Convergence of discretized attractors for parabolic equations on the line

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Abstract

We show that, for a semilinear parabolic equation on the real line satisfying a dissipativity condition, global attractors of time-space discretizations converge (with respect to the Hausdorff semi-distance) to the attractor of the continuous system as the discretization steps tend to zero. The attractors considered correspond to pairs of function spaces (in the sense of Babin-Vishik) with weighted and locally uniform norms (taken from Mielke-Schneider) used for both the continuous and the discrete system.

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1 Introduction

Let Σ be the evolutionary system generated by a semilinear parabolic equation

$$u_t = \Delta u + f(u), \quad t \geq 0, \quad x \in \mathbb{R}, \quad \Delta u = \frac{\partial^2 u}{\partial x^2}\ , \tag{1}$$

so that $\Sigma(t, u_0)$, $t \geq 0$, is the solution with initial value $u_0 = u_0(x)$.

Let $\mathcal{S} = \mathcal{S}_{h,d}$ be the dynamical system generated by the implicit space-time discretization

$$(u^{n+1} - u^n)/h = Au^{n+1} + \overline{f}(u^{n+1}) \tag{2}$$

of Eq. (1) with time step h and space step d, where A is the standard three point difference approximation of the operator Δ (see the exact definition below) and $(\overline{f}(u))_i = f(u_i)$.

In this paper, we study the convergence of the global attractors $\mathcal{A}(h, d)$ of the systems $\mathcal{S}_{h,d}$ (for their existence see [3]) to the global attractor $\mathcal{A}$ of the system Σ as the stepsizes $h, d \to 0$.

For bounded domains the problem of convergence of "approximate" global attractors to the "exact" attractor has been studied extensively for various approximations in x and t (see, for example, [5],[10],[11]). The main new feature for an unbounded domain compared to the bounded domain is that the evolutionary system lacks compactness properties.

We adopt the approach of Babin and Vishik [1] and consider global attractors corresponding to pairs of function spaces. Our choice of weighted spaces follows Mielke and Schneider [9].

Let us first describe the spaces we work with. We fix $\varepsilon > 0$ (to be specified later) and a weight function

$$\rho(x) = (1 + \varepsilon^2 x^2)^{-\gamma}, \quad \text{where} \quad \gamma \geq \frac{1}{2}\,.$$

For shifts and finite differences we use the following notation: if $u = \{u_k : k \in \mathbb{Z}\}$ is a sequence and $v(x), x \in \mathbb{R}$, is a function, then

$$(\partial_+ u)_k = (u_{k+1} - u_k)/d, \quad (\partial_- u)_k = (u_k - u_{k-1})/d,$$

$$(\partial_+ v)(x) = (v(x+d) - v(x))/d, \quad (\partial_- v)(x) = (v(x) - v(x-d))/d,$$

$$(T_y u)_k = u_{k+y}, \text{ and } (T_y v)(x) = v(x+y),$$

where $y \in \mathbb{Z}$ in the first case and $y \in \mathbb{R}$ in the second case.

With this notation the operator A in Eq. (2) is defined by the formula $Au = \partial_+\partial_- u$.

We consider two Hilbert spaces of sequences $u = \{u_k : k \in \mathbb{Z}\}$: H_ρ with the norm defined by

$$\|u\|_{0,\rho}^2 = d \sum_{k\in\mathbb{Z}} \rho_k u_k^2,$$

where $\rho_k = \rho(kd)$, and Z_ρ with the norm defined by

$$\|u\|_{1,\rho}^2 = \|u\|_{0,\rho}^2 + \|\partial_- u\|_{0,\rho}^2,$$

and two Hilbert spaces of functions $u = u(x), x \in \mathbb{R}$: $\mathcal{H}_\rho$ with the norm defined by

$$\|u\|_{0,\rho}^2 = \int_{\mathbb{R}} \rho(x) u(x)^2 \, dx$$

and $\mathcal{Z}_\rho$ with the norm defined by

$$\|u\|_{1,\rho}^2 = \|u\|_{0,\rho}^2 + \|\nabla u\|_{0,\rho}^2.$$

Note that, for any of the spaces above, the norms corresponding to two different choices of ε are equivalent.

Finally, we introduce the space Z_u of sequences $u = \{u_k : k \in \mathbb{Z}\}$ with the norm defined by

$$\|u\|_{1,u} = \sup_{y\in\mathbb{Z}} \|T_y u\|_{1,\rho}$$

and the space $\mathcal{Z}_u$ of functions $u = u(x), x \in \mathbb{R}$, with the norm defined by

$$\|u\|_{1,u} = \sup_{y\in\mathbb{R}} \|T_y u\|_{1,\rho}.$$

We assume that the nonlinearity f in Eq. (1) satisfies the following main conditions:

(AI) f is in $C^1(\mathbb{R})$ with globally bounded derivative;

(AII) for some $a, b > 0$ the function f satisfies the dissipativity condition

$$uf(u) \le -au^2 + b, \quad u \in \mathbb{R} \tag{3}$$

Condition (3) implies that $f(c) = 0$ for some $c \in \mathbb{R}$ and therefore, by the change of variables $u := u + c$, we may assume that $f(0) = 0$.

Let Φ be a general evolutionary system on a Banach space $\mathcal{R}$ and let $\mathcal{R}$ contain a Banach space $\mathcal{R}'$ with continuous embedding. Following [1], we say that $I \subset \mathcal{R}$ is the global $(\mathcal{R}', \mathcal{R})$-attractor of Φ if

(i) I is a compact set in $\mathcal{R}$;

(ii) I is positively invariant with respect to Φ, i.e., $\Phi(t, I) = I$ for $t \geq 0$;

(iii) I attracts bounded subsets of $\mathcal{R}'$ with respect to the topology of $\mathcal{R}$.

It is proved in [6] that, under conditions (AI),(AII), the system Σ has the global $(\mathcal{Z}_u, \mathcal{Z}_\rho)$ attractor $\mathcal{A}$. Note that, for a different pair of function spaces, the existence of a global attractor of the system Σ has been established by Babin and Vishik in a pioneering paper [2]. The proof in [6] uses the choice of spaces and an abstract result from [9].

The main result of [3] shows that if conditions (AI) and (AII) are satisfied and h and d are small enough, then the system $\mathcal{S}$ has the global (Z_u, Z_ρ)-attractor $\mathcal{A}(h, d)$ and this attractor has a bound in Z_u that is uniform in h and d. In addition, the attractor $\mathcal{A}(h, d)$ is invariant under $\mathcal{S}$ in the sense

$$\mathcal{S}^t(\mathcal{A}(h, d)) = \mathcal{A}(h, d) \quad \text{for} \quad t \in \mathbb{Z}. \tag{4}$$

Notice that this property can be shown for all $t \in \mathbb{Z}$ due to invertibility of the system $\mathcal{S}$. For noninvertible systems such as Σ this property is replaced by the fact that the global attractor consists of complete orbits [10].

We embed the space H_ρ into $\mathcal{H}_\rho$ as follows. Define a partition of unity $\{\omega_k\}_{k\in\mathbb{Z}}$ where the hat functions $\omega_k(x), x \in \mathbb{R}$ are given by

$$\omega_k(x) = \begin{cases} (x - (k-1)d)/d, & x \in [(k-1)d, kd], \\ ((k+1)d - x)/d, & x \in [kd, (k+1)d], \\ 0 & \text{otherwise.} \end{cases}$$

Then define the interpolation operator $\mathcal{T} : H_\rho \to \mathcal{H}_\rho$ by

$$\mathcal{T}\{u_k\} = \sum_{k\in Z} \omega_k(x) u_k. \tag{5}$$

For two sets $B_1, B_2 \subset \mathcal{Z}_\rho$, we introduce the Hausdorff semi-distance, which we call the deviation for short, by

$$\operatorname{dev}(B_1, B_2) = \sup_{u\in B_1} \operatorname{dist}(u, B_2).$$

Here 'dist' is generated by the norm of the space $\mathcal{Z}_\rho$.

The main result of this paper is the following statement.

Theorem 1 *Under the assumptions (AI), (AII) the attractors converge in the following sense*

$$\operatorname{dev}(\mathcal{T}\mathcal{A}(h, d), \mathcal{A}) \to 0 \quad \text{as } h, d \to 0. \tag{6}$$

We have stated this theorem for scalar parabolic equations. It is, however, quite straightforward to extend the result to systems along the lines of [3].

The structure of the paper is as follows. In Sec. 2, we summarize regularity estimates from [6] for solutions of Eq. (1) on finite time intervals and we state error estimates for projectors and interpolation operators in weighted norms that have been derived in [7]. Moreover, we set up the basic technical results that are used in Sec. 3 to prove the main theorem. Then sections 4 and 5 are devoted to the proof of the technical results from Sec. 2 – a finite time error estimate between solutions of Eq. (1) and the corresponding finite difference solution, a regularity estimate for the discrete solution and an asymptotic compactness result. In all cases we use the weighted and uniform norms defined above.

2 Preliminary Estimates

In our reasoning below, we apply the following regularity result for solutions of Eq. (1) on finite time intervals (see [6]). Note that all derivatives of a solution mentioned in Proposition 2 exist for almost all (t, x).

Proposition 2 *Assume that the nonlinearity f in Eq. (1) is Lipschitz continuous. Let $u(t, x)$ be a solution of Eq. (1) such that $u_0 \in \mathcal{Z}_\rho$. For any $T > 0$, there exists a constant $C(T) > 0$ such that the following estimates hold:*

$$\|\Delta u(t)\|_{0,\rho}^2 + \|u_t(t)\|_{0,\rho}^2 \le C(T)t^{-1}\|u_0\|_{1,\rho}^2, \quad 0 < t \le T; \tag{7}$$

$$\|\nabla u_t(t)\|_{0,\rho}^2 \le C(T)t^{-2}\|u_0\|_{1,\rho}^2, \quad 0 < t \le T; \tag{8}$$

$$\int_0^T \|u_t(t)\|_{0,\rho}^2 dt \le C(T)\|u_0\|_{1,\rho}^2; \tag{9}$$

$$\int_0^T t^2\|\Delta u_t(t)\|_{0,\rho}^2 dt \le C(T)\|u_0\|_{1,\rho}^2; \tag{10}$$

$$\int_0^T t^2\|u_{tt}(t)\|_{0,\rho}^2 dt \le C(T)\|u_0\|_{1,\rho}^2. \tag{11}$$

For the discrete system in Eq. (1) we need a certain analog that yields estimates up to second order.

Proposition 3 *For h and d sufficiently small Eq. (1) defines a solution operator $\mathcal{S}$ on H_ρ and on Z_ρ with Lipschitz constant $1 + Ch$ for both norms.*

For any fixed $T > 0$ there exists a constant $C = C(T) > 0$ such that solutions $u^{(n+1)} = \mathcal{S}(u^{(n)})$ of the discrete system (1) satisfy for $0 < nh \le T$ the following estimates

$$\left\|u^{(n)}\right\|_{0,\rho}^2 + h\sum_{k=1}^{n}\left\|\partial_+ u^{(k)}\right\|_{0,\rho}^2 \le C\left\|u^{(0)}\right\|_{0,\rho}^2, \tag{12}$$

$$\left\|\partial_+\partial_- u^{(n)}\right\|_{0,\rho}^2 \le \frac{C}{(nh)^2}\left\|u^{(0)}\right\|_{0,\rho}^2. \tag{13}$$

The first part essentially follows from [3] while a detailed proof of (13) will be given in Section 5.

In the following it will be convenient to use second order spaces and norms

$$\mathcal{Y}_\rho = \{u \in \mathcal{Z}_\rho : \|u\|_{2,\rho}^2 = \|u\|_{0,\rho}^2 + \|\nabla u\|_{1,\rho}^2 < \infty\},$$

$$Y_\rho = \{u \in Z_\rho : \|u\|_{2,\rho}^2 = \|u\|_{0,\rho}^2 + \|\partial_- u\|_{1,\rho}^2 < \infty\}$$

with their uniform counterparts denoted by $(Y_u, \|\cdot\|_{2,u})$ and $(\mathcal{Y}_u, \|\cdot\|_{2,u})$.

Now consider the interpolation operator $\mathcal{T}$ defined by (6). The following Lemmas 4 - 7 are proved in [7].

Lemma 4 *There exists a constant $C > 0$ such that*

$$C^{-1}\|u\|_{0,\rho} \le \|\mathcal{T}u\|_{0,\rho} \le C\|u\|_{0,\rho} \quad \text{and} \quad C^{-1}\|u\|_{1,\rho} \le \|\mathcal{T}u\|_{1,\rho} \le C\|u\|_{1,\rho}$$

for any $u \in H_\rho$ and $u \in Z_\rho$, respectively.

Comment Here and below, we denote by C various constants that are independent of h and d but may depend on the parameter ε in the weight function ρ.

Lemma 5 *The operator $\mathcal{T}$ is uniformly (in d) bounded from Z_u into $\mathcal{Z}_u$.*

Introduce the subspace $\mathcal{V}_d = \mathcal{T}(H_\rho) \subset \mathcal{H}_\rho$ of piecewise linear functions. Lemma 4 implies that $\mathcal{V}_d$ is a closed subspace of $\mathcal{H}_\rho$ and that $\mathcal{T}$ is a homeomorphism between H_ρ and $\mathcal{V}_d$.

Let $\mathcal{P}_d$ be the orthogonal projector onto $\mathcal{V}_d$ in the space $\mathcal{H}_\rho$.

Lemma 6 *If $u \in \mathcal{Z}_\rho$, then*

$$\|\mathcal{P}_d u\|_{1,\rho} \le C\|u\|_{1,\rho} \tag{14}$$

and

$$\|(I - \mathcal{P}_d)u\|_{0,\rho} \le Cd\|u\|_{1,\rho}.$$

Note that the error estimates above as well as (15) below are classical in finite element analysis for the case of bounded domains and without weights (see e.g. [4]).

Lemma 7 *If $u \in \mathcal{Y}_\rho$, then*

$$\|(I - \mathcal{P}_d)u\|_{1,\rho} \le Cd\|u\|_{2,\rho}. \tag{15}$$

and for any $K > 0$ there exists a constant $C(K) > 0$ such that

$$\|\mathcal{P}_d T_{kd}(I - \mathcal{P}_d)\|_{0,\rho} \le C(K)d^3\|u\|_{2,\rho}$$

for all integers k such that $|k| \le K$.
If, in addition, $u \in \mathcal{Y}_u$ then for all $|y| \le 1$

$$\|T_y \mathcal{T} u - \mathcal{T} u\|_{1,u} \le C\sqrt{|y|}\ \|u\|_{2,u}.$$

An easy consequence of these Lemmas is the following (see Section 5).

Lemma 8 *The attractor $\mathcal{A}$ is contained in $\mathcal{Y}_\rho$ and*

$$\mathrm{dev}(\mathcal{P}_d\mathcal{A}, \mathcal{A}) \le Cd.$$

For the following finite time estimate we remind the reader that the operators $\mathcal{T}$ and $\mathcal{S}$ depend on d and on d, h, respectively.

Proposition 9 *If we fix $v \in \mathcal{Z}_\rho$ and a number $T > 0$, then*

$$\sup_{0<nh\le T} \left\|\mathcal{T}\,\mathcal{S}^n(\mathcal{T}^{-1}\mathcal{P}_d v) - \mathcal{P}_d\Sigma(nh, v)\right\|_{1,\rho} \to 0 \quad as \quad h, d \to 0.$$

Let us note that, for the case of a parabolic equation on a bounded (in x) domain, explicit (in terms of the steps) estimates of finite-time discretization errors were obtained, for example, in [8]. Proposition 9 will be proved in Section 4.

Finally, consider a sequence (h_m, d_m) of discretization steps such that $h_m, d_m \to 0$ as $m \to \infty$ and let $\mathcal{T}_m$ denote the interpolation operator corresponding to $d = d_m$.

Proposition 10 *If $u_m \in \mathcal{A}(h_m, d_m)$, then the sequence $v_m = \mathcal{T}_{d_m} u_m$ is precompact in $\mathcal{Z}_\rho$.*

The essential tool in the proof of Proposition 10 (cf. Section 5) is the following compactness result from [6].

Proposition 11 *Any bounded set $B \subset \mathcal{Z}_u$ that satisfies*

$$\sup_{u\in B} \|T_y u - u\|_{1,u} \to 0 \quad as \quad y \to 0$$

is precompact in $\mathcal{Z}_\rho$.

3 Proof of the main theorem

To prove the main theorem, let us assume that relation (6) does not hold. In this case, there exists a positive number c and a sequence $(h_m, d_m) \to (0,0)$ such that

$$\operatorname{dev}(\mathcal{T}_m \mathcal{A}(h_m, d_m), \mathcal{A}) \geq 2c.$$

Find points $u'_m \in \mathcal{A}(h_m, d_m)$ such that

$$\operatorname{dist}(\mathcal{T}_m u'_m, \mathcal{A}) \geq c. \tag{16}$$

Since $\mathcal{T}_m$ is uniformly (in d) bounded from Z_u into $\mathcal{Z}_u$ (see Lemma 4) and the Z_u-size of the attractors $\mathcal{A}(h_m, d_m)$ is uniformly bounded for large m [3], there exists a closed bounded ball B of the space $\mathcal{Z}_u$ such that

$$\mathcal{T}_m \mathcal{A}(h_m, d_m) \subset B.$$

Find a number $T > 1$ such that

$$\operatorname{dist}(\Sigma(t, B), \mathcal{A}) < c/C \quad \text{for} \quad t \geq T - 1,$$

where C is from (14) in Lemma 6.

If $h_m < 1$, we can find integers $\tau(m)$ such that $T - 1 \leq \tau(m) h_m \leq T$. Let

$$u_m := \mathcal{S}_m^{-\tau(m)}(u'_m) \in \mathcal{A}(h_m, d_m) \quad \text{and} \quad v_m := \mathcal{T}_m u_m,$$

where $\mathcal{S}_m$ is the solution operator for $d = d_m, h = h_m$.

Since $u_m \in \mathcal{A}(h_m, d_m)$, it follows from Proposition 10 that the sequence v_m contains a subsequence convergent in $\mathcal{Z}_\rho$; we assume that $v_m \to v$ as $m \to \infty$. It is easy to show that $v \in B \subset \mathcal{Z}_u$.

Thus, there exist points $w_m \in \mathcal{A}$ such that

$$\|\Sigma(\tau(m) h_m, v) - w_m\|_{1,\rho} < c/C. \tag{17}$$

Note that $\mathcal{T}_m \mathcal{S}_m^{\tau(m)}(u_m) = \mathcal{T}_m u'_m$.

Let us estimate

$$\operatorname{dist}(\mathcal{T}_m u'_m, \mathcal{A}) \leq \left\| \mathcal{T}_m \mathcal{S}_m^{\tau(m)}(u_m) - \mathcal{T}_m \mathcal{S}_m^{\tau(m)}(\mathcal{T}_m^{-1} \mathcal{P}_{d_m} v) \right\|_{1,\rho} +$$

$$+ \left\| \mathcal{T}_m \mathcal{S}_m^{\tau(m)}(\mathcal{T}_m^{-1} \mathcal{P}_{d_m} v) - \mathcal{P}_{d_m} \Sigma(\tau(m) h_m, v) \right\|_{1,\rho} +$$

$$+ \|\mathcal{P}_{d_m} \Sigma(\tau(m) h_m, v) - \mathcal{P}_{d_m} w_m\|_{1,\rho} + \operatorname{dist}(\mathcal{P}_{d_m} w_m, \mathcal{A}).$$

By Proposition 3 the mapping $\mathcal{S}_m$ has a Lipschitz constant of the form $1+Ch_m$ with C independent of m. Hence, the mappings $\mathcal{S}_m^{\tau(m)}$ have uniform Lipschitz constants for small h_m and d_m. Since $u_m = \mathcal{T}_m^{-1}\mathcal{P}_{d_m}v_m$ and the operators $\mathcal{T}_m$, $\mathcal{T}_m^{-1}$ and $\mathcal{P}_{d_m}$ are uniformly bounded, the first term on the right in the above inequality tends to 0 as $m \to \infty$.

By Proposition 9, the second term on the right tends to 0 as $m \to \infty$. It follows from inequality (17) that $\|\mathcal{P}_{d_m}\Sigma(\tau(m)h_m, v) - \mathcal{P}_{d_m}w_m\|_{1,\rho} < c$.

By Lemma 8 we have $\text{dist}(\mathcal{P}_{d_m}w_m, \mathcal{A}) \to 0$ as $m \to \infty$. Thus

$$\text{dist}(\mathcal{T}_m u_m', \mathcal{A}) < c$$

for large m, and we obtain a contradiction with inequalities (16). This completes the proof.

4 An error estimate with weighted norms

This section is devoted to the proof of Proposition 10. Let $\mathcal{L}$ be a Lipschitz constant of f. Denote

$$A_d = \mathcal{T}(\partial_+\partial_-)\mathcal{T}^{-1}$$

and note that $A_d = \partial_+\partial_-$ holds on $\mathcal{V}_d$. We further define

$$f_d(u) = \mathcal{T}\{f((\mathcal{T}^{-1}u)_k)\} \text{ for } u \in \mathcal{V}_d.$$

and use the notation $f(u)(x) = f(u(x))$ for $u \in \mathcal{H}_\rho$.

Fix a function $u_0 \in \mathcal{Z}_\rho$ (this function will play the role of v). Take $u^0 = \mathcal{T}^{-1}\mathcal{P}_d u_0$ and consider the corresponding trajectory $\{u^n : n \geq 0\}$ of the discretized equation (2). Denote $v^n = \mathcal{T}u^n \in \mathcal{V}_d \subset \mathcal{H}_\rho$. Applying $\mathcal{T}$ to (2), we see that the functions v^n satisfy the following equation:

$$(v^{n+1} - v^n)/h = \mathcal{T}(\partial_+\partial_-)u^{n+1} + \mathcal{T}\overline{f}(u^{n+1}) = A_d v^{n+1} + f_d(v^{n+1}).$$

Let us write this equation as follows:

$$(v^{n+1} - v^n)/h = A_d v^{n+1} + \mathcal{P}_d f(v^{n+1}) + \sigma_1^{n+1}, \qquad (18)$$

where

$$\sigma_1^{n+1} = f_d(v^{n+1}) - \mathcal{P}_d f(v^{n+1}).$$

Let $u(t,x)$ be the solution of Eq. (1) with inital value $u_0(x)$ at $t = 0$. Denote $u^{(n)}(x) = \mathcal{P}_d u(nh, x)$. Applying $\mathcal{P}_d$ to (1) at $t = (n+1)h$, we see that

$$(u^{(n+1)}(x) - u^{(n)}(x))/h = A_d u^{(n+1)}(x) + \mathcal{P}_d f(u((n+1)h, x)) - \sigma_2^{n+1}(x) - \sigma_3^{n+1}(x), \qquad (19)$$

where

$$\sigma_2^{n+1}(x) = A_d u^{(n+1)}(x) - \mathcal{P}_d \Delta u((n+1)h, x)$$

and

$$\sigma_3^{n+1}(x) = \mathcal{P}_d u_t((n+1)h, x) - (u^{(n+1)}(x) - u^{(n)}(x))/h.$$

Let $\Theta^{(n)} = v^n - u^{(n)}$. Subtracting (19) from (18), we see that

$$(\Theta^{(n+1)} - \Theta^{(n)})/h =$$

$$= A_d\Theta^{(n+1)} + \mathcal{P}_d(f(v^{(n+1)}) - f(u((n+1)h, x))) + \sigma_1^{n+1} + \sigma_2^{n+1} + \sigma_3^{n+1} \quad (20)$$

and $\Theta^{(0)} = v^0 - \mathcal{P}_d u_0$. Below we take into account that $\Theta^{(0)} = 0$ due to our choice of u^0.

Now we fix $T > 0$ and estimate $\left\|\Theta^{(n)}\right\|_{1,\rho}$ for $0 < nh \leq T$. Let us begin with preliminary estimates.

Estimation of σ_1^{n+1}.

Fix $x = (k + \theta)d$, where $\theta \in [0, 1]$. Since $\mathcal{P}_d f_d = f_d$,

$$|\sigma_1^{n+1}(x)| = |\mathcal{P}_d(f_d(v^{n+1}) - f(v^{n+1}))(x)|.$$

Let us estimate

$$|f(u_{k+1}^{n+1})\theta + f(u_k^{n+1})(1-\theta) - f(u_{k+1}^{n+1}\theta + u_k^{n+1}(1-\theta))| \leq$$

$$\leq |f(u_{k+1}^{n+1}\theta + u_k^{n+1}(1-\theta)) - f(u_{k+1}^{n+1})|\theta +$$

$$+|f(u_{k+1}^{n+1}\theta + u_k^{n+1}(1-\theta)) - f(u_k^{n+1})|(1-\theta) \leq$$

$$\leq 2\mathcal{L}\theta(1-\theta)|u_{k+1}^{n+1} - u_k^{n+1}| \leq \mathcal{L}|u_{k+1}^{n+1} - u_k^{n+1}|/2 = \mathcal{L}d|(\partial_+ u^{n+1})_k|/2.$$

Since $\mathcal{P}_d$ is an orthogonal projector, $||\mathcal{P}_d|| = 1$, and we get the following estimate:

$$\left\|\sigma_1^{n+1}\right\|_{0,\rho}^2 \leq \int_{\mathbb{R}} \rho(x)|(f_d(v^{n+1}) - f(v^{n+1}))(x)|^2 dx \leq$$

(recall that $\rho(x + \theta d) \leq C\rho(x)$ for $|\theta| \leq 1$, [8])

$$\leq C\mathcal{L}^2 d^3 \sum_{k\in Z} \rho_k |(\partial_+ u^{n+1})_k|^2 \leq Cd^2 \|\partial_+ u^{n+1}\|_{0,\rho}^2.$$

Finally, we arrive at the estimate

$$\left\|\sigma_1^{n+1}\right\|_{0,\rho}^2 \leq Cd^2 \|\partial_+ u^{n+1}\|_{0,\rho}^2. \quad (21)$$

Estimation of σ_2^{n+1}.

Let us transform

$$\sigma_2^{n+1}(x) = A_d u^{(n+1)} - \mathcal{P}_d \Delta u((n+1)h, x) =$$

$$= \partial_+ \partial_- \mathcal{P}_d u((n+1)h, x) - \mathcal{P}_d \Delta u((n+1)h, x) =$$

$$= \partial_+ \partial_- \mathcal{P}_d u((n+1)h, x) - \mathcal{P}_d \partial_+ \partial_- u((n+1)h, x) +$$

$$+ \mathcal{P}_d (\partial_+ \partial_- u((n+1)h, x) - \Delta u((n+1)h, x)).$$

We denote

$$\sigma_{2,1}^{n+1} := \partial_+ \partial_- \mathcal{P}_d u((n+1)h, x) - \mathcal{P}_d \partial_+ \partial_- u((n+1)h, x),$$

$$\sigma_{2,2}^{n+1} := \mathcal{P}_d (\partial_+ \partial_- u((n+1)h, x) - \Delta u((n+1)h, x)). \qquad (22)$$

Let us estimate the term $\sigma_{2,1}^{n+1}$. The following equalities hold:

$$\sigma_{2,1}^{n+1} = \mathcal{P}_d \partial_+ \partial_- (I - \mathcal{P}_d) u((n+1)h, x) =$$

$$= \frac{1}{d^2} \mathcal{P}_d (T_d - 2I + T_{-d})(I - \mathcal{P}_d) u((n+1)h, x) =$$

$$= \frac{1}{d^2} \mathcal{P}_d (T_d + T_{-d})(I - \mathcal{P}_d) u((n+1)h, x) - \frac{2}{d^2} \mathcal{P}_d (I - \mathcal{P}_d) u((n+1)h, x) =$$

$$= \frac{1}{d^2} \mathcal{P}_d (T_d + T_{-d})(I - \mathcal{P}_d) u((n+1)h, x).$$

The second estimate in Lemma 7 implies that if $u(x) = u((n+1)h, x)$ satisfies the inequality

$$\|u\|_{2,\rho}^2 = \|u\|_{0,\rho}^2 + \|\nabla u\|_{1,\rho}^2 < \infty,$$

then

$$\left\|\sigma_{2,1}^{n+1}\right\|_{0,\rho} \le \frac{1}{d^2} \Big(\|\mathcal{P}_d T_d (I - \mathcal{P}_d) u\|_{0,\rho} +$$

$$+ \|\mathcal{P}_d T_{-d} (I - \mathcal{P}_d) u\|_{0,\rho} \Big) \le 2Cd\|u\|_{2,\rho}. \qquad (23)$$

Now let us estimate the term $\sigma_{2,2}^{n+1}$. Since

$$|\partial_+ \partial_- u((n+1)h, x)| = |\textstyle\int_0^1 \partial_- \nabla u((n+1)h, x + \theta d) d\theta| =$$

$$= |\textstyle\int_0^1 \int_0^1 \Delta u((n+1)h, x + (\theta + \theta_1) d - d) d\theta d\theta_1| =$$

(introduce $\theta_2 = \theta + \theta_1 - 1$)

$$= |\int_0^1 \int_{\theta-1}^{\theta} \Delta u((n+1)h, x + \theta_2 d) d\theta_2 d\theta|, \tag{24}$$

we obtain the following equalities:

$$|\partial_+ \partial_- u((n+1)h, x) - \Delta u((n+1)h, x)| =$$

$$= |\int_0^1 \int_{\theta-1}^{\theta} (\Delta u((n+1)h, x + \theta_2 d) - \Delta u((n+1)h, x)) d\theta_2 d\theta| =$$

$$= |\int_{-1}^0 \int_0^{1+\theta_2} \dots d\theta d\theta_2 + \int_0^1 \int_{\theta_2}^1 \dots d\theta d\theta_2| =$$

$$= d|\int_{-1}^1 \int_0^{\theta_2} u'''((n+1)h, x + \theta_3 d)(1 - |\theta_2|) d\theta_3 d\theta_2| =$$

$$= d| - \int_{-1}^0 \int_{\theta_2}^0 \dots + \int_0^1 \int_0^{\theta_2} \dots | =$$

$$= d|\int_0^1 \int_{\theta_3}^1 u'''((n+1)h, x + \theta_3 d)(1 - |\theta_2|) d\theta_2 d\theta_3 + \int_{-1}^0 \int_{-1}^{\theta_3} \dots | =$$

$$= d|\int_0^1 u'''(\dots)(1 - |\theta_3|)^2/2 d\theta_3 - \int_{-1}^0 u'''(\dots)(1 - |\theta_3|)^2/2 d\theta_3|.$$

It follows that

$$\left\|\sigma_{2,2}^{n+1}\right\|_{0,\rho}^2 = \|\partial_+ \partial_- u((n+1)h, x) - \Delta u((n+1)h, x)\|_{0,\rho}^2 = \int_{\mathbb{R}} \rho |\dots|^2 dx \leq$$

$$\leq d \int_{\mathbb{R}} \int_{-d}^{d} \rho |u'''((n+1)h, x + \theta_4)|^2 d\theta_4 dx \leq$$

(we differentiate Eq. (1))

$$\leq 2d \int_{\mathbb{R}} \int_{-d}^{d} \rho |\nabla u_t((n+1)h, x + \theta_4)|^2 d\theta_4 dx +$$

$$+ 2d \int_{\mathbb{R}} \int_{-d}^{d} \rho |\nabla f(u((n+1)h, x + \theta_4))|^2 d\theta_4 dx \leq$$

(apply Proposition 2)

$$\leq C d^2 ((n+1)h)^{-2} \|u_0\|_{1,\rho}^2 + 2\mathcal{L}^2 d \int_{\mathbb{R}} \int_{-d}^{d} \rho |\nabla u((n+1)h, x + \theta_4)|^2 d\theta_4 dx.$$

Finally, by applying Proposition 2 once more we have

$$\left\|\sigma_{2,2}^{n+1}\right\|_{0,\rho}^2 \leq C d^2 (1 + ((n+1)h)^{-2}) \|u_0\|_{1,\rho}^2. \tag{25}$$

Estimation of σ_3^{n+1}.

The following estimates hold:

$$\left\|\sigma_3^{n+1}\right\|_{0,\rho}^2 = \int_{\mathbb{R}} \rho |(u^{(n+1)} - u^{(n)})/h - \mathcal{P}_d u_t((n+1)h, x)|^2 dx =$$

$$= \int_{\mathbb{R}} \rho |\mathcal{P}_d \int_0^1 (u_t((n+\theta)h, x) - u_t((n+1)h, x) d\theta|^2 dx \leq$$

$$\leq h^2 \int_{\mathbb{R}} \rho |\int_0^1 \int_\theta^1 u_{tt}((n+\theta_1)h, x) d\theta d\theta_1|^2 dx =$$

$$= h^2 \int_{\mathbb{R}} \rho |\int_0^1 u_{tt}((n+\theta_1)h, x) \theta_1 d\theta_1|^2 dx \leq$$

$$\leq h^2 \int_0^1 \int_{\mathbb{R}} \rho |u_{tt}((n+\theta_1)h, x)|^2 \theta_1^2 dx d\theta_1.$$

Hence,

$$\left\|\sigma_3^{n+1}\right\|_{0,\rho}^2 \leq h^2 \int_0^1 \int_{\mathbb{R}} \rho |u_{tt}((n+\theta_1)h, x)|^2 dx d\theta_1. \tag{26}$$

Estimation of $\Theta^{(n)}$.

Multiplying Eq. (20) by $\Theta^{(n+1)}$, we get the following estimate:

$$(|\Theta^{(n+1)}|^2 - |\Theta^{(n)}|^2)/(2h) \leq (|\Theta^{(n+1)}|^2 - \Theta^{(n+1)}\Theta^{(n)})/h =$$

$$= A_d \Theta^{(n+1)} \Theta^{(n+1)} - \mathcal{P}_d(f(v^{(n+1)}) - f(u((n+1)h, x)))\Theta^{(n+1)} +$$

$$+ (\sigma_1^{(n+1)} + \sigma_2^{(n+1)} + \sigma_3^{(n+1)})\Theta^{(n+1)}. \tag{27}$$

Now we multiply (27) by ρ and integrate over $\mathbb{R}$:

$$(\left\|\Theta^{(n+1)}\right\|_{0,\rho}^2 - \left\|\Theta^{(n)}\right\|_{0,\rho}^2)/(2h) \leq \int_{\mathbb{R}} \rho A_d \Theta^{(n+1)} \Theta^{(n+1)} dx -$$

$$- \int_{\mathbb{R}} \rho \mathcal{P}_d (f(v^{(n+1)}) - f(u((n+1)h, x))\Theta^{(n+1)} dx +$$

$$+ \int_{\mathbb{R}} \rho (\sigma_1^{(n+1)} + \sigma_2^{(n+1)} + \sigma_3^{(n+1)})\Theta^{(n+1)} dx \leq$$

$$\leq \int_{\mathbb{R}} \rho \partial_+ \partial_- \Theta^{(n+1)} \Theta^{(n+1)} dx -$$

$$- \int_{\mathbb{R}} \rho \mathcal{P}_d (\int_0^1 f'(u((n+1)h, x)(1-\chi) + v^{(n+1)}\chi) \times$$

$$\times (v^{(n+1)} - u((n+1)h, x)) d\chi) \Theta^{(n+1)} dx +$$

$$+ \left\|\Theta^{(n+1)}\right\|_{0,\rho}^2 / 2 + (\left\|\sigma_1^{(n+1)}\right\|_{0,\rho}^2 + \left\|\sigma_2^{(n+1)} + \sigma_3^{(n+1)}\right\|_{0,\rho}^2)/2. \tag{28}$$

Let us estimate the terms separately:

$$\int_{\mathbb{R}} \rho \partial_+ \partial_- \Theta^{(n+1)} \Theta^{(n+1)} dx = - \int_{\mathbb{R}} \partial_- (\rho \Theta^{(n+1)}) \partial_- \Theta^{(n+1)} dx =$$

$$= - \int_{\mathbb{R}} \rho |\partial_- \Theta^{(n+1)}|^2 dx - \int_{\mathbb{R}} (\partial_- \rho)(T_{-d} \Theta^{(n+1)}) \partial_- \Theta^{(n+1)} dx \leq$$

$$\leq - \int_{\mathbb{R}} \rho |\partial_- \Theta^{(n+1)}|^2 dx + C\varepsilon \int_{\mathbb{R}} \rho |\Theta^{(n+1)}|^2 dx + C\varepsilon \int_{\mathbb{R}} \rho |\partial_- \Theta^{(n+1)}|^2 dx \leq$$

$$\leq - \left\|\partial_- \Theta^{(n+1)}\right\|_{0,\rho}^2 / 2 + C\varepsilon \left\|\Theta^{(n+1)}\right\|_{0,\rho}^2 \tag{29}$$

by a proper choice of ε.

Further,

$$|\int_{\mathbb{R}}\rho\mathcal{P}_d(\int_0^1 f'(u((n+1)h,x)(1-\chi)+v^{(n+1)}\chi)\times$$

$$\times(v^{(n+1)}-u((n+1)h,x))d\chi)\Theta^{(n+1)}dx|\leq$$

$$\leq\int_{\mathbb{R}}\rho|\mathcal{P}_d(\int_0^1 f'(u((n+1)h,x)(1-\chi)+v^{(n+1)}\chi)(v^{(n+1)}-u((n+1)h,x))d\chi|^2dx+$$

$$+\left\|\Theta^{(n+1)}\right\|_{0,\rho}^2\leq$$

$$\leq\mathcal{L}\int_{\mathbb{R}}\rho|v^{(n+1)}-\mathcal{P}_d u((n+1)h,x))|^2dx+$$

(recall that $v^{(n+1)}-\mathcal{P}_d u((n+1)h,x))=\Theta^{(n+1)}$)

$$+\mathcal{L}\int_{\mathbb{R}}\rho|\mathcal{P}_d u((n+1)h,x))-u((n+1)h,x)|^2dx+\left\|\Theta^{(n+1)}\right\|_{0,\rho}^2\leq$$

$$\leq Cd^2\|u_0\|_{1,\rho}^2+C\left\|\Theta^{(n+1)}\right\|_{0,\rho}^2. \tag{30}$$

In the last step we applied Lemma 6 to the second term and used that

$$\|u(kh,\cdot)\|_{1,\rho}^2\leq C\|u_0\|_{1,\rho}^2$$

holds for $0\leq kh\leq T$ with a constant C depending on T.

It follows from inequalities (28)–(30) and equality (22) that

$$(\left\|\Theta^{(n+1)}\right\|_{0,\rho}^2-\left\|\Theta^{(n)}\right\|_{0,\rho}^2)/h+\left\|\partial_-\Theta^{(n+1)}\right\|_{0,\rho}^2\leq$$

$$\leq C\left\|\Theta^{(n+1)}\right\|_{0,\rho}^2+Cd^2\|u_0\|_{1,\rho}^2+2\left\|\sigma_1^{n+1}\right\|_{0,\rho}^2+2\left\|\sigma_{2,1}^{n+1}\right\|_{0,\rho}^2+2\sigma^{n+1},$$

where

$$\sigma^{n+1}=\left\|\sigma_{2,2}^{n+1}+\sigma_3^{n+1}\right\|_{0,\rho}^2.$$

Summing the latter inequalities, we see that

$$\left\|\Theta^{(n+1)}\right\|_{0,\rho}^2\leq\left\|\Theta^{(0)}\right\|_{0,\rho}^2+Ch\sum_{k=1}^{n+1}\left\|\Theta^{(k)}\right\|_{0,\rho}^2+CTd^2\|u_0\|_{1,\rho}^2+$$

$$+2h\sum_{k=1}^{n+1}(\sigma^{(k)}+\left\|\sigma_1^k\right\|_{0,\rho}^2+\left\|\sigma_{2,1}^{n+1}\right\|_{0,\rho}^2)\leq$$

(we apply estimates (21) and (23))

$$\leq\left\|\Theta^{(0)}\right\|_{0,\rho}^2+Ch\sum_{k=1}^{n+1}\left\|\Theta^{(k)}\right\|_{0,\rho}^2+CTd^2\|u_0\|_{1,\rho}^2+$$

$$+Chd^2\sum_{k=1}^{n+1}\|u(kh,x)\|_{2,\rho}^2+Chd^2\sum_{k=1}^{n+1}\left\|\partial_+u^{(k)}\right\|_{0,\rho}^2+2h\sum_{k=1}^{n+1}\sigma^{(k)}. \qquad (31)$$

Note that Proposition 3 gives the estimate

$$h\sum_{k=1}^{n+1}\left\|\partial_+u^k\right\|_{0,\rho}^2\le C\|u^0\|_{0,\rho}^2. \qquad (32)$$

Next we estimate the term $hd^2\sum_{k=1}^{n+1}\|u(kh,x)\|_{2,\rho}^2$:

$$hd^2\sum_{k=1}^{n+1}\|u(kh,x)\|_{2,\rho}^2=hd^2\sum_{k=1}^{n+1}\int_{\mathbb{R}}\rho|\Delta u(kh,x)|^2+hd^2\sum_{k=1}^{n+1}\|u(kh,x)\|_{1,\rho}^2\le$$

$$\le hd^2\sum_{k=1}^{n+1}(\int_{\mathbb{R}}\rho|\Delta u(kh,x)|^2dx-\int_0^1\int_{\mathbb{R}}\rho|\Delta u((k+\theta)h,x)|^2dxd\theta)+$$

$$+d^2\int_0^{T+h}\int_{\mathbb{R}}\rho|\Delta u(t,x)|^2dxdt+CTd^2\|u_0\|_{0,\rho}^2=$$

$$=hd^2\sum_{k=1}^{n+1}\int_{\mathbb{R}}\rho\int_0^1(|\Delta u(kh,x)|^2-|\Delta u((k+\theta)h,x)|^2)d\theta dx+$$

$$+d^2\int_0^{T+h}\int_{\mathbb{R}}\rho|\Delta u(t,x)|^2dxdt+CTd^2\|u_0\|_{0,\rho}^2=$$

$$=-hd^2\sum_{k=1}^{n+1}\int_{\mathbb{R}}\rho\int_0^1\int_0^\theta 2h\Delta u_t((k+\theta_1)h,x)\Delta u((k+\theta_1)h,x)d\theta_1d\theta dx+$$

$$+d^2\int_0^{T+h}\int_{\mathbb{R}}\rho|\Delta u(t,x)|^2dxdt+CTd^2\|u_0\|_{0,\rho}^2\le$$

$$\le hd^2\sum_{k=1}^{n+1}\int_{\mathbb{R}}\rho\int_0^1\int_0^\theta(h^2|\Delta u_t((k+\theta_1)h,x)|^2+|\Delta u((k+\theta_1)h,x)|^2)d\theta_1d\theta dx+$$

$$+d^2\int_0^{T+h}\int_{\mathbb{R}}\rho|\Delta u(t,x)|^2dxdt+CTd^2\|u_0\|_{0,\rho}^2=$$

$$=hd^2\sum_{k=1}^{n+1}\int_{\mathbb{R}}\rho\int_0^1(h^2|\Delta u_t((k+\theta_1)h,x)|^2+|\Delta u((k+\theta_1)h,x)|^2)(1-\theta_1)d\theta_1dx+$$

$$+d^2\int_0^{T+h}\int_{\mathbb{R}}\rho|\Delta u(t,x)|^2dxdt+CTd^2\|u_0\|_{0,\rho}^2\le$$

$$\le d^2\int_h^{T+h}\int_{\mathbb{R}}\rho(h^2|\Delta u_t(t,x)|^2+|\Delta u(t,x)|^2)dxdt+$$

$$+d^2\int_0^{T+h}\int_{\mathbb{R}}\rho|\Delta u(t,x)|^2dxdt+CTd^2\|u_0\|_{0,\rho}^2\le$$

$$\leq d^2\int_0^{T+h}\int_{\mathbb{R}}\rho t^2|\Delta u_t(t,x)|^2dxdt+$$
$$+2d^2\int_0^{T+h}\int_{\mathbb{R}}\rho|\Delta u(t,x)|^2dxdt + CTd^2\|u_0\|^2_{0,\rho} \leq Cd^2\|u_0\|^2_{0,\rho},$$
where Proposition 2 was employed in the last step.

Now we fix an arbitrary $\alpha > 0$ and denote

$$\Sigma' = \sum_{h\leq kh\leq \alpha} \text{ and } \Sigma'' = \sum_{\alpha<kh\leq(n+1)h}.$$

Applying estimates (25) and (26) (and taking into account the expressions for σ_2 and σ_3 considering Σ'), we see that

$$h\sum_{k=1}^{n+1}\sigma^{(k)} = h(\Sigma'\sigma^{(k)} + \Sigma''\sigma^{(k)}) \leq$$

$$= h\Sigma'\left\|\mathcal{P}_d\Delta u(kh,x) - \mathcal{P}_d u_t(kh,x) + (u^{(k)} - u^{(k-1)})/h - P_dA_du(kh,x)\right\|^2_{0,\rho}+$$
$$+hd^2\Sigma''(1+(kh)^{-2})\|u_0\|^2_{1,\rho} + h^3\Sigma''\int_0^1\int_{\mathbb{R}}\rho|u_{tt}((k+\theta_1-1)h,x)|^2d\theta_1dx) \leq$$
$$\leq h\Sigma'\|\mathcal{P}_df(u(kh,x))\|^2_{0,\rho} + h\Sigma'\left\|\int_0^1 u_t((k-1+\theta)h,x)d\theta\right\|^2_{0,\rho}+$$
$$+h\Sigma'\|P_dA_du(kh,x)\|^2_{0,\rho} + CTd^2(1+\alpha^{-2})\|u_0\|^2_{1,\rho}+$$
$$+h^2\int_{\alpha-h}^T\int_{\mathbb{R}}\rho|u_{tt}(t,x)|^2dtdx. \tag{33}$$
We estimate the terms separately.

From the Lipschitz continuity of f we obtain:

$$h\Sigma'\|\mathcal{P}_df(u(kh,x))\|^2_{0,\rho} \leq h\Sigma'\|f(u(kh,x))\|^2_{0,\rho} \leq C\alpha\|u_0\|^2_{0,\rho}. \tag{34}$$

Further,

$$h\Sigma'\left\|\int_0^1 u_t((k-1+\theta)h,x)d\theta\right\|^2_{0,\rho} =$$
$$= h\Sigma'\int_{\mathbb{R}}\rho|\int_0^1 u_t((k-1+\theta)h,x)d\theta|^2dx \leq$$
$$\leq h\Sigma'\int_0^1\int_{\mathbb{R}}\rho|u_t((k-1+\theta)h,x)|^2d\theta dx \leq$$
$$\leq \int_0^\alpha\|u_t(t,\cdot)\|^2_{0,\rho}dt. \tag{35}$$

Further,

$$h\Sigma'\|P_dA_du(kh,x)\|^2_{0,\rho} \leq h\Sigma'\int_{\mathbb{R}}\rho|A_du(kh,x)|^2 \leq$$

(according to equality (24))

$$\leq h\Sigma'\int_{\mathbb{R}}\rho|\int_{-1}^1\Delta u(kh,x+\theta d)(1-|\theta|)d\theta|^2dx \leq$$

$$\leq 2h\Sigma' \int_{\mathbb{R}} \rho \int_{-1}^{1} |\Delta u(kh, x+\theta d)|^2 (1-|\theta|)^2 d\theta dx \leq$$

$$\leq 2h\Sigma' \int_{-1}^{1} \int_{\mathbb{R}} \rho |\int_0^1 \Delta u((k+\theta_1)h, x+\theta d) d\theta_1 +$$

$$+\int_0^1 (\Delta u(kh, x+\theta d) - \Delta u((k+\theta_1)h, x+\theta d)) d\theta_1|^2 dx d\theta \leq$$

$$\leq 4h\Sigma' \int_{-1}^{1} \int_{\mathbb{R}} \rho \int_0^1 |\Delta u((k+\theta_1)h, x+\theta d)|^2 dx d\theta_1 d\theta +$$

$$+4h\Sigma' \int_{-1}^{1} \int_{\mathbb{R}} \rho |h \int_0^1 \int_0^{\theta_1} \Delta u_t((k+\theta_2)h, x+\theta d) d\theta_2 d\theta_1|^2 dx d\theta \leq$$

$$\leq 4h\Sigma' \int_{-1}^{1} \int_0^1 \int_{\mathbb{R}} \rho |\Delta u((k+\theta_1)h, x+\theta d)|^2 dx d\theta_1 d\theta +$$

$$+4h\Sigma' \int_{-1}^{1} \int_0^1 \int_{\mathbb{R}} \rho (1-\theta_2)^2 h^2 |\Delta u_t((k+\theta_2)h, x+\theta d)|^2 dx d\theta_2 d\theta \leq$$

(we introduce $x_1 = x + \theta d$)

$$\leq 4Ch\Sigma' \int_{-1}^{1} \int_0^1 \int_{\mathbb{R}} \rho(x_1) |\Delta u((k+\theta_1)h, x_1)|^2 dx_1 d\theta_1 d\theta +$$

$$+4Ch\Sigma' \int_{-1}^{1} \int_0^1 \int_{\mathbb{R}} \rho(x_1)(1-\theta_2)^2 h^2 |\Delta u_t((k+\theta_2)h, x_1)|^2 dx_1 d\theta_1 d\theta \leq$$

$$\leq 4C \int_{-1}^{1} \int_h^{\alpha+h} \int_{\mathbb{R}} \rho |\Delta u(t, x_1)|^2 dx_1 dt d\theta +$$

$$+4C \int_{-1}^{1} \int_h^{\alpha+h} \int_{\mathbb{R}} \rho h^2 |\Delta u_t(t, x_1)|^2 dx_1 dt d\theta \leq$$

$$\leq 8C \int_0^{\alpha+h} (\|\Delta u(t,\cdot)\|_{0,\rho}^2 + t^2 \|\Delta u_t(t,\cdot)\|_{0,\rho}^2) dt. \tag{36}$$

We estimate the remaining term in (33) as follows:

$$h^2 \int_{\alpha-h}^{T} \int_{\mathbb{R}} \rho |u_{tt}(t,x)|^2 dx dt \leq \frac{h^2}{(\alpha-h)^2} \int_{\alpha-h}^{T} \int_{\mathbb{R}} t^2 \rho |u_{tt}(t,x)|^2 dx dt \leq$$

$$\leq \frac{Ch^2}{(\alpha-h)^2} \|u_0\|_{1,\rho}^2. \tag{37}$$

Now we fix a bounded ball in $\mathcal{Z}_\rho$ and take the initial function u_0 for Eq. (1) from this ball. Below, the constants C depend on T and the size of this ball, i.e., they "accumulate" the terms $\|u_0\|_{0,\rho}^2$ and $\|u_0\|_{1,\rho}^2$.

It follows from (33) and our estimates that

$$h \sum_{k=1}^{n+1} \sigma^{(k)} \leq C\alpha + \int_0^{\alpha} \|u_t(t,\cdot)\|_{0,\rho}^2 dt +$$

$$+\int_0^{\alpha+h} (\|\Delta u(t,\cdot)\|_{0,\rho}^2 + t^2 \|\Delta u_t(t,\cdot)\|_{0,\rho}^2) dt +$$

$$+Cd^2 + \frac{Cd^2}{\alpha^2} + \frac{Ch^2}{(\alpha-h)^2}, \tag{38}$$

where the constant C does not depend on α, d, and h. Taking, for example, $\alpha = (d^2 + h^2)^{1/4}$, we see that the value

$$h \sum_{k=1}^{n+1} \sigma^{(k)}$$

tends to 0 as $h, d \to 0$, note that the integrands in (38) are summable due to Proposition 2.

Thus, it follows from (31) and the Gronwall lemma that if we take $u^0 = \mathcal{T}^{-1}(\mathcal{P}_d u_0)$ (so that $\Theta^{(0)} = 0$), then

$$\left\|\Theta^{(n)}\right\|_{0,\rho}^2 \leq (\left\|\Theta^{(0)}\right\|_{0,\rho}^2 + Cd^2 + 2h \sum_{k=1}^{n+1} \sigma^{(k)}) \exp(CT) \to 0 \tag{39}$$

as $h, d \to 0$.

Estimation of $\left\|\Theta^{(n)}\right\|_{1,\rho}$.

For this term we use arguments similar to those for $\left\|\Theta^{(n)}\right\|_{0,\rho}$.

Multiply equality (20) by $\rho A_d \Theta^{(n+1)}$ and integrate over $\mathbb{R}$:

$$(\textstyle\int_{\mathbb{R}} \rho(\Theta^{(n+1)} - \Theta^{(n)})\partial_+\partial_- \Theta^{(n+1)} dx)/h = \int_{\mathbb{R}} \rho |A_d \Theta^{(n+1)}|^2 dx +$$

$$+ \textstyle\int_{\mathbb{R}} \rho(\sigma_1^{n+1} + \sigma_2^{n+1} + \sigma_3^{n+1} + \mathcal{P}_d(f(v^{(n+1)} - f(u(n+1)h, x)))) A_d \Theta^{(n+1)} dx. \tag{40}$$

Let us "integrate by parts" on the left in (40):

$$(\textstyle\int_{\mathbb{R}} \rho(\Theta^{(n+1)} - \Theta^{(n)})\partial_+\partial_- \Theta^{(n+1)} dx)/h =$$

$$= -(\textstyle\int_{\mathbb{R}} \partial_-(\rho(\Theta^{(n+1)} - \Theta^{(n)}))\partial_- \Theta^{(n+1)} dx)/h =$$

$$= -(\textstyle\int_{\mathbb{R}} \rho(\partial_- \Theta^{(n+1)} - \partial_- \Theta^{(n)})\partial_- \Theta^{(n+1)} dx)/h -$$

$$-(\textstyle\int_{\mathbb{R}} (\partial_- \rho)(T_{-d}(\Theta^{(n+1)} - \Theta^{(n)}))\partial_- \Theta^{(n+1)} dx)/h \leq$$

(we change variables in the second integral and take (20) into account)

$$\leq -(\left\|\partial_- \Theta^{(n+1)}\right\|_{0,\rho}^2 - \left\|\partial_- \Theta^{(n)}\right\|_{0,\rho}^2)/(2h) -$$

$$- \textstyle\int_{\mathbb{R}} (\partial_+ \rho(A_d \Theta^{(n+1)} + \sigma_1^{n+1} + \sigma_2^{n+1} + \sigma_3^{n+1} +$$

$$+ \mathcal{P}_d(f(v^{(n+1)} - f(u(n+1)h, x))))) \partial_+ \Theta^{(n+1)} dx. \tag{41}$$

It follows from (40) and (41) that

$$(\left\|\partial_- \Theta^{(n+1)}\right\|_{0,\rho}^2 - \left\|\partial_- \Theta^{(n)}\right\|_{0,\rho}^2)/(2h) + \left\|A_d \Theta^{(n+1)}\right\|_{0,\rho}^2 \leq$$

$$\leq -\textstyle\int_{\mathbb{R}} (\partial_- \rho) A_d \Theta^{(n+1)} \partial_+ \Theta^{(n+1)} dx +$$

$$+ \textstyle\int_{\mathbb{R}} (\sigma_1^{n+1} + \sigma_2^{n+1} + \sigma_3^{n+1})(\rho A_d \Theta^{(n+1)} - (\partial_+ \rho) \partial_+ \Theta^{(n+1)}) dx -$$

$$- \textstyle\int_{\mathbb{R}} \mathcal{P}_d (f(v^{(n+1)} - f(u(n+1)h, x))))((\partial_+ \rho) \partial_+ \Theta^{(n+1)} - \rho A_d \Theta^{(n+1)}) dx \leq$$

(we take $\beta = 1/(2Cd)$ and apply the usual $2ab \leq \beta a^2 + b^2/\beta$ trick)

$$\leq C\beta d \left\| A_d \Theta^{(n+1)} \right\|_{0,\rho}^2 + Cd(1 + 1/\beta) \left\| \partial_+ \Theta^{(n+1)} \right\|_{0,\rho}^2 +$$

$$+ C(1 + 1/\beta) \left\| \sigma_1^{n+1} + \sigma_2^{n+1} + \sigma_3^{n+1} \right\|_{0,\rho}^2 +$$

$$+ C(1 + 1/\beta) \left\| \mathcal{P}_d (f(v^{(n+1)}) - f(u(n+1)h, x))) \right\|_{0,\rho}^2.$$

Estimating the squared norm in the latter term by

$$\mathcal{L}^2 \left\| \Theta^{(n+1)} \right\|_{0,\rho}^2 + Cd^2 \| u_0 \|_{1,\rho}^2,$$

we arrive at the following estimate:

$$(\left\| \partial_- \Theta^{(n+1)} \right\|_{0,\rho}^2 - \left\| \partial_- \Theta^{(n)} \right\|_{0,\rho}^2)/(2h) + \left\| A_d \Theta^{(n+1)} \right\|_{0,\rho}^2 / 2 \leq$$

$$\leq Cd \left\| \partial_- \Theta^{(n+1)} \right\|_{0,\rho}^2 + C \left\| \sigma_1^{n+1} + \sigma_2^{n+1} + \sigma_3^{n+1} \right\|_{0,\rho}^2 + C \left\| \Theta^{(n+1)} \right\|_{0,\rho}^2 + Cd^2.$$

By the Gronwall lemma,

$$\left\| \partial_- \Theta^{(n)} \right\|_{0,\rho}^2 \leq C(\left\| \partial_- \Theta^{(0)} \right\|_{0,\rho}^2 +$$

$$\sum_{k=1}^{n} \left\| \sigma_1^k + \sigma_2^k + \sigma_3^k \right\|_{0,\rho}^2 + h \sum_{k=1}^{n} \left\| \Theta^{(k)} \right\|_{0,\rho}^2 + d^2) \exp(CT).$$

The first term in parentheses on the right vanishes, while the remaining terms tend to 0 as $h, d \to 0$, see (38) and (39).

Finally, we apply Lemma 4 to show that

$$\left\| \nabla \Theta^{(n)} \right\|_{0,\rho}^2 =$$

$$= \sum_{k \in Z} \textstyle\int_{kd}^{(k+1)d} \rho |((\mathcal{T}^{-1} \Theta^{(n)})_{k+1} - (\mathcal{T}^{-1} \Theta^{(n)})_k)/d|^2 dx \leq$$

$$\leq C \sum_{k \in Z} \rho(kd) |((\mathcal{T}^{-1} \Theta^{(n)})_{k+1} - (\mathcal{T}^{-1} \Theta^{(n)})_k)/d|^2 =$$

$$= C \left\| \mathcal{T} \partial_- \mathcal{T}^{-1} \Theta^{(n)} \right\|_{0,\rho}^2 = C \left\| \partial_- \Theta^{(n)} \right\|_{0,\rho}^2 \to 0$$

as $h, d \to 0$.

To complete the proof of Proposition 9, it remains to note that if $T > 0$, then the estimates obtained above hold for any $nh \in (0, T]$.

5 Regularity estimates and compactness

Let us start with the

Proof of Proposition 3

The proof of Lemma 2.3 in [3], with inequality (2.15) replaced by the inequality $\langle B^*Bv, v\rangle_\rho \geq (1 - Ch)\langle v, v\rangle_\rho)$, shows that the operator $\mathcal{S}$ has a Lipschitz constant $1 + Ch$. For the a-priori estimate (12) we use the energy estimate from Lemma 2.1 in [3]

$$\langle Av, \rho v\rangle \leq -C\|\partial_- v\|_{0,\rho}^2 + C\varepsilon\|v\|_{0,\rho}^2, \quad v \in H_\rho. \tag{42}$$

Here we used the inner product $\langle u, v\rangle = \sum_{k=0}^\infty u_k v_k$. Multiply (2) by ρu^{n+1} and use (AI) and (42) to obtain

$$\begin{aligned} \frac{1}{2h}(\|u^{(n+1)}\|_{0,\rho}^2 - \|u^{(n)}\|_{0,\rho}^2) &\leq \frac{1}{2h}\langle u^{(n+1)} - u^{(n)}, \rho(u^{(n+1)} - u^{(n)} + u^{(n+1)} + u^{(n)})\rangle \\ &= \langle \rho u^{n+1}, \overline{f}(u^{n+1})\rangle + \langle \rho u^{n+1}, Au^{n+1}\rangle \\ &\leq C\|u^{(n+1)}\|_{0,\rho}^2 - C\|\partial_- u^{(n+1)}\|_{0,\rho}^2. \end{aligned}$$

Summing up leads to

$$\|u^{(n+1)}\|_{0,\rho}^2 + Ch\sum_{j=1}^n \|\partial_- u^{(j)}\|_{0,\rho}^2 \leq \|u^{(0)}\|_{0,\rho}^2 + Ch\sum_{j=1}^n \|u^{(j)}\|_{0,\rho}^2,$$

from which (12) follows by a discrete Gronwall estimate.

For the proof of (13) consider $w_k^{(n)} := \partial_+ u_k^{(n)}$ which satisfies the following equation:

$$\frac{w_k^{(n+1)} - w_k^{(n)}}{h} = \partial_+\partial_- w_k^{(n+1)} - \frac{f(u_{k+1}^{(n+1)}) - f(u_k^{(n+1)})}{h} =$$

$$= \partial_+\partial_- w_k^{(n+1)} - \int_0^1 f'(u_{k+1}^{(n+1)}\theta + u_k^{(n+1)}(1-\theta))d\theta \cdot w_k^{(n+1)}.$$

Therefore, the sequence $v_k^{(n)} := hn\partial_+ u_k^{(n)}$, $n \geq 0$, $k \in \mathbb{Z}$, is a solution of the equation

$$\frac{v_k^{(n+1)} - v_k^{(n)}}{h} = \partial_+\partial_- v_k^{(n+1)} - \int_0^1 f'(u_{k+1}^{(n+1)}\theta + u_k^{(n+1)}(1-\theta))v_k^{(n+1)}d\theta + \partial_+ u_k^{(n)};$$

$$v^{(0)} = 0.$$

As usual, we multiply the last equation by $\rho_k \partial_+ \partial_- v_k^{(n+1)}$, sum the expressions obtained for all $k \in \mathbb{Z}$, and result in the following inequalities:

$$\frac{\left\|\partial_+ v^{(n+1)}\right\|_{0,\rho}^2 - \left\|\partial_+ v^{(n)}\right\|_{0,\rho}^2}{2h} = d \sum_{k\in\mathbb{Z}} \rho_k \frac{|\partial_+ v_k^{(n+1)}|^2 - |\partial_+ v_k^{(n)}|^2}{2h} \leq$$

$$\leq d \sum_{k\in\mathbb{Z}} \rho_k \frac{\partial_+ v_k^{(n+1)} (\partial_+ v_k^{(n+1)} - \partial_+ v_k^{(n)})}{h} =$$

$$= -d \sum_{k\in\mathbb{Z}} \partial_- (\rho_k \partial_+ v_k^{(n+1)}) \frac{v_k^{(n+1)} - v_k^{(n)}}{h} =$$

$$= -d \sum_{k\in\mathbb{Z}} \Big(\partial_- \rho_k \partial_- v_k^{(n+1)} + \rho_k \partial_+ \partial_- v_k^{(n+1)} \Big) \frac{v_k^{(n+1)} - v_k^{(n)}}{h} =$$

$$= -d \sum_{k\in\mathbb{Z}} \Big(\partial_- \rho_k \partial_- v_k^{(n+1)} + \rho_k \partial_+ \partial_- v_k^{(n+1)} \Big) \Big(\partial_+ \partial_- v_k^{(n+1)} +$$

$$+ \int_0^1 f'(u_{k+1}^{(n+1)} \theta + u_k^{(n+1)} (1-\theta)) d\theta \cdot v_k^{(n+1)} + \partial_+ u_k^{(n)} \Big) =$$

$$= -d \sum_{k\in\mathbb{Z}} \rho_k |\partial_+ \partial_- v_k^{(n+1)}|^2 - \sum_{k\in\mathbb{Z}} \partial_+ \partial_- v_k^{(n+1)} \Big(\partial_- \rho_k \partial_- v_k^{(n+1)} +$$

$$+ \rho_k \int_0^1 f'(u_{k+1}^{(n+1)} \theta + u_k^{(n+1)} (1-\theta)) d\theta \cdot v_k^{(n+1)} + \rho_k \partial_+ u_k^{(n)} \Big) -$$

$$-d \sum_{k\in\mathbb{Z}} \partial_- \rho_k \partial_- v_k^{(n+1)} \Big(\int_0^1 f'(u_{k+1}^{(n+1)} \theta + u_k^{(n+1)} (1-\theta)) d\theta \cdot v_k^{(n+1)} + \partial_+ u_k^{(n)} \Big) \leq$$

$$\leq -d \sum_{k\in\mathbb{Z}} \rho_k |\partial_+ \partial_- v_k^{(n+1)}|^2 + C\varepsilon d \sum_{k\in\mathbb{Z}} \rho_k |\partial_+ \partial_- v_k^{(n+1)}|^2 +$$

$$+ C\varepsilon^{-1} d \sum_{k\in\mathbb{Z}} \rho_k \Big(|\partial_+ v_k^{(n+1)}|^2 + |v_k^{(n+1)}|^2 + |\partial_+ u_k^{(n)}|^2 \Big) +$$

$$+ Cd \sum_{k\in\mathbb{Z}} \rho_k \Big(|\partial_+ v_k^{(n+1)}|^2 + |v_k^{(n+1)}|^2 + |\partial_+ u_k^{(n)}|^2 \Big).$$

Taking $\varepsilon < C^{-1}$ we can continue for $(n+1)h \leq T$

$$\leq Cd \sum_{k\in\mathbb{Z}} \rho_k \Big(|\partial_+ v_k^{(n+1)}|^2 + |v_k^{(n+1)}|^2 + |\partial_+ u_k^{(n)}|^2 \Big) =$$

$$= Cd\sum_{k\in\mathbb{Z}}\rho_k\Big(|\partial_+ v_k^{(n+1)}|^2 + |h(n+1)\partial_+ u_k^{(n+1)}|^2 + |\partial_+ u_k^{(n)}|^2\Big) \leq$$

$$\leq Cd\sum_{k\in\mathbb{Z}}\rho_k\Big(|\partial_+ v_k^{(n+1)}|^2 + |\partial_+ u_k^{(n+1)}|^2 + |\partial_+ u_k^{(n)}|^2\Big).$$

Summarizing, we have shown

$$\frac{\left\|\partial_+ v^{(n+1)}\right\|_{0,\rho}^2 - \left\|\partial_+ v^{(n)}\right\|_{0,\rho}^2}{2h} \leq C\left(\left\|\partial_+ v^{(n+1)}\right\|_{0,\rho}^2 + \left\|\partial_+ u^{(n+1)}\right\|_{0,\rho}^2 + \left\|\partial_+ u^{(n)}\right\|_{0,\rho}^2\right). \tag{43}$$

Finally we multiply inequality (43) by $2h$, take the sum over all $n = 0,\dots,N-1$, $Nh \leq T$, and with $v^{(0)} = 0$ obtain the following inequality:

$$\left\|\partial_+ v^{(N)}\right\|_{0,\rho}^2 0 \leq \left\|\partial_+ v^{(0)}\right\|_{0,\rho}^2 + Ch\sum_{n=1}^{N}\left\|\partial_+ v^{(n)}\right\|_{0,\rho}^2 + Ch\sum_{n=0}^{N}\left\|\partial_+ u^{(n)}\right\|_{0,\rho}^2 \leq$$

$$\leq Ch\sum_{n=1}^{N}\left\|\partial_+ v^{(n)}\right\|_{0,\rho}^2 + C\left\|u^{(0)}\right\|_{0,\rho}^2.$$

Applying the discrete Gronwall inequality, we get the estimate:

$$\left\|\partial_+ v^{(N)}\right\|_{0,\rho}^2 \leq C\left\|u^{(0)}\right\|_{0,\rho}^2$$

for all numbers N such that $0 < Nh \leq T$. This means that

$$\left\|\partial_+\partial_- u^{(N)}\right\|_{0,\rho}^2 \leq C(Nh)^{-2}\left\|u^{(0)}\right\|_{0,\rho}^2.$$

and Proposition 3 is proved. □

Proof of Proposition 10

We apply Proposition 11 to the sequence $v_m = \mathcal{T}_m u_m$. Since the attractors $\mathcal{A}(h_m, d_m)$ are uniformly bounded in the space $\mathcal{Z}_u$, Lemma 5 shows that $\|\mathcal{T}_m u_m\|_{1,u}$ is also bounded. Moreover, using the invariance of the attractor $\mathcal{A}(h_m, d_m)$ under translation and iteration, Proposition 3 implies that $\|u_m\|_{2,u}$ is uniformly bounded as well (use (13) with $nh = 1$). Therefore, we can apply Lemma 7 and obtain, uniformly in m,

$$\|T_y v_m - v_m\|_{1,\rho} \leq \sqrt{|y|}\,\|u_m\|_{2,u} \to 0 \quad \text{as} \quad y \to 0.$$

Then Proposition 11 yields the assertion. □

Proof of Lemma 8

We know that the attractor $\mathcal{A}$ is bounded with respect to $\|\cdot\|_{1,u}$ and therefore - similar to the discrete case - the regularity estimate (7) in Proposition 2 together with the translation invariance of the attractor shows that $\mathcal{A}$ is bounded with respect to $\|\cdot\|_{2,u}$. Lemma 8 then implies

$$\sup_{u\in\mathcal{A}} \|\mathcal{P}_d u - u\|_{1,\rho} \leq Cd.$$

□

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