The Schwarzschild solution in a Kaluza-Klein theory with two times

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Abstract

A new spherically-symmetric solution is determined in a noncompactified Kaluza-Klein theory in which a time character is ascribed to the fifth coordinate. This solution contains two independent parameters which are related with mass and electric charge. The solution exhibits a Schwarzschild radius and represents a generalization of the Schwarzschild solution in four dimensions. The parameter of the solution connected with the electric charge depends on the derivative of the fifth (second time) coordinate with respect to the ordinary time coordinate. It is shown that the perihelic motion in four-dimensional relativity has a counterpart in five dimensions in the perinucleic motion of a negatively-charged particle. If the quantization conditions of the older quantum theory are applied to that motion, an analogue of the fine-structure formula of atomic spectra is obtained.

1 Introduction

The notion of a second time variable is not alien to physics. In the papers of Horwitz and his coworkers [1, 2, 3, 4], a five-dimensional theory of electromagnetism was presented. In that theory the system develops on the four-dimensional space-time manifold $(\vec{x}, t)$, according to an evolution parameter, "universal time" τ. The signatures $(4, 1)$ and $(3, 2)$ of the five-dimensional metric are considered, and the reasons for preference of the metric with signature $(3, 2)$ are explained. The five-dimensional field equations derived in [2] are called "pre-Maxwell" equations, and the respective fields which obey those equations are called "pre-Maxwell" fields. The "pre-Maxwell" equations, when referred to the Cartesian system of axes, formally represent a counterpart of Maxwell-Nordström equations (see Section 4 in [2] and [5]), with imaginary fifth coordinate, although Nordström's paper is not quoted in [2]. A possible physical meaning of the evolution parameter τ was not discussed in [2, 3, 4].

In general relativity a second time variable was introduced and discussed in the quality of a universal parametric "historical time" by Horwitz and Piron [1]. That idea was developed by Burakovsky and Horwitz [6] in their study of a five-dimensional cosmological model of Kaluza-Klein type.

In the Kaluza-Klein theory three objections were formulated against the timelike signature of the fifth coordinate [7, 8, 9]: (1) When in the five-dimensional action the integration over the fifth coordinate is performed, provided that all derivatives with respect to the fifth coordinate are omitted and the cylinder condition is accepted, the Maxwell action comes out with an opposite sign to that of the Einstein action. This is considered to be incorrect. (2) The existence of tachyons follows from the accepted cylinder condition. (3) There would appear closed time curves. The problem of closed time curves in four-dimensional gravity was investigated in a series of papers of Friedman, Thorne and their coworkers [10, 11, 12]. A particular attention was paid to the question whether closed time curves violate the causality principle. The available answer does not seem to be conclusive in that respect. In the case of the five-dimensional gravity with two time coordinates, an analogous investigation has not been undertaken. It is an open question whether the second objection concerning the existence of tachyons is relevant for noncompactified Kaluza-Klein theories. As to the first objection, it seems that the relative sign of Einstein and Maxwell actions should be the outcome of a five- or higher-dimensional relativity theory, and that an answer to the question which relative sign is correct is not yet finally settled. The attitude towards a timelike signature of the fifth coordinate is less restrained in the book of Wesson [13], in which a spacelike or a timelike signature of the fifth coordinate is admitted, depending on the physical problem in question.

The idea of two times in a physical theory is a leit-motiv in the investigations of I. Bars and his coworkers [14], [15], [16]. They have shown that two-time physics provides a new perspective for the understanding of the one-time dynamics, from a higher-dimensional point of view; from a single action formula of two-time physics, with the application of gauge theory, diverse one-time dynamical systems can be obtained.

We observe that the second time variable was employed in [5] in a discussion of the consequences following from Maxwell-Nordström equations with two time variables. These equations then describe electromagnetic phenomena together with hypothetical gravitomagnetic phenomena, in particular, gravitomagnetic waves as a counterpart of electromagnetic waves. The main results of this paper were presented in [17].

In Section 2 we touch upon the controversy concerning the relative sign of Einstein and Maxwell actions.

In Section 3 we start from the line element which formally is identical with that of Chodos and Detweiler [18]. The difference consists in the spatial character of the fifth coordinate x^5 in [18] and the time character ascribed to this coordinate in the present case. We determine a static five-dimensional spherically-symmetric solution on the basis of the line element in which there are two time coordinates. This solution exhibits a Schwarzschild radius and represents a generalization of the four-dimensional Schwarzschild solution. The solution depends on four parameters of which two are independent and can be related with gravitational mass and electric charge. This is accomplished in Section 3 where we discuss the geodesic equation in a nearly flat space. From the geodesic equa-

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tion there follows a linear relation between the ordinary time t and the second time variable u.

In Section 4 we solve the problem of the perinucleic motion of a negatively charged test particle (electron) moving in the field of the central positively charged mass. Including into the five-dimensional geometry the quantization conditions of the older quantum theory [19] we derive an analogue of Sommerfeld's relativistic energy-level formula [19]. The adiabatic invariance principle of Ehrenfest with which Bohr-Sommerfeld quantization rules are related, already played an important role in the determination of the mass spectrum of black holes [20, 21, 22, 23]. The present application of Bohr-Sommerfeld quantization rules yields another example of the role of Ehrenfest's principle in bridging the gap between general relativity and quantum theory.

2 The five-dimensional line element in the Kaluza-Klein theory

In the Kaluza-Klein theory the five-dimensional line element is split into the four-dimensional part and the term depending on the fifth dimension. Referring to [24] we write the line element in the five-dimensional Riemann space

$$dS^2 = \gamma_{ab}\, dx^a dx^b \quad a, b = 0, 1, 2, 3, 4 \tag{1}$$

where the indices $a, b = 0, 1, 2, 3$ refer to four dimensions, the label 0 refers to the time dimension, and the label 4 refers to the fifth dimension. Spacelike or timelike character of the fifth coordinate is admitted. The signature of the metric tensor therefore is $(+, -, -, -, \epsilon)$ with $\epsilon = +1$ for timelike and $\epsilon = -1$ for spacelike fifth coordinate.

A new metric tensor is defined by

$$g_{ab} = \gamma_{ab} - \frac{\gamma_{4a}\gamma_{4b}}{\gamma_{44}} \tag{2}$$

with the property

$$g_{44} = g_{4a} = 0 \tag{3}$$

We refer to [24] for this transformation, while an analogous transformation for the case of four dimensions appears with Jordan [25].

The original metric tensor can now be rewritten in the form

$$\gamma_{ab} = g_{ab} + \frac{\gamma_{4a}\gamma_{4b}}{\gamma_{44}} \tag{4}$$

Since $g_{ab} \neq 0$ only for $a, b = 0, 1, 2, 3$ the line element in Eq. (1) takes the form

$$dS^2 = g_{\mu\nu}\, dx^\mu dx^\nu + \gamma_{44}\Big(dx^4 + \frac{\gamma_{4\alpha}}{\gamma_{44}}\, dx^\alpha\Big)^2 \quad \mu, \nu, \alpha = 0, 1, 2, 3 \tag{5}$$

We next define the electromagnetic four-potential

$$A_\mu = \frac{\gamma_{\mu 4}}{\gamma_{44}} \tag{6}$$

and the Brans-Dicke scalar field

$$\Phi^2 = \epsilon\,\gamma_{44} \tag{7}$$

where $\epsilon = +1$ for timelike signature and $\epsilon = -1$ for spacelike signature. The line element now takes the form

$$dS^2 = ds^2 + \epsilon\,\Phi^2\big(dx^4 + A_\mu dx^\mu\big)^2 \tag{8}$$

where

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu\,, \quad \mu,\nu = 0,1,2,3 \tag{9}$$

A calculation of the components of the Riemann tensor R_{abcd} in five dimensions on the basis of the metric defined in Eq. (8) is largely facilitated when the calculus of exterior forms is applied. This was done by Thiry [26, 27]. The five dimensional line element in [26] is written in the form

$$d\sigma^2 = \gamma_{\mu\nu}\,dx^\mu dx^\nu = \sum_{\alpha=0}^{4}(\sigma_\alpha)^2 \tag{10}$$

where $\mu,\nu = 0,1,2,3,4$ and the index 0 labels the fifth dimension. Thiry introduces the new metric g_{ab} in Eq. (2) in which index 4 is replaced by index 0. In this metric the line element is written in the form

$$d\sigma^2 = g_{ij}dx^i dx^j + V^2(dx^0 + \beta\varphi_i dx^i)^2\,, \quad i,j = 1,2,3,4 \tag{11}$$

where

$$\varphi_i = \frac{\gamma_{i0}}{\beta V^2}\,, \quad V^2 = \gamma_{00} \tag{12}$$

These quantities correspond to those in Eq. (6) and (7), where β is a constant. This expression for the line element is the expression in Eq. (8) in the case when $\epsilon = +1$, which means timelike fifth dimension. In [26] Thiry directly refers to the Kaluza-Klein theory, hence the spatial character of the fifth coordinate is implicit. In [27] the spatial character of the fifth coordinate in [26] is explicitly asserted. On the basis of the preceding argument we conclude that the spatial character assigned to the fifth coordinate in [26] and [27] is incompatible with the expression for the line element in Eq. (11) which appears in [26].

3   A spherically-symmetric solution for a two-time line element

We consider the line element in Eq. (1), in which, however, we change the meaning of the indices a, b. We define $a, b = 1, \ldots, 5$, where $1, 2, 3$ label the space coordinates, 4 labels the ordinary time coordinate, and 5 labels the second-time coordinate.

In a flat space we introduce the Cartesian coordinates $x^1, x^2, x^3 = x, y, z$, $x^4 = ct$, $x^5 = cu$ where c denotes the speed of light in the vacuum while t and u are expressed in the units of time. The non-zero components of the metric tensor are: $\gamma_{11} = \gamma_{22} = \gamma_{33} = -1$, $\gamma_{44} = 1$ and $\gamma_{55} = 1$ since time character is ascribed to the fifth coordinate.

With

$$(x^1, x^2, x^3, x^4, x^5) = (r, \theta, \varphi, ct, cu) \tag{13}$$

the spherically-symmetric line element in a flat space has the form

$$dS^2 = -dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + c^2 dt^2 + c^2 du^2 \tag{14}$$

A general form of the spherically symmetric line element in a curved space is the following:

$$
\begin{aligned}
dS^2 = {} & A(r, t, u)\, c^2 dt^2 + B(r, t, u)\, dr^2 + C(r, t, u)\, cdrdt + \\
& + D(r, t, u)\, (d\theta^2 + \sin^2\theta\, d\varphi^2) + E(r, t, u)\, c^2 du^2 \\
& + F(r, t, u)\, cdrdu + G(r, t, u)\, c^2 dudt
\end{aligned}
\tag{15}
$$

In an appropriate coordinate system r', t', u' we can assume that

$$C(r', t', u') = F(r', t', u') = 0 \tag{16}$$

and

$$D(r', t', u') = -r'^2 \tag{17}$$

In the following we shall omit the "prime" of the new coordinates r', t' and u' and we assume that the functions A, B, E, and G in Eq. (4) are of the form:

$$
\begin{aligned}
A(r, t, u) = e^{\nu(r)} \qquad & B(r, t, u) = -e^{\lambda(r)} \\
E(r, t, u) = e^{\mu(r)} \qquad & G(r, t, u) = \sigma(r)
\end{aligned}
\tag{18}
$$

with $\mu, \nu, \lambda, \sigma \to 0$ when $r \to \infty$. The line element in Eq. (15) now takes the form:

$$dS^2 = -e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + e^\nu c^2 dt^2 + e^\mu c^2 du^2 + \sigma c^2 dudt \tag{19}$$

With the speed of light in the vacuum c absorbed by the variables t and u, the respective metric tensor components are

$$
\begin{array}{llll}
\gamma_{11} = -e^\lambda \ , & \gamma_{22} = -r^2 \ , & \gamma_{33} = -r^2 \sin^2\theta \ , \\
\gamma_{44} = e^\nu \ , & \gamma_{55} = e^\mu \ , & \gamma_{45} = \gamma_{54} = \sigma
\end{array}
\tag{20}
$$

The determinant γ of this metric tensor is given by

$$
\gamma = r^4 \, e^\lambda \, \sin^2\theta \left(\sigma^2 - e^{\mu+\nu} \right)
\tag{21}
$$

The line element in Eq. (19) is analogous to that considered by Chodos and Detweiler in Eq. (17) of [18], however, in that paper the fifth coordinate is a space coordinate, while we assign a time character to the fifth coordinate. Consequently, the spherically symmetric solution connected with the line element in Eq. (19) will be of a different form that in [18].

When all derivatives with respect to the times t and u are omitted, denoting by a "prime", the derivative d/dr we obtain the following non-zero components of the contracted Riemann tensor

$$
\begin{aligned}
R_{11} = \frac{1}{4\,r\,(\sigma^2 - e^{\mu+\nu})^2} & \left\{ 2\,r\,\left(-\sigma^2 - e^{\mu+\nu}\right)\sigma'^{\,2} \right. \\
- \left(\sigma^2 - e^{\mu+\nu}\right)\lambda'\,&\left(4\,\sigma^2 - 4\,e^{\mu+\nu} - e^{\mu+\nu}\,r\,\mu' - e^{\mu+\nu}\,r\,\nu'\right) \\
-2\,\sigma\,r\,\sigma'\,&\left[\left(\sigma^2 - e^{\mu+\nu}\right)\lambda' - 2\,e^{\mu+\nu}\,(\mu' + \nu')\right] \\
+r\Big[e^{\mu+\nu}(e^{\mu+\nu} - &2\sigma^2)(\mu'^2 + \nu'^2) - 2\,e^{\mu+\nu}\,\sigma^2\,\mu'\,\nu' \\
+4\,\sigma^3\,\sigma'' - 4\,e^{\mu+\nu}\,\sigma\,\sigma'' + &\left. 2e^{\mu+\nu}(e^{\mu+\nu} - \sigma^2)(\mu'' + \nu'')\Big]\right\}
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
R_{22} = -\frac{e^{-\lambda}}{2\,(\sigma^2 - 2e^{\mu+\nu})}& \left[-2\,\sigma\,r\,\sigma' + \sigma^2\left(-2 + 2\,e^\lambda + r\,\lambda'\right) \right. \\
& \left. - e^{\mu+\nu}\left(-2 + 2\,e^\lambda + r\,\lambda' - r\,\mu' - r\,\nu'\right) \right]
\end{aligned}
\tag{23}
$$

$$
R_{33} = R_{22}\,\sin^2\theta
\tag{24}
$$

$$
\begin{aligned}
R_{44} = \frac{-e^{-\lambda+\nu}}{4\,r\,(\sigma^2 - e^{\mu+\nu})}& \left[2\,r\,\sigma'^{\,2} - 2\,\sigma\,r\,\sigma'\,\nu' \right. \\
+ \left[4\,\sigma^2 - 4\,e^{\mu+\nu} - \sigma^2\,r\,\lambda' + e^{\mu+\nu}\,r\,(\lambda' - \mu')\right]&\,\nu' \\
+r\left(2\,\sigma^2 - e^{\mu+\nu}\right)\nu'^2 + &\left. 2\,r\left(\sigma^2 - e^{\mu+\nu}\right)\nu'' \right]
\end{aligned}
\tag{25}
$$

$$R_{45} = R_{54} = \frac{1}{4\,e^\lambda\,r\,(\sigma^2 - e^{\mu+\nu})}\left[\sigma^2\,\sigma'\,(-4 + r\,\lambda')\right.$$
$$- e^{\mu+\nu}\,\sigma'\,(-4 + r\,\lambda' + r\,\mu' + r\,\nu')$$
$$\left. +2\,r\,\left(e^{\mu+\nu}\,\sigma\,\mu'\,\nu' - \sigma^2\,\sigma'' + e^{\mu+\nu}\,\sigma''\right)\right] \quad (26)$$

$$R_{55} = \frac{-e^{-\lambda+\mu}}{4\,r\,(\sigma^2 - e^{\mu+\nu})}\left[2\,r\,\sigma'^2 - 2\,\sigma\,r\,\sigma'\,\mu' + r\,\left(2\,\sigma^2 - e^{\mu+\nu}\right)\,\mu'^2\right.$$
$$+\mu'\,\left(4\,\sigma^2 - 4\,e^{\mu+\nu} - \sigma^2\,r\,\lambda' + e^{\mu+\nu}\,r\,\lambda' - e^{\mu+\nu}\,r\,\nu'\right)$$
$$\left. +2\,r\,\left(\sigma^2 - e^{\mu+\nu}\right)\,\mu''\right] \quad (27)$$

These are equated to zero, yielding five equations for the four unknown functions $\mu, \nu, \lambda, \sigma$ in Eq. (18).

It can be verified that a solution of the equations

$$R_{11} = R_{22} = R_{44} = R_{55} = R_{45} = 0 \quad (28)$$

is given by the following functions:

$$e^\nu = 1 - \frac{\mathcal{G}}{r}, \quad e^\mu = 1 - \frac{\mathcal{C}}{r}, \quad e^\lambda = \left(1 - \frac{\mathcal{R}}{r}\right)^{-1}, \quad \sigma = \frac{\mathcal{P}}{r} \quad (29)$$

where the real parameters $\mathcal{G}, \mathcal{C}, \mathcal{R}$ and $\mathcal{P}$ satisfy the conditions

$$\mathcal{R} = \mathcal{G} + \mathcal{C}, \quad \mathcal{P}^2 = \mathcal{G}\mathcal{C} \quad (30)$$

The parameters $\mathcal{G}$ and $\mathcal{C}$ will be related with gravitational mass and electric charge, respectively, on the basis of the linearized form of the geodesic equation.

4   The parameters in the sperically-symmetric solution

We consider the two-time-independent metric tensor of the form:

$$\gamma_{ab} = \gamma_{ab}^{(5)} + \eta_{ab}, \quad a, b = 1, \ldots, 5 \quad (31)$$

where $\gamma_{ab}^{(5)}$ is the five-dimensional flat-space metric tensor specified as at the beginning of Section 3, and η_{ab} represents a small perturbation, due to the presence of a gravitating body with an electric charge. The perturbation vanishes very far from the body. To show that the η_{ab} terms are the agents of gravitational and electrostatic forces we consider the geodesic equation of motion in the Riemann space with the above metric. We assume that the velocity of a test particle (with mass and electric charge) along the geodesic line is much smaller

than the speed of light c. Using the nearly flat-space metric tensor in Eq. (31) we then obtain the line element

$$dS^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2 + (dx^5)^2 + \eta_{ab}dx^a dx^b \qquad (32)$$

from which we obtain:

$$\left(\frac{dS}{dt}\right)^2 = c^2\left[1 - \frac{v^2}{c^2} + \left(\frac{du}{dt}\right)^2 + \frac{1}{c^2}\,\eta_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}\right] \qquad (33)$$

where $v^j = dx^j/dt$, $j = 1, 2, 3$. Retaining in Eq. (33) the terms of the first order in v/c we obtain:

$$\left(\frac{dS}{dt}\right)^2 \approx c^2\left[1 + \left(\frac{du}{dt}\right)^2(1 + \eta_{55}) + \eta_{44} + 2\eta_{45}\frac{du}{dt}\right] \qquad (34)$$

We had to assume that $v^2/c^2 \ll (du/dt)^2$, since otherwise we would have to omit $(du/dt)^2$ together with v^2/c^2.

We next apply the same approximations to the geodesic equation

$$\frac{d^2x^a}{dS^2} + \Gamma^a_{bc}\frac{dx^b}{dS}\frac{dx^c}{dS} = 0 \qquad (35)$$

From the form of the metric in Eq. (32) follows that each Christoffel symbol linearly depends on the perturbation η_{ab}. With the accuracy to terms of order v/c we obtain the equality

$$\Gamma^a_{bc}\frac{dx^b}{dS}\frac{dx^c}{dS} = c^2\left[\Gamma^a_{44} + 2\frac{du}{dt}\Gamma^a_{45} + \left(\frac{du}{dt}\right)^2\Gamma^a_{55}\right]\left(\frac{dt}{dS}\right)^2 \qquad (36)$$

From Eq. (35) and (36) follows the equation

$$\frac{d^2x^a}{dt^2} = c^2\left[\Gamma^a_{44} + 2\frac{du}{dt}\Gamma^a_{45} + \left(\frac{du}{dt}\right)^2\Gamma^a_{55}\right] \qquad (37)$$

Since η_{ab} are independent of t and u the Christoffel symbols in Eq. (37) vanish for $a = 4, 5$. For $a = 1, 2, 3$ we have

$$\Gamma^a_{44} = \frac{1}{2}\frac{\partial\eta_{44}}{\partial x^a}\,, \quad \Gamma^a_{45} = \frac{1}{2}\frac{\partial\eta_{45}}{\partial x^a}\,, \quad \Gamma^a_{55} = \frac{-1}{2}\frac{\partial\eta_{55}}{\partial x^a} \qquad (38)$$

and from Eq. (37), for $a = 5$ we find that $d^2u/dt^2 = 0$, hence

$$u = wt + u_0 \qquad (39)$$

where w and u_0 are constants. With $u_0 = 0$, $w > 0$, considering Eqs. (38), and with $w = du/dt$ we obtain from Eq. (37) the equation

$$\frac{d^2x^a}{dt^2} = -\frac{c^2}{2}\left[\frac{\partial\eta_{44}}{\partial x^a} + 2w\frac{\partial\eta_{45}}{\partial x^a} + w^2\frac{\partial\eta_{55}}{\partial x^a}\right] \qquad (40)$$

On the basis of this equation we will determine the parameters $\mathcal{G}$ and $\mathcal{C}$ in Eqs. (29) and (30).

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We assume that the test particle is an electron with mass m. Multiplying both sides of Eq. (40) with m, we can identify the first term on the r.h.s. with the gravitational force acting on the electron mass m, and to one of the two remaining terms we can ascribe the meaning of the Coulomb force acting on the electron charge e. It will appear that we have to relate the third term on the r.h.s. of Eq. (40) with the Coulomb force.

We begin with the first term on the r.h.s. of Eq. (40) and write:

$$\left(m\frac{d^2x^a}{dt^2}\right)_{\text{mechanical}} = -m\frac{c^2}{2}\frac{\partial\eta_{44}}{\partial x^a} = -m\frac{\partial\psi}{\partial x^a} \tag{41}$$

with ψ denoting the gravitational potential of the mass M, where κ is the gravitational constant,

$$\psi = -\kappa\frac{M}{r} \tag{42}$$

thus obtaining the equality

$$\eta_{44} = \frac{2}{c^2}\psi \tag{43}$$

and from this and from Eq. (32), the equality

$$\gamma_{44} = 1 + \eta_{44} \tag{44}$$

From the first of Eqs. (29) and from Eqs. (42), (43) and (44) we then find that

$$\mathcal{G} = \frac{2\kappa M}{c^2} \tag{45}$$

as in the case of the Schwarzschild solution in 4 dimensions.

We next consider the third term on the r.h.s of Eq. (40). Let M denote the proton mass and let φ denote the electrostatic potential of the proton charge Q

$$\varphi = \frac{Q}{4\pi\varepsilon_0 r} \tag{46}$$

where ϵ_0 denotes the vacuum electric permeability. On the basis of Eq. (40) we write

$$\left(m\frac{d^2x^a}{dt^2}\right)_{\text{electrical}} = -\frac{1}{2}mc^2w^2\frac{\partial\eta_{55}}{\partial x^a} = -e\frac{\partial\varphi}{\partial x^a} \tag{47}$$

for the electrostatic force acting on the electric charge e connected with the mass m. From Eq. (47) and (48) we obtain

$$\frac{1}{2}c^2w^2\eta_{55} = \frac{e}{m}\varphi = \frac{Qe}{4\pi\varepsilon_0 rm} \tag{48}$$

and hence

$$\eta_{55} = \frac{2eQ}{4\pi\varepsilon_0 mc^2w^2 r} \tag{49}$$

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On the other hand from the second of Eqs. (29) and from Eq. (32) we obtain

$$\gamma_{55} = 1 + \eta_{55} = 1 - \frac{\mathcal{C}}{r} \tag{50}$$

From Eqs. (49) and (50) we find that

$$\mathcal{C} = \frac{-2eQ}{4\pi\varepsilon_0 mc^2 w^2} \tag{51}$$

From the second of Eqs. (30) and from Eq. (46) follows that $\mathcal{C}$ must be positive. This implies that when the charge Q is assumed positive, the charge e must be negative and vice versa. When e and Q are of the same sign the second condition in Eq. (30) cannot be fulfilled. This formula depends on the parameter $w^2 = (du/dt)^2$, which in connection with Eq. (34) has to be much larger than v^2/c^2.

We now can determine the parameter $\mathcal{P}$ in Eq. (30). Owing to Eq. (51) we obtain

$$\mathcal{P} = \sqrt{\mathcal{G}\mathcal{C}} = \frac{1}{wc^2}\sqrt{\frac{\kappa M |eQ|}{\pi\varepsilon_0 m}} \tag{52}$$

where $-eQ$ in Eq. (51) has been replaced by $|eQ|$, since e and Q must have opposite signs. With the gravitational constant $\kappa = 6.673 \times 10^{-11}\,\mathrm{N} \cdot \mathrm{M}^2 \cdot \mathrm{K}$, the vacuum electric permeability $\varepsilon_0 = 0.885 \times 10^{-11}\,\mathrm{Q} \cdot \mathrm{V}^{-1} \cdot \mathrm{M}^{-1}$, with $e/m = 1.76 \times 10^{11}\,\mathrm{Q} \cdot \mathrm{K}$, and the absolute value of electron charge $|e| = 1.6 \times 10^{-19}\,\mathrm{Q}$, where N=newton, M=meter, K=kilogram, Q=coulomb, V=volt [28] identifying the mass M in Eq. (45) with the proton mass we find from Eq. (45) that

$$\mathcal{G} \approx 2.4 \times 10^{-54}\,\mathrm{M} \tag{53}$$

With Q denoting the proton charge, from Eq. (51) we find that

$$\mathcal{C} \approx 5.6\,w^{-2} \times 10^{-15}\,\mathrm{M} \tag{54}$$

and then with $Q = |e|$ from Eq. (52) we obtain

$$\mathcal{P} \approx 1.2\,w^{-1} \times 10^{-34}\,\mathrm{M} \tag{55}$$

We now can answer the question for the Schwarzschild radius. From Eqs. (30), (45) and (51) we find that in Eq. (19)

$$\mathcal{R} = \mathcal{G} + \mathcal{C} = 2.4 \times 10^{-54}\,\mathrm{M} + 5.6\,w^{-2} \times 10^{-15}\,\mathrm{M} \tag{56}$$

If w^{-2} is not extremely small, the Schwarzschild radius is determined by the parameter $\mathcal{C}$ connected with the electric charge of the proton.

We now explain why it is not possible to relate the term $\mathcal{P}/r$ with the Coulomb potential. If $\mathcal{P}/r$ were related with the Coulomb potential, from Eq. (40) we then would obtain:

$$\left(m\frac{d^2 x^a}{dt^2}\right)_{\text{electrical}} = -mc^2 w\frac{\partial\eta_{45}}{\partial x^a} = -e\frac{\partial\varphi}{\partial x^a} \tag{57}$$

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with φ defined in Eq. (46). From Eqs. (46) and (57) we would obtain:

$$\eta_{45} = \frac{eQ}{4\pi\varepsilon_0 rmc^2 w} \tag{58}$$

and since from Eqs. (19) and (32) we have

$$\gamma_{45} = \eta_{45} = \frac{\mathcal{P}}{r} \tag{59}$$

we would find

$$\mathcal{P} = \frac{eQ}{4\pi\varepsilon_0 mc^2 w} \tag{60}$$

The parameter $\mathcal{C}$ now is obtained from Eq. (30)

$$\mathcal{C} = \mathcal{P}^2\mathcal{G}^{-1} \tag{61}$$

The parameter $\mathcal{R}$ is determined in Eq. (30). Inserting into Eq. (61) the numerical values for $\mathcal{P}$ and $\mathcal{G}$ given in Eqs. (55) and (53) we find that $\mathcal{C} \approx 3\, w^{-2} \times 10^{24}$ M, hence, unless w^{-2} is extremely small, the respective $\mathcal{R} = \mathcal{G} + \mathcal{C}$ is unacceptable as a candidate for the Schwarzschild radius of the proton. This seems to indicate that we cannot identify the force connected with η_{45} with the Coulomb force between the central charge Q and the charge e of a test particle.

5 The relativistic energy-level formula

We set up the variational problem for the square of the interval in Eq. (19) (see for example [29]) in the form

$$\delta \int \left[\left(1 - \frac{\mathcal{G}}{r}\right)c^2\,\dot{t}^2 + \left(1 - \frac{\mathcal{C}}{r}\right)c^2\,\dot{u}^2 + \frac{\sqrt{\mathcal{G}\mathcal{C}}}{r}c^2\,\dot{t}\dot{u} - \right.$$
$$\left. \left(1 - \frac{\mathcal{G} + \mathcal{C}}{r}\right)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2 + \dot{\varphi}^2\,\sin^2\theta) \right] dS = 0 \tag{62}$$

where the "dot" denotes d/dS. The Euler-Lagrange equations for θ, φ, t and u yield the equalities:

$$\frac{d}{dS}(r^2\,\dot{\theta}) - r^2\dot{\varphi}^2\sin\theta\cos\theta = 0 \tag{63}$$

$$\frac{d}{dS}(r^2\dot{\varphi}\sin^2\theta) = 0 \tag{64}$$

$$\frac{d}{dS}\left[2\left(1 - \frac{\mathcal{G}}{r}\right)\dot{t} + \frac{\sqrt{\mathcal{G}\mathcal{C}}}{r}\dot{u}\right] = 0 \tag{65}$$

$$\frac{d}{dS}\left[2\left(1 - \frac{\mathcal{C}}{r}\right)\dot{u} + \frac{\sqrt{\mathcal{G}\mathcal{C}}}{r}\dot{t}\right] = 0 \tag{66}$$

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Dividing the expression for the interval in Eq. (19) by dS^2 we obtain the equation for $\dot{r}$

$$1 = \left(1 - \frac{\mathcal{G}}{r}\right)c^2\dot{t}^2 + \left(1 - \frac{\mathcal{C}}{r}\right)c^2\dot{u}^2 + \frac{\sqrt{\mathcal{G}\mathcal{C}}}{r}c^2\dot{t}\dot{u} - \left(1 - \frac{\mathcal{G} + \mathcal{C}}{r}\right)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) \tag{67}$$

We assign the charge $Z|e|$ to the central mass and will determine the perinucleic motion of an electron in the Coulomb field. We observe that the term with $\sqrt{\mathcal{G}\mathcal{C}}$ determines the main influence of the gravitational field, connected with the central singularity, on the energy levels. This influence is very small in comparison with the influence of the Coulomb field represented by the parameter $\mathcal{C}$. In Eqs. (65), (66), and (67) we omit the terms containing the factors $\mathcal{G}/r$ or $\sqrt{\mathcal{G}\mathcal{C}}/r$, since they are small in comparison with the terms containing $\mathcal{C}/r$, thus obtaining from Eq. (67) the equation

$$1 = c^2\dot{t}^2 + \left(1 - \frac{\mathcal{C}}{r}\right)c^2\dot{u}^2 - \left(1 - \frac{\mathcal{C}}{r}\right)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) \tag{68}$$

By an appropriate orientation of the coordinate axes we can make $\theta = \pi/2$ and $d\theta/dS = \dot{\theta} = 0$, for some initial value of S. From Eq. (63) then follows that for all values of S we have $\theta = \pi/2$. Substituting this value of θ into Eq. (64) we obtain

$$r^2\frac{d\varphi}{dS} = k = \text{const} \tag{69}$$

while from Eqs. (65) and (66) we obtain

$$\frac{dt}{dS} = \tau = \text{const} \tag{70}$$

and

$$\frac{du}{dS}\left(1 - \frac{\mathcal{C}}{r}\right) = \rho = \text{const} \tag{71}$$

Substituting the expressions in Eqs. (69), (70) and (71) into Eq. (68) we obtain the equation for $r = r(S)$

$$\left(\frac{dr}{dS}\right)^2 = [c^2(\tau^2 + \rho^2) - 1] + (1 - c^2\tau^2)\frac{\mathcal{C}}{r} - \frac{k^2}{r^2} + \frac{\mathcal{C}k^2}{r^3} \tag{72}$$

From this equation in the new variable $v = 1/r$, in the customary way [29] we obtain the equation

$$v'' + v = \mathcal{A} + \frac{\beta}{\mathcal{A}}v^2 \tag{73}$$

where

$$\mathcal{A} = \frac{(1 - c^2\tau^2)\,\mathcal{C}}{2k^2} \qquad \text{and} \qquad \beta = \frac{3}{2}\mathcal{A}\mathcal{C} \tag{74}$$

Eq. (73) determines the perinucleic motion.

We now intend to determine a formula for the energy levels of the test particle. From Eq. (72) we obtain the expression

$$\frac{dr}{dS} = \frac{dr}{dt}\frac{dt}{dS} = \tau\frac{dr}{dt} =$$
$$\sqrt{[c^2(\tau^2 + \rho^2) - 1] + (1 - c^2\tau^2)\frac{\mathcal{C}}{r} - \frac{k^2}{r^2} + \frac{\mathcal{C}k^2}{r^3}} \tag{75}$$

We take over from Sommerfeld (p. 277 in [19]) the quantization conditions

$$\oint m\frac{dr}{dt}dr = n_r h\,, \quad n_r = 0, 1, 2, \ldots \tag{76}$$

$$\int\limits_0^{2\pi} mr^2\frac{d\varphi}{dt}d\varphi = n_\varphi h\,, \quad n_\varphi = 1, 2, 3, \ldots \tag{77}$$

where m denotes the electron mass, and h is Planck's constant. From Eq. (75) we find that

$$m\frac{dr}{dt} = \sqrt{\frac{m^2}{\tau^2}\left\{[c^2(\tau^2 + \rho^2) - 1] + (1 - c^2\tau^2)\frac{\mathcal{C}}{r} - \frac{k^2}{r^2} + \frac{\mathcal{C}k^2}{r^3}\right\}} \tag{78}$$

From Eqs. (69) and (70) and from Eq. (77) we find that

$$k = \frac{\tau}{m}n_\varphi\hbar \tag{79}$$

where $\hbar = h/2\pi$. The integral in Eq. (76) with the integrand given in Eq. (78) was calculated in [19]. It has the value

$$I = \oint \sqrt{A_0 + 2\frac{A_1}{r} + \frac{A_2}{r^2} + \frac{A_3}{r^3}}\,dr =$$
$$-2\pi i\left(\sqrt{A_2} - \frac{A_1}{\sqrt{A_0}} - \frac{A_1 A_3}{2A_2\sqrt{A_2}}\right) \tag{80}$$

where $\sqrt{A_2}$ is negative imaginary. Comparing Eq. (78) with the integrand on the l.h.s. of Eq. (80) we find that

$$A_0 = \frac{m^2}{\tau^2}\left[c^2(\tau^2 + \rho^2) - 1\right] \qquad A_1 = \frac{m^2}{2\tau^2}(1 - c^2\tau^2)\,\mathcal{C}$$
$$A_2 = -\frac{m^2 k^2}{\tau^2} = -n_\varphi^2\hbar^2 \qquad A_3 = \frac{m^2}{\tau^2}\mathcal{C}k^2 = \mathcal{C}n_\varphi^2\hbar^2 \tag{81}$$

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The three terms on the r.h.s. of Eq. (80) take the form

$$-2\pi i \sqrt{A_2} = -2\pi n_\varphi \hbar \tag{82}$$

$$2\pi i \frac{A_1}{\sqrt{A_0}} = 2\pi i \frac{m(1-c^2\tau^2)\mathcal{C}}{2\tau\sqrt{c^2(\tau^2+\rho^2)-1}} \tag{83}$$

$$\pi i \frac{A_1 A_3}{A_2\sqrt{A_2}} = 2\pi \frac{(1-c^2\tau^2)\mathcal{C}^2 m^2}{4n_\varphi\hbar\tau^2} \tag{84}$$

From Eqs. (75), (78), (80) and (82) through (84), introducing the expression for $\mathcal{C}$ given in Eq. (51), with $\alpha = e^2/4\pi\varepsilon_0\hbar c$ we obtain the equality

$$\frac{-iZ\alpha(1-c^2\tau^2)}{\tau c\sqrt{c^2(\tau^2+\rho^2)-1}}\left(\frac{dt}{du}\right)^2 = n_r + n_\varphi - \left[\frac{1-c^2\tau^2}{c^2\tau^2}\left(\frac{dt}{du}\right)^4\right]\frac{\alpha^2 Z^2}{n_\varphi} \tag{85}$$

Writing $du/dt = w$ we find from Eqs. (33), (70) and (71) with $v^2/c^2 \ll w^2$, as it was assumed in Section 4, that

$$c^2\tau^2 \approx \frac{1}{1+w^2} \tag{86}$$

and

$$c^2\rho^2 \approx \frac{w^2}{1+w^2}\left(1-\frac{\mathcal{C}}{r}\right) \tag{87}$$

Hence w^2 is approximately expressed through the constant of motion τ. Utilizing these equalities we can rewrite Eq. (85) in the form

$$1 + w^2\left(2\frac{\mathcal{C}}{r} - \frac{\mathcal{C}^2}{r^2}\right) = 1 + \frac{\alpha^2 Z^2}{\left(n_r + n_\varphi - \frac{\alpha^2 Z^2}{w^2 n_\varphi}\right)^2} \tag{88}$$

and compare it with Sommerfeld's formula for energy levels of an electron in the relativistic Kepler motion in a flat space in Eq. (26) on p. 278 of [19] which is

$$\left(1 + \frac{W}{m_0 c^2}\right) = \left[1 + \frac{\alpha^2 Z^2}{\left(n_r + \sqrt{n_\varphi^2 - \alpha^2 Z^2}\right)^2}\right]^{-\frac{1}{2}} \tag{89}$$

where $W = E - m_0 c^2$ and $E = $ total energy. The inverse square root of the l.h.s. of Eq. (88) can be equalized to the total energy E. On the r.h.s. of Eq. (89) the first two terms of a series expansion of the square root are equal to the respective two terms in Eq. (88), when we put $w^2 = 2$. This means that for $\alpha^2 Z^2/2n_\varphi^2 \ll 1$ and $w^2 = 2$ the r.h.s. of Sommerfeld's formula in Eq. (89) coincides with the inverse square root of the r.h.s. of Eq. (88). Eq. (88) therefore represents an analogue of Sommerfeld's formula for energy levels of a hydrogen-like atom for sufficiently small values of Z.

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6 Conclusions and discussion

There exists an extensive literature concerning spherically-symmetric solutions in the Kaluza-Klein theory. We only name the five-dimensional spherically symmetric solutions determined by Chodos and Detweiler [18], by Ponce de Leon and Wesson [30] and by Wesson [13]. Those solutions are based on the assumption of a spatial character of the fifth coordinate. A thorough discussion of those solutions is given in Overduin and Wesson [9] and in Wesson [13].

Assuming that in a non-compactified Kaluza-Klein theory the fifth coordinate x^5 has time character, we have determined a Schwarzschild type solution of the five-dimensional Einstein equations in the vacuum. The two independent parameters of that solution have been related with mass and electric charge, respectively. The solution exhibits a Schwarzschild radius whose magnitude is predominantly determined by the electric-charge parameter.

The perihelic motion in four-dimensional relativity has a counterpart in the perinucleic motion of an electron in a Kaluza-Klein theory with two times. If the quantization conditions of the older quantum theory are included into the five-dimensional geometry, the perinucleic motion of an electron leads to the fine structure of line spectra, which in the limit of $Z^2\alpha^2/n_\varphi^2 \ll 1$ is analogous to that determined by Sommerfeld's formula for hydrogen-like atoms.

The parameter $\mathcal{C}$ which determines the Schwarzschild radius and the parameter $\mathcal{P}$ connected with a new force depend on the derivative du/dt of the second time coordinate u with respect to the ordinary time coordinate t. Their numerical values therefore hinge on the magnitude of that quantity. The indicated physical meaning of those parameters, however, is not impaired by the lack of knowledge of the magnitude of the quantity du/dt as long as it is not extremely large or extremely small in comparison with 1.

Acknowledgment

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