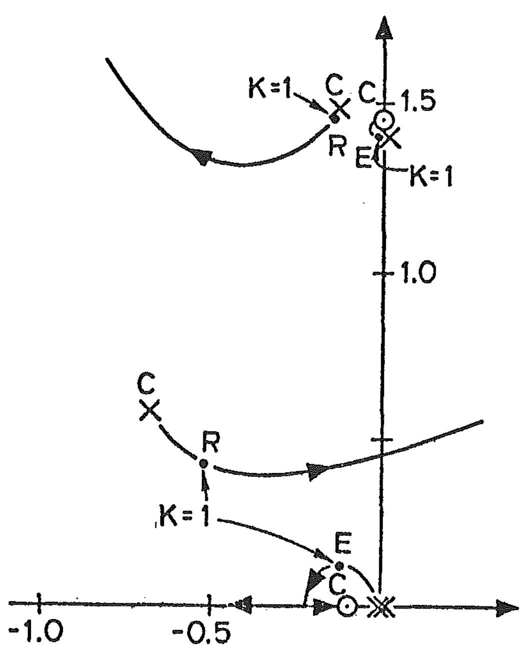

Fig. 9 Root locus versus overall gain K for compensated system, $a/b = 1$, $q/r = 1/256$. Optimal compensation → $K = 1$.

If we write the Laplace transform of the closed-loop system in the form

$$y(s) = Y(s)u(s) + Z(s)w(s), \tag{65}$$

$$u(s) = -T_c(s)[y(s) + v(s)], \tag{66}$$

then it is straightforward to show that

$$\begin{bmatrix} y(s) \\ u(s) \end{bmatrix} = (-)\frac{\begin{bmatrix} \Delta_c(s) & -N(s) \\ N_c(s) & \Delta(s) \end{bmatrix}}{\Delta_R(s)\Delta_E(s)} \begin{bmatrix} N_w(s)w(s) \\ N_c(s)v(s) \end{bmatrix} \tag{67}$$

where

$$T_c(s) = N_c(s)/\Delta_c(s), \tag{68}$$

$$\Delta_R(s) = |sI - F + gc^T|, \tag{69}$$

$$\Delta_E(s) = |sI - F + kh^T|, \tag{70}$$

and we used

$$\Delta_R(s)\Delta_E(s) \equiv \Delta(s)\Delta_c(s) + N(s)N_c(s). \tag{71}$$

Figs. 6 and 7 show the pole-zero configuration of these four transfer functions for the example problem with $q/r = 1$, $a/b = 1$. Again, poles marked with an R or an E are regulator or estimator poles, respectively (from $\Delta_R(s) = 0$ or $\Delta_E(s) = 0$). Zeros marked with a P or C are associated with plant or compensator transfer functions respectively. The figures show that the y and u responses to an impulse in w and the u response to an impulse in v are primarily at $s \cong -0.5 \pm 0.5j$ since there are two sets of complex zeros close to the two sets of complex poles near $-0.2 \pm 1.4j$. They also show that the y response to an impulse in v is at both frequencies and there will be a long-lasting response at $s \cong -0.2 \pm 1.4j$, since there is only one set of nearby complex zeros to offset the two sets of complex poles there.

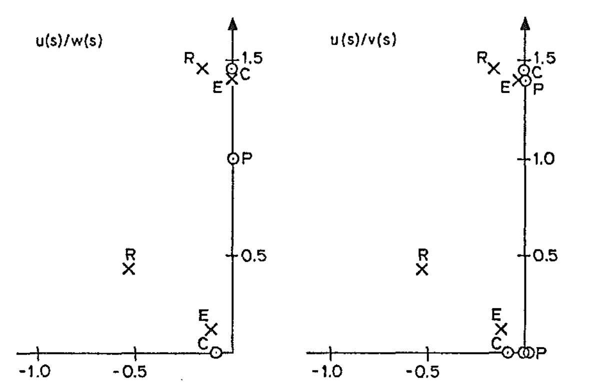

Fig. 11 Pole-zero configurations of the closed-loop transfer functions from disturbances w and v to control input u for $a/b = 1$, $q/r = 1/256$

From equation (67) the spectral densities of y and u due to white noise w and v, are given by:

$$\begin{bmatrix} S_y(\omega) \\ S_u(\omega) \end{bmatrix} = \frac{\begin{bmatrix} \Delta_c\tilde{\Delta}_c & N\tilde{N} \\ N_c\tilde{N}_c & \Delta\tilde{\Delta} \end{bmatrix}\begin{bmatrix} N_w\tilde{N}_w q \\ N_c\tilde{N}_c r \end{bmatrix}_{s=j\omega}}{\Delta_R\tilde{\Delta}_R\Delta_E\tilde{\Delta}_E} \tag{72}$$

where ($\tilde{\ }$) indicates complex conjugate and (q, r) are the spectral densities of w and v, respectively. Fig. 8 shows these two spectral densities for the example problem with $a/b = q/r = 1$. Note the control spectral density has a peak around $\omega = 0.7$, whereas the output spectral density peaks at $\omega = 0$ with a lower peak near the open-loop resonant frequency, $\omega \cong 1.4$.

Limits as Process Noise and/or Measurement Noise Tend to Zero

If the process noise v is negligible compared to the measurement noise v, then $q/r \to 0$. Fig. 9 shows the root locus versus K for the optimally compensated system for the example problem with $q/r = 1/256$, $a/b = 1$. Note the estimator poles are very close to the open-loop poles and the compensator "notch" is very close to the open-loop vibration pole; in effect the compensator zero is nearly cancelling the vibration pole and replacing it with a compensator pole that is more stable. Such a design is very sensitive to the location of the vibration pole. If the vibration frequency were uncertain, it would be prudent to move the compensator "notch" farther away from the location of the nominal vibration pole. This could be done directly or, alternatively, the observer design could be modified with an eigenvalue constraint, say $Rl(s_i) < 0.2$ (see [9]).

Figs. 10 and 11 show the pole-zero configurations of the closed-loop transfer functions from disturbances w and v to the output y and the input u with $q/r = 1/256$, $a/b = 1$. Since v is much larger than w, the optimal controller places zeros close to the lightly-damped estimator poles in $y(s)/v(s)$ and $u(s)/v(s)$ but not in $y(s)/w(s)$.

At the other limit, where the measurement noise is negligible compared to the process noise, then $q/r \to \infty$. Fig. 12 shows the root locus versus K for the optimally compensated system of the example problem with $q/r = 256$, $a/b = 1$. Note the observer bandwidth is high, which is reasonable since the measurement contains very little noise. One set of estimator poles is very close to $s = \pm j$; the reason for this is clear from Figs. 13 and 14, where the pole-zero configurations of the closed-loop transfer functions from w and v to y and u are shown. The estimator poles are placed so as to nearly cancel with the plant zeros at $s = \pm j$ in $y(s)/w(s)$ and $u(s)/w(s)$. Another set of zeros is close to the regulator poles at $s = -0.16 \pm 1.45j$, so that the dominant y response to w is at $s = -0.53 \pm 0.43j$. However, prudence would dictate that the complex estimator pole be placed slightly farther to the left of $s = \pm j$ to avoid an instability through small errors in setting the controller gains which might move these poles into the right half plane.

The pole-zero configurations of $y(s)/v(s)$ and $u(s)/v(s)$ do not look too good until one remembers that v is very small compared to w, so the controller is properly concentrating on reducing the effects of w and not worrying about v.

Another, rather academic, limit is when *both* measurement noise v and process noise w are negligible, i.e., $q = r = 0$; this is the case treated in the first section of this paper. A "compensator" interpretation of the optimal controller must treat the "observer" as a *differentiator:*

$$\begin{aligned} \hat{y} &= z \\ \hat{y}_1 &= \dot{z} \\ \hat{y}_2 &= \ddot{z} \\ \hat{y}_3 &= \dddot{z} \end{aligned} \tag{73}$$

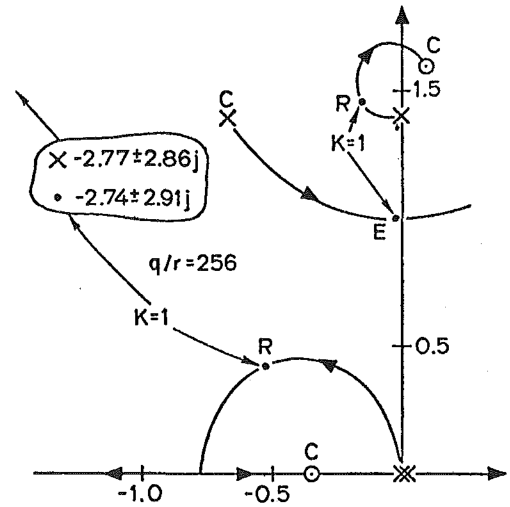

Fig. 12 Root locus versus overall gain K for compensated system, $a/b = 1$, $q/r = 256$. Optimal compensation → $K = 1$.

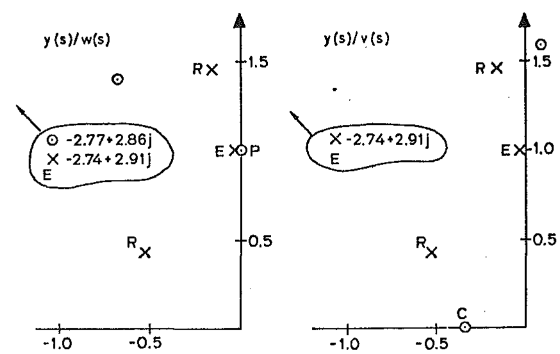

Fig. 13 Pole-zero configurations of the closed-loop transfer functions from disturbances w and v to output error y for $a/b = 1$, $q/r = 256$

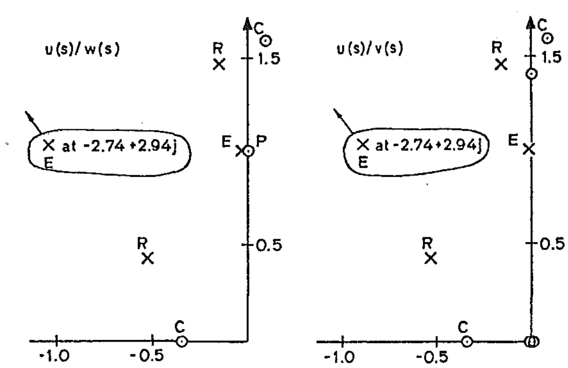

Fig. 14 Pole-zero configurations of the closed-loop transfer functions from disturbances w and v to control input u for $a/b = 1$, $q/r = 256$

Differentiating the measurements is not unreasonable if there is no noise. The control law (22) with (73) becomes

$$u = -c_1y - c_2\dot{y} - c_3\ddot{y} - c_4\dddot{y}$$

or

$$u(s) = -[c_4s^3 + c_3s^2 + c_2s + c_1]y(s) \tag{74}$$

Substituting the optimal gains (25) into (74) gives a compensator transfer function that has only zeros:

$$u(s)/y(s) = -1.374(s + 0.439)[(s + 0.124)^2 + (1.281)^2]. \tag{75}$$

This may be interpreted as a "proportional plus derivatives" feedback controller.

Multiplying this "transfer function" by K, a root locus versus K is plotted in Fig. 15 for the compensated system; $K = 1$ gives the optimal regulator poles. This plot is obviously similar to those in Figs. 5, 9, and 12.

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Conclusions

Use of *symmetric root locus* helps to visualize the closed-loop pole locations for (*a*) the optimal regulator as a function of the ratio of weighting parameters a/b in the quadratic performance index, and (*b*) the minimum variance observer as a function of the ratio of process noise spectral density to measurement noise spectral density q/r.

A good stochastic controller is often obtained by feeding back the estimated states from the minimum variance observer using the optimal regulator gains and it may be interpreted as a *classical compensator*. However, this not always the case since the optimal controller, in general, has an estimator with time-varying gains (the Kalman filter); in order to use constant gains, a minimum variance observer with eigenvalue constraints is suggested (cf. [9]).

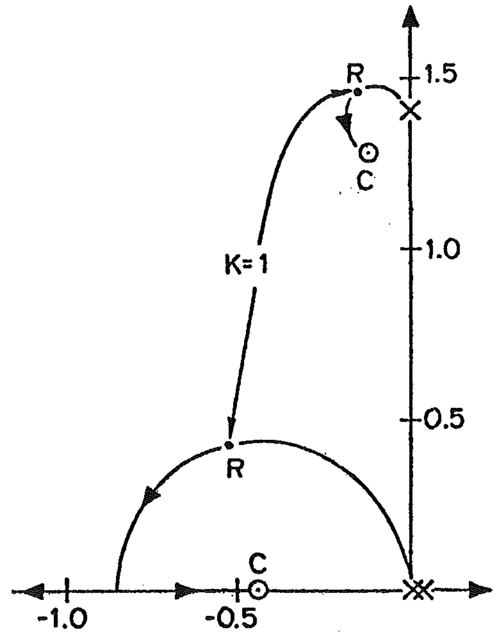

Fig. 15 Root locus versus overall gain K for compensated system, $a/b = 1$, $q = r = 0$. Optimal compensator $\rightarrow K = 1$.

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