Entry and Exit Decisions under Uncertainty for a Generalized Class of One-Dimensional Diffusions

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Abstract

We consider the optimal entry and exit policy of a firm in the presence of output price uncertainty and costly reversibility of investment under a generalized class of one-dimensional diffusions accommodating different drift and volatility specifications. This will allow us to analyze how output price uncertainty and costly reversibility affects the optimal entry and exit policy of a competitive price-taking firm, and how the hysteretic band is affected by the choice of the appropriate stochastic process.
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1. Introduction

It is broadly accepted by academics and corporate managers that traditional valuation techniques based on discounted cash flows (e.g. the standard net present value method) are not the most appropriate tool in decision making, especially in the presence of uncertainty, complete irreversibility or costly reversibility of investment, and when there is some leeway for conducting a flexible management. The main reason for this observation is that, in

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the presence of irreversibility, the firm is unable to instantaneously adjust its operations to a desired optimal level if market conditions unexpectedly deteriorate and change in an unfavorable direction after an investment decision has been made. As expected, the presence of uncertainty augments this effect and raises the required investment premium associated with the irreversible decision by increasing the option value of waiting.

Given its analytical attractiveness, the process most used in the literature of real options is the geometric Brownian motion (GBM henceforth). However, it is well documented in the literature that the GBM assumption embodies some unrealistic implications for the dynamic behavior of real asset prices. Namely, there is empirical evidence indicating that this assumption is not rich enough to capture the volatility smiles or skews found in the equity options market - see, for example Jackwerth and Rubinstein (2001). To overcome this issue, alternative stochastic processes have been considered in the real options literature. Such alternatives beyond the classic GBM assumption and the vast range of possible applications for a practitioner, bring up a number of important issues deserving a detailed examination.

One of the first attempts to investigate the biases in value provoked by the use of traditional methods of valuation was done using a mean-reverting process, which can be more suitable under equilibrium conditions. Bhattacharya (1978) studies the accuracy of traditional valuation methods when cash flows follow a mean-reverting process, as opposed to the standard GBM, and the ensuing biases in value. There are several works in the real options literature where the stochastic processes used are the mean-reverting type. For example, Sarkar (2003) assumes that the stochastic costs (not stochastic revenues) follow a mean-reverting process, concluding that the mean reversion, in general, have a significant impact on investment. Thus, it is generally inappropriate to use a GBM process to approximate a mean-reverting process. This work does not consider reversibility nor disinvestment and is only concentrated on irreversible entry. To overcome this situation, an extension has been proposed by Tsekrekos (2010). Dias and Shackleton (2011) also examine the investment and divestment decisions problem assuming that the stochastic interest rate follows a mean-reverting process.

The constant elasticity of variance (CEV hereafter) model of Cox (1975) is another stochastic process used to overcome the drawbacks pointed out to the GBM process. This model is consistent with two well known facts that have found empirical support in the literature: the existence of a negative correlation between stock returns and realized volatility (leverage effect), as observed, for instance, in Bekaert and Wu (2000); and the inverse relation between the implied volatility and the strike price of an option contract (implied volatility skew)—see, for example, Dennis and Mayhew (2002).

To our knowledge, there are only a few empirical studies on real options where the CEV model has been used. Nevertheless, there is evidence supporting the use of this stochastic process. Choi and Longstaff (1985) have examined the stochastic behavior of soybean future quasi-returns. Their empirical study suggests that the CEV process is theoretical superior to the GBM process for pricing options on soybean futures. The dynamic of crude oil prices by region, time period, and observation frequency using the Chan et al. (1992) general diffusion formulation has been examined by Lee and Heo (2008), where they have conclude that the CEV model is the most suitable process to explain the dynamics of crude oil prices. An interesting study was performed by Geman and Shih (2009), where they analyze the performance of the CEV process (the mean-reverting CEV process is also considered) to model the crude oil, coal, copper, and gold prices. They conclude that the CEV exponent plays an important role in metal and energy commodities after the year 2000. Recently, Dias and Nunes (2011) derive analytical solutions for perpetual American-style call and put options under the CEV model. Their results strongly highlight the case for moving beyond the simplistic real models based on the GBM assumption to more realistic models incorporating volatility smile effects.

In this paper, we reconsider the problem originally addressed by Dixit (1989a) and Tsekrekos (2010), and analyze how output price uncertainty and costly reversibility affects the optimal entry and exit policy of a competitive price-taking firm. We extend these previous studies in two ways. First, we assume that the underlying output price dynamics follows a generalized one-dimensional diffusion which takes the modeling assumptions of Dixit (1989a) and Tsekrekos (2010) as two special cases. Second, we analyze the impact of costly reversibility on the dynamic entry and exit problem. This latter issue as also been considered by Dias and Shackleton (2011), but in a real options model where uncertainty stems from the interest rate uncertainty.

Hence, our analysis covers a broad class of descriptions both for the reversibility degree and for the underlying stochastic price dynamics which, within our generalized class of one-dimensional diffusions, includes most typically applied mean-reverting models as well as different volatility specifications. These issues should be important for academics and practitioners, since our modeling framework admits the analysis of the general properties of entry and exit decisions under alternative underlying driving stochastic factor dynamics and characterizes the circumstances under which the obtained results are significantly different or remain qualitatively valid, depending on the assumption made for the underlying output price dynamics.

The structure of the paper is organized as follows. Section 2 presents the firm's policy, the general output price dynamics, the value-matching condition, the smooth-pasting condition, and define the hysteretic band. Section 3 specializes the architecture modeling framework for the GBM, CEV, and the mean-reverting CEV processes. Section 4 compares the optimal entry-exit policy under the several processes. In Section 5, we compute the *ex ante* probabilities of entry and exit and compare the results, and finally Section 6 concludes.

2. Modeling architecture

For the analysis to remain self-contained, the next four subsections provide the necessary building blocks for modeling entry and exit decisions under alternative output price dynamics.

2.1. The firm's policy

Following Dixit (1989a) and Tsekrekos (2010), we shall consider a price-taking firm that has the possibility to invest (at any time) a lump-sum entry cost $\overline{K}$ to enter in a market (i.e. it needs to pay $\overline{K}$ to switch from the idle or inactive state to the operating state). As usual, the entry mode is in the form of a discrete unit of investment, namely a single project of a given size. While active in the market, the firm can produce a unit flow of output at a variable cost C. Moreover, the firm can decide to suspend operations (at any time) if market conditions deteriorate.

Similarly to Abel et al. (1996), Abel and Eberly (1996), Alvarez (2011), and Dias and Shackleton (2011), and to accommodate the generalization for different costly reversibility levels, we assume that by divesting the firm receives the disinvestment proceeds $\underline{K}$, i.e. there is a fraction α of the invested capital, with $\alpha := \underline{K}/\overline{K}$ (the ratio of the *direct* switching costs), that a firm can recoup when divesting.[1] Such prescription for the α parameter encompasses different reversibility degrees contemplated in the literature, namely:

- $\alpha = 1$ represents the traditional costlessly reversible investment case in which the wedge between the investment cost and the divestment proceeds is zero, and the optimal investment policy of a firm maintains the marginal revenue product of capital equal to the Jorgenson (1963) marginal user cost of capital. As expected, such standard myopic investment rule is unrealistic since, in the presence of irreversibility and uncertainty, it is not expected that a firm can divest at no cost due to the so-called *lemons* problem of Akerlof (1970).

- $\alpha = 0$ stands for the completely irreversible investment case in which the sale price of capital is zero (so that the wedge is 100% of the purchase price of capital) initiated by Arrow (1968), and then employed in much of the subsequent work on optimal investment under uncertainty.

- There are also more realistic investment cases characterized by costly reversibility in which a firm can purchase capital at a given price and sell capital at a lower price, i.e. with $\alpha \in (0, 1)$. In other words, even though capital has resale value, it is below its acquisition cost, thus making part of the initial entry costs sunk. For example, this modeling specification has been considered by Abel et al. (1996), Abel and Eberly

[1] As in Dixit (1989a), Tsekrekos (2010), or Dias and Shackleton (2011), we assume that entry and exit takes place immediately after the decision to invest or divest has been made, thus ignoring the so-called time to build (or investment lags) effects discussed in Majd and Pindyck (1987), Bar-Ilan and Strange (1996), and Milne and Whalley (2000).

(1996), Alvarez (2011), and Dias and Shackleton (2011). Such partial reversibility case is of paramount importance because, as it was shown by Keswani and Shackleton (2006), a project's option value increases with incremental levels of investment and disinvestment flexibility.

- In the previous case capital can be abandoned at a cost since only a fraction of the entry cost can be recovered on exit. There may be, however, situations where it is necessary to pay a lump-sum cost to close a project, such as the cases of a copper mine or a nuclear power station where environmental clean costs may have to be supported. In our modeling framework, this is equivalent to assume $\alpha < 0$. For instance, such assumption was taken by Dixit (1989a) and Tsekrekos (2010).

In order to simplify the exposition and keep our generalized modeling framework similar (and thus comparable) to the work of Dixit (1989a) and Tsekrekos (2010), we assume that the parameter values $\overline{K}$, α, and C are constant and non-stochastic. Moreover, uncertainty stems from the output equilibrium price P which is assumed to be exogenous to the firm (i.e. the firm is a price-taker as already stated).

Let $V_0(P)$ be the expected net present value of the firm (with an initial output price P in the idle state) and following dynamic optimal entry-exit policies. The optimal entry and exit policy is determined through two time independent values of the state variable P, one upper threshold price $\overline{P}$ (reached from below) and one lower trigger $\underline{P}$ (reached from above), with $\overline{P} > \underline{P}$, at which a firm optimally switches from the idle to the operating state and vice versa. At the optimal entry threshold $\overline{P}$, the idle firm exercises its entry option by paying $\overline{K}$ in order to receive an "underlying asset" of value $V_1(\overline{P})$, which includes both an option to exit and a flow reward component. Similarly, at the optimal exit threshold $\underline{P}$, the active firm exercises its exit option in favour of regaining an "underlying asset" worth $V_0(\underline{P})$ (i.e. an option to enter the market again) and a cash amount $\underline{K} := \alpha\overline{K}$ (positive, if $\alpha \in (0, 1)$, or negative, if $\alpha < 0$).[2]

[2]In other words, we are assuming a two-sided discrete regulator problem with lump-sum costs (or discrete

2.2. Output price dynamics

Hereafter, we assume the equilibrium output time-t price P_t, evolving on the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, is characterized by the following one-dimensional Itô diffusion:

$$dP_t = \mu(P_t)dt + \sigma(P_t)dW_t^{\mathbb{P}}, \;\; P_0 = P \in \mathbb{R}_+, \tag{1}$$

where $W_t^{\mathbb{P}}$ is a standard Brownian motion under the physical probability measure $\mathbb{P}$.[3] Furthermore, we assume that the drift coefficient $\mu : \mathbb{R}_+ \to \mathbb{R}$ and the volatility coefficient $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and satisfy the conditions $\sigma(P) > 0$, $\forall\, P \in (0, \infty)$, and

$$\int_{P-\varepsilon}^{P+\varepsilon} \frac{1 + |\mu(y)|}{\sigma^2(y)} dy < \infty,$$

for some $\varepsilon > 0$, $\forall\, P \in (0, \infty)$. As shown by Karatzas and Shreve (1991, pp. 342-351), these conditions guarantee the existence of a weak solution for the stochastic differential equation (1).

Using Itô's lemma for our *infinite-horizon stochastic problem*, it follows that

$$\begin{aligned} dV_0(P) &= V_0^{'}(P)dP + \frac{1}{2}V_0^{''}(P)(dP)^2 \\ &= \left[\frac{1}{2}\sigma^2(P)V_0^{''}(P) + \mu(P)V_0^{'}(P)\right] dt + \sigma(P)V_0^{'}(P)dW^{\mathbb{P}}. \end{aligned} \tag{2}$$

Note that the time partial derivative usually appearing in (2) is zero due to the perpetual nature of the problem. The expected return and the standard deviation of the return of the firm are respectively given by:

$$\mathbb{E}\left[R\right] = \frac{\frac{1}{2}\sigma^2(P)V_0^{''}(P) + \mu(P)V_0^{'}(P)}{V_0(P)}, \tag{3}$$

adjustments) $\overline{K}$ and $\underline{K}$, in which controls are applied only when the state variable P hits the threshold levels $\overline{P}$ and $\underline{P}$, thus making our economic application resembling a *stochastic optimal impulse control problem* in the Constantinides and Richard (1978), Harrison et al. (1983), Dixit (1991), and Dumas (1991) sense.

[3] To lighten notation, the subscript 't' is dropped in the remainder of the paper.

and

$$\mathbb{D}\left[R\right] = \frac{\sigma(P)V_0^{'}(P)}{V_0(P)}. \tag{4}$$

As usual, the firm value must satisfy the following risk-return relationship

$$\mathbb{E}\left[R\right] = r + \lambda^*(P)\,\mathbb{D}\left[R\right] = r + \lambda^*(P)\,\frac{\sigma(P)V_0^{'}(P)}{V_0(P)}, \tag{5}$$

where $\lambda^*(P)$ is the compensation per unit risk above the (constant) riskless rate r. Note that the functional form of the market price of risk $\lambda^*(P)$ will depend on the respective stochastic process that is chosen for modeling the output price dynamics.

Substituting equation (3) into equation (5), multiplying both sides by $V_0(P)$, and then rearranging terms yields the following ordinary differential equation (ode):

$$\frac{1}{2}\sigma^2(P)V_0^{''}(P) + \left[\mu(P) - \lambda^*(P)\sigma(P)\right]V_0^{'}(P) - rV_0(P) = 0. \tag{6}$$

This is the ordinary differential equation that the value of the firm must satisfy over the range of output prices that is optimal for an idle firm to remain in the inactive state, i.e. for $P \in (0, \overline{P})$.

Similarly, over the range of prices where it is optimal for an active firm to continue in the operating state, i.e. for $P \in (\underline{P}, \infty)$, the total return of the expected net present value of the firm, $V_1(P)$, comprises the expected capital gain $\mathbb{E}\left[dV_1(P)\right]/dt$, plus a cash inflow $(P - C)$ per unit of time. Following the same line of reasoning, the value $V_1(P)$ must satisfy the following ordinary differential equation (ode):

$$\frac{1}{2}\sigma^2(P)V_1^{''}(P) + \left[\mu(P) - \lambda^*(P)\sigma(P)\right]V_1^{'}(P) - rV_1(P) + P - C = 0. \tag{7}$$

Solving equations (6) and (7) subject to appropriate boundary conditions yields the value functions for an idle and an active firm.

2.3. Solutions of the value functions $V_0(P)$ and $V_1(P)$

Omitting the term $f(P) := P - C$ in the ode (7), one notes that both (6) and (7) are linear differential equations possessing the same general solution for the homogeneous equation, which can be expressed as a linear combination of any two independent solutions. In other words, the corresponding linearly independent complementary functions are similar, and can thus be solved together. What determines the difference between the two contingent solutions and the options they represent is their boundary conditions.

As usual, the firm's option value to enter the market should be nearly worthless as the output price P becomes very small. To ensure such economic rationale, the ode (6) must be solved subject to the following boundary condition:

$$\lim_{P \to 0^+} V_0(P) = 0. \tag{8}$$

However, the general solution of the non-homogeneous ode (7) should be expressed as the sum of two parts: The general solution of the homogeneous equation neglecting the flow reward function $f(P)$ and an arbitrarily chosen particular solution of the full equation (7). As shown by Dixit (1991), a very convenient particular solution of (7) is the expected discounted flow payoff

$$F(P) := \mathbb{E}\left[\int_0^{+\infty} e^{-rs} f(P_s) ds | P_0 = P\right], \tag{9}$$

that is calculated ignoring both (upper and lower) barriers on the one-dimensional diffusion process P.[4]

As expected, the firm's option value to exit the market (while in the active state) should be nearly worthless as the output price P becomes very high. Thus, to rule out any explosive growth of firm value with high output price, we must impose the so-called *no-bubbles condition* which implies that, for high equilibrium output prices, the exit option becomes worthless and the value function $V_1(P)$ converges to the expected present value of operating in the market perpetually given in equation (9), that is

[4]The particular solution $F(P)$ can be interpreted as the expected present value payoff when the (uncontrolled) state variable P is allowed to fluctuate without regulation, while the corresponding full solutions are interpreted similarly, but when the stochastic process is assumed to be regulated using the impulse form of control.

$$\lim_{P \to +\infty} V_1(P) = \mathbb{E}\left[\int_0^{+\infty} (P_s - C)\, e^{-rs} ds | P_0 = P\right]. \tag{10}$$

2.4. Boundary and first order conditions

The optimal switching policy (i.e. the displacement strategies idle state → active state and active state → idle state) is determined through two time independent trigger prices $\overline{P}$ and $\underline{P}$. Each threshold level is similar in spirit to the critical asset price (or early exercise boundary) that separates the *continuation* and *stopping* (or exercise) regions of an American-style option contract, thus turning the dynamic entry and exit decision of a firm an *optimal stopping* problem with two barriers.

Such entry and exit thresholds are determined numerically through a set of *value-matching* and *smooth-pasting* conditions. The former are stated as

$$V_0(\overline{P}) + \overline{K} = V_1(\overline{P}) \tag{11}$$

$$V_0(\underline{P}) + \underline{K} = V_1(\underline{P}), \tag{12}$$

which ensure that the gain in value from exercising the option is exactly equal to the cost of doing so. These value-matching conditions reflect an intuitive requirement for continuity at the optimal thresholds.

The optimality condition of such (optimal impulse control) two-sided discrete regulator problems arise, however, from the so-called smooth-pasting (also known as high-contact or first-order) conditions[5]

$$V_0'(\overline{P}) = V_1'(\overline{P}) \tag{13}$$

$$V_0'(\underline{P}) = V_1'(\underline{P}), \tag{14}$$

[5]Samuelson (1965), McKean (1965), and Merton (1973) established conditions of optimality for such *optimal stopping* problems. A rigorous exposition of these conditions is provided by Dixit (1991) and Dumas (1991).

requiring that the first derivative of the firm value function must take the same value before and after the option (to enter or exit) has been exercised. In other words, these conditions require that *marginal utility* should take the same value before and after the action has been taken. This is equivalent to say that, at the optimum thresholds $\overline{P}$ and $\underline{P}$, the marginal cost of discounting the payoff function that is obtained by exercising the switching option equals the marginal net benefit from further waiting. As shown by Shackleton and Sødal (2005), such conditions guarantee the equalization of the rate of return of the firm both prior and after the decision to invest or divest has been taken.

To sum up, equations (11)-(14) constitute a set of four highly non-linear equations with four unknowns, represented in matrix form and denoted by $F(X)$, whose optimal solution $X = [\overline{P}, \underline{P}, A, B]'$ is uniquely determined by (numerically) solving the system $F(X) = 0$. Such solution highlights the optimal entry and exit policy of a firm acting dynamically in the aforementioned generalized stochastic environment by simultaneously determining both the thresholds $\overline{P}$ and $\underline{P}$ and the pair of constants A and B (to be determined from the boundary conditions) associated, respectively, to the idle and operating states of the firm.

2.5. Hysteresis

Whenever the underlying output price is between the two critical boundaries $\overline{P}$ and $\underline{P}$, the firm remains in its current state (idle or active). Thus, the firm takes no action at all over the region of the state space $(\underline{P}, \overline{P})$. In other words, the firm's actions (to enter or exit) are only triggered when the state variable P reaches the boundary of the *region of no action* (or *zone of no intervention*). This range of inaction results in hysteresis (i.e. permanent effects of temporary shifts). The duration of these hysteretic periods depends on the expected growth rate of the underlying price and on its volatility. As shown by Dixit (1989a), the presence of fixed entry and exit costs under uncertainty widens the hysteretic band, since

$$\overline{W} := C + r\overline{K} \quad < \quad \overline{P} \tag{15}$$

$$\underline{W} := C + r\underline{K} \quad > \quad \underline{P}, \tag{16}$$

with $\overline{W}$ and $\underline{W}$ being, respectively, the Marshallian investment and divestment trigger prices based on the standard myopic investment rule. One of the purposes of this article is to show if this zone of no intervention changes substantially under alternative assumption for modeling the output price dynamics.

3. Applications

In this section we specialize the architecture modeling framework for some special cases, namely: The classic GBM process, the CEV process, and the mean-reverting CEV process.

3.1. The classic geometric Brownian motion process

Definition 1. *The classic GBM process underlying most of the real options literature can be nested into the general framework described by equations* (1) *to* (7) *through the following restrictions:* $\mu(P) = \mu P$, $\sigma(P) = \sigma P$, *and* $\lambda^*(P) = \lambda\rho$, *where* μ *and* σ *denote, respectively, the (constant) growth rate and the (constant) volatility of the market price* P, *and* $\lambda = \left(\mathbb{E}\left[R_m\right] - r\right)/\mathbb{D}\left[R_m\right]$ *is the market price of risk (with* $\mathbb{E}\left[R_m\right]$ *and* $\mathbb{D}\left[R_m\right]$ *being, respectively, the expected return and standard deviation of the market portfolio), and* ρ *is the correlation between the output price* P *and the market portfolio, i.e.* $dW^{\mathbb{P}}dW_m^{\mathbb{P}} = \rho dt$. *Both* λ *and* ρ *are assumed constant.*

Proposition 1. *Under the restrictions stated in Definition 1, the optimal solution* $\boldsymbol{X} = [\overline{P}, \underline{P}, A_0, B_1]'$ *is uniquely determined by solving the system* $\boldsymbol{F(X)} = \boldsymbol{0}$, *where*

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -A_0\overline{P}^{\xi_1} + B_1\overline{P}^{\xi_2} + \varphi\overline{P} - \overline{X} \\ -A_0\underline{P}^{\xi_1} + B_1\underline{P}^{\xi_2} + \varphi\underline{P} - \underline{X} \\ -A_0\xi_1\overline{P}^{\xi_1} + B_1\xi_2\overline{P}^{\xi_2} + \varphi\overline{P} \\ -A_0\xi_1\underline{P}^{\xi_1} + B_1\xi_2\underline{P}^{\xi_2} + \varphi\underline{P} \end{bmatrix}, \tag{17}$$

with

$$\xi_1 = \frac{1}{2} - \frac{(\mu - \lambda\rho\sigma)}{\sigma^2} + \sqrt{\left(\frac{(\mu - \lambda\rho\sigma)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \tag{18}$$

$$\xi_2 = \frac{1}{2} - \frac{(\mu - \lambda\rho\sigma)}{\sigma^2} - \sqrt{\left(\frac{(\mu - \lambda\rho\sigma)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} < 0, \tag{19}$$

$$\varphi = (r + \lambda\rho\sigma - \mu)^{-1}, \tag{20}$$

$$\overline{X} = \frac{C}{r} + \overline{K}, \tag{21}$$

and

$$\underline{X} = \frac{C}{r} + \underline{K}. \tag{22}$$

Proof. The proof of this proposition is standard in the literature and can be found, for example, in Dixit (1989a) and Tsekrekos (2010, Appendix A). ■

3.2. The constant elasticity of variance process

The CEV model of Cox (1975) was originally studied to the case where the elasticity parameter, β, is less than two ($\beta < 2$), and then extended to the case where $\beta > 2$ by Emanuel and MacBeth (1982). While Cox (1975) has restricted the β parameter to the range $0 \leq \beta \leq 2$, Jackwerth and Rubinstein (2001) document that typical values of β implicit in the S&P 500 stock index option prices are as low as $\beta = -6$ in the post-crash of 1987. Elasticity values of $\beta < 2$ (i.e. with a direct leverage effect) are expected for stock index options and crude oil prices, whereas values of $\beta > 2$ (i.e. with an inverse leverage effect) are characteristic of some commodity spot prices and futures options with upward sloping implied volatility smiles (see, for instance, Davydov and Linetsky (2001), Geman and Shih (2009), and Dias and Nunes (2011)).

The CEV process assumption has been used in many different contexts, e.g. by Dias and Nunes (2011) to compute the analytical solutions for perpetual American-style call and put options, and Lee and Heo (2008) where these authors have concluded that the CEV process is the most suitable to explain the dynamics of crude oil prices.

Definition 2. *The class of CEV processes can be nested into the general framework described by equations* (1) *to* (7) *through the following restrictions:* $\mu(P) = \mu P$, $\sigma(P) = \delta P^{\beta/2}$, *and* $\lambda^*(P) = \lambda\rho P^{1-\beta/2}$, *where* β *represents the elasticity parameter,* μ *and* δ *denote, respectively, the (constant) growth rate and the (constant) scale parameter fixing the initial instantaneous volatility at time* $t = 0$, $\sigma_0 = \sigma(P_0) = \delta P_0^{\beta/2}$, *and* $\lambda = (\mathbb{E}[R_m] - r) / \mathbb{D}[R_m]$ *is* *the market price of risk (with $\mathbb{E}[R_m]$ and $\mathbb{D}[R_m]$ being, respectively, the expected return and standard deviation of the market portfolio), and ρ is the correlation between the output price P and the market portfolio, i.e. $dW^{\mathbb{P}}dW_m^{\mathbb{P}} = \rho dt$. Both λ and ρ are assumed constant.*

To obtain the optimal solution for the CEV process, we need to distinguish six situations: for the cases $\beta < 2$ and $\beta > 2$, we have to consider three situations: $(\mu - \lambda\rho\delta) > 0$, $(\mu - \lambda\rho\delta) < 0$ and $(\mu - \lambda\rho\delta) = 0$. The next proposition gives the optimal solution when $\beta < 2$ for the cases: A) $(\mu - \lambda\rho\delta) > 0$ and B) $(\mu - \lambda\rho\delta) < 0$.

Proposition 2. *Under the restrictions stated in Definition 2, and for $\beta < 2$, the optimal solution $\boldsymbol{X} = [\overline{P}, \underline{P}, A_0, B_1]'$ is uniquely determined by solving the system $\boldsymbol{F(X) = 0}$:*
A) In case $(\mu - \lambda\rho\delta) > 0$,

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -A_0 M_1(\overline{P}) + B_1 U_1(\overline{P}) + \varphi\overline{P} - \overline{X} \\ -A_0 M_1(\underline{P}) + B_1 U_1(\underline{P}) + \varphi\underline{P} - \underline{X} \\ -A_0 W_1(\overline{P}) + B_1 V_1(\overline{P}) + \varphi\overline{P} \\ -A_0 W_1(\underline{P}) + B_1 V_1(\underline{P}) + \varphi\underline{P} \end{bmatrix}, \tag{23}$$

with

$$M_1(y) = \zeta_1\, y e^{-x(y)} M(a_1, b_1, x(y)), \tag{24}$$

$$U_1(y) = \zeta_1\, y e^{-x(y)} U(a_1, b_1, x(y)), \tag{25}$$

$$W_1(y) = M_1(y) - \vartheta y^{2-\beta}\left(M_1(y) - \zeta_1\, y e^{-x(y)} \frac{a_1}{b_1} M(a_1+1, b_1+1, x(y))\right), \tag{26}$$

$$V_1(y) = U_1(y) - \vartheta y^{2-\beta}\left(U_1(y) + \zeta_1\, y e^{-x(y)} a_1\, U(a_1+1, b_1+1, x(y))\right), \tag{27}$$

$$x(y) = \frac{2|\mu - \lambda\rho\delta|}{\delta^2|\beta - 2|} y^{2-\beta}, \tag{28}$$

$$a_1 = 1 - \frac{r}{(\mu - \lambda\rho\delta)(\beta - 2)}, \tag{29}$$

$$b_1 = 1 - \frac{1}{\beta - 2}, \tag{30}$$

$$\vartheta = \frac{2(\mu - \lambda\rho\delta)}{\delta^2}, \tag{31}$$

$$\zeta_1 = \left[-\frac{\vartheta}{(\beta-2)} \right]^{\frac{1}{2}-\frac{1}{2(\beta-2)}}, \tag{32}$$

and φ*,* $\overline{X}$*, and* $\underline{X}$ *as defined as in* (20)*,* (21)*, and* (22)*, respectively, and where* $M(a,b,x)$ *and* $U(a,b,x)$ *are the Kummer functions, as defined by Abramowitz and Stegun (1972, expressions 13.1.2 and 13.1.3).*

B) In case $(\mu - \lambda\rho\delta) < 0$*,*

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -A_0 M_2(\overline{P}) + B_1 U_2(\overline{P}) + \varphi \overline{P} - \overline{X} \\ -A_0 M_2(\underline{P}) + B_1 U_2(\underline{P}) + \varphi \underline{P} - \underline{X} \\ -A_0 W_2(\overline{P}) + B_1 V_2(\overline{P}) + \varphi \overline{P} \\ -A_0 W_2(\underline{P}) + B_1 V_2(\underline{P}) + \varphi \underline{P} \end{bmatrix}, \tag{33}$$

with

$$M_2(y) = \zeta_2\, y\, M(a_2, b_2, x(y)), \tag{34}$$

$$U_2(y) = \zeta_2\, y\, U(a_2, b_2, x(y)), \tag{35}$$

$$W_2(y) = M_2(y) - \zeta_2\, \vartheta y^{3-\beta} \frac{a_2}{b_2} M\big(a_2+1, b_2+1, x(y)\big), \tag{36}$$

$$V_2(y) = U_2(y) + \zeta_2\, \vartheta y^{3-\beta} a_2\, U\big(a_2+1, b_2+1, x(y)\big), \tag{37}$$

$$a_2 = \frac{r}{(\mu-\lambda\rho\delta)(\beta-2)} - \frac{1}{\beta-2}, \tag{38}$$

$$b_2 = 1 - \frac{1}{\beta-2}, \tag{39}$$

$$\zeta_2 = \left[\frac{\vartheta}{(\beta-2)} \right]^{\frac{1}{2}-\frac{1}{2(\beta-2)}}, \tag{40}$$

and φ*,* $\overline{X}$*,* $\underline{X}$*,* $x(y)$*, and,* ϑ *as defined as in* (20)*,* (21)*,* (22)*,* (28)*, and,* (31)*, respectively.*

Proof. To prove this proposition we will apply the results of Davydov and Linetsky (2001) and the same methodology as used in the GBM process. ■

The next proposition gives the optimal solution to the $\beta > 2$ for the cases: A) $(\mu - \lambda\rho\delta) > 0$ and B) $(\mu - \lambda\rho\delta) < 0$.

Proposition 3. *Under the restrictions stated in Definition 2, and for $\beta > 2$, the optimal solution $\boldsymbol{X} = [\overline{P}, \underline{P}, A_1, B_0]'$ is uniquely determined by solving the system $\boldsymbol{F(X)} = \boldsymbol{0}$:*
A) In case $(\mu - \lambda\rho\delta) > 0$,

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -B_0 U_3(\overline{P}) + A_1 M_3(\overline{P}) + \varphi\overline{P} - \overline{X} \\ -B_0 U_3(\underline{P}) + A_1 M_3(\underline{P}) + \varphi\underline{P} - \underline{X} \\ -B_0 V_3(\overline{P}) + A_1 W_3(\overline{P}) + \varphi\overline{P} \\ -B_0 V_3(\underline{P}) + A_1 W_3(\underline{P}) + \varphi\underline{P} \end{bmatrix}, \tag{41}$$

with

$$M_3(y) = \zeta_3\, M(a_3, b_3, x(y)), \tag{42}$$

$$U_3(y) = \zeta_3\, U(a_3, b_3, x(y)), \tag{43}$$

$$W_3(y) = -\zeta_3\, \vartheta y^{2-\beta} \frac{a_3}{b_3}\, M\big(a_3 + 1, b_3 + 1, x(y)\big), \tag{44}$$

$$V_3(y) = \zeta_3\, \vartheta y^{2-\beta} a_3\, U\big(a_3 + 1, b_3 + 1, x(y)\big), \tag{45}$$

$$a_3 = \frac{r}{(\mu - \lambda\rho\delta)(\beta - 2)}, \tag{46}$$

$$b_3 = 1 + \frac{1}{\beta - 2}, \tag{47}$$

$$\zeta_3 = \left[\frac{\vartheta}{(\beta - 2)}\right]^{\frac{1}{2} + \frac{1}{2(\beta-2)}}, \tag{48}$$

and φ, $\overline{X}$, $\underline{X}$, $x(y)$, and, ϑ as defined as in (20), (21), (22), (28), *and,* (31), *respectively.*

B) In case $(\mu - \lambda\rho\delta) < 0$,

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -B_0 U_4(\overline{P}) + A_1 M_4(\overline{P}) + \varphi\overline{P} - \overline{X} \\ -B_0 U_4(\underline{P}) + A_1 M_4(\underline{P}) + \varphi\underline{P} - \underline{X} \\ -B_0 V_4(\overline{P}) + A_1 W_4(\overline{P}) + \varphi\overline{P} \\ -B_0 V_4(\underline{P}) + A_1 W_4(\underline{P}) + \varphi\underline{P} \end{bmatrix}, \tag{49}$$

with

$$M_4(y) = \zeta_4 e^{-x(y)} M(a_4, b_4, x(y)), \tag{50}$$

$$U_4(y) = \zeta_4 e^{-x(y)} U(a_4, b_4, x(y)), \tag{51}$$

$$W_4(y) = -\vartheta y^{2-\beta}\left(M_4(y) - \zeta_4 e^{-x(y)}\frac{a_4}{b_4}\,M(a_4+1, b_4+1, x(y))\right), \tag{52}$$

$$V_4(y) = -\vartheta y^{2-\beta}\Big(U_4(y) + \zeta_4 e^{-x(y)} a_4\,U(a_4+1, b_4+1, x(y))\Big), \tag{53}$$

$$a_4 = 1 + \frac{1}{\beta-2} - \frac{r}{(\mu-\lambda\rho\delta)(\beta-2)}, \tag{54}$$

$$b_4 = 1 + \frac{1}{\beta-2}, \tag{55}$$

$$\zeta_4 = \left[-\frac{\vartheta}{(\beta-2)}\right]^{\frac{1}{2}-\frac{1}{2(\beta-2)}}, \tag{56}$$

and φ, $\overline{X}$, $\underline{X}$, $x(y)$, *and,* ϑ *as defined as in* (20), (21), (22), (28), *and* (31), *respectively.*

Proof. The proof of this proposition is similar to the proof of the Proposition 2 ■

We analyze now the particular case of the CEV model where $(\mu - \lambda\rho\delta) = 0$. In the next proposition we will give the optimal solution $\mathbf{X} = [\overline{P}, \underline{P}, A_\phi, B_\phi]'$ to the cases: A) $\beta < 2$ and B) $\beta) > 2$.

Proposition 4. *Under the restrictions stated in Definition 2, and for* $(\mu-\lambda\rho\delta) = 0$, *the optimal solution* $\boldsymbol{X} = [\overline{P}, \underline{P}, A_\phi, B_\phi]'$ *is uniquely determined by solving the system* ***F(X) = 0****:*
A) In case $\beta < 2$,

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -A_0 I(\overline{P}) + B_1 K(\overline{P}) + r^{-1}\overline{P} - \overline{X} \\ -A_0 I(\underline{P}) + B_1 K(\underline{P}) + r^{-1}\underline{P} - \underline{X} \\ -A_0 S_1(\overline{P}) + B_1 T_1(\overline{P}) + r^{-1}\overline{P} \\ -A_0 S_1(\underline{P}) + B_1 T_1(\underline{P}) + r^{-1}\underline{P} \end{bmatrix}, \tag{57}$$

with

$$I(y) = y^{\frac{1}{2}} I_v(z(y)), \tag{58}$$

$$K(y) = y^{\frac{1}{2}} K_v(z(y)), \tag{59}$$

$$S_1(y) = y^{\frac{1}{2}} I_v(z(y)) + \frac{\sqrt{2r}}{\delta} y^{\frac{3}{2}-\frac{\beta}{2}} I_{v+1}(z(y)), \tag{60}$$

$$T_1(y) = y^{\frac{1}{2}} K_v(z(y)) - \frac{\sqrt{2r}}{\delta} y^{\frac{3}{2}-\frac{\beta}{2}} K_{v+1}(z(y)), \tag{61}$$

$$z(y)=\frac{2\sqrt{2r}}{\delta|\beta-2|}y^{1-\beta/2}. \tag{62}$$

$$\overline{X}=\frac{C}{r}+\overline{K}, \tag{63}$$

$$\underline{X}=\frac{C}{r}+\underline{K}, \tag{64}$$

and, where $I_v(z)$ and $K_v(z)$ are the modified Bessel functions of the first and second kind of order v as defined by Abramowitz and Stegun (1972, expressions 9.6.3 and 9.6.4).

B) In case $\beta>2$, where

$$\mathbf{F}(\mathbf{X})=\begin{bmatrix}-B_0K(\overline{P})+A_1I(\overline{P})+r^{-1}\overline{P}-\overline{X}\\ -B_0K(\underline{P})+A_1I(\underline{P})+r^{-1}\underline{P}-\underline{X}\\ -B_0T_2(\overline{P})+A_1S_2(\overline{P})+r^{-1}\overline{P}\\ -B_0T_2(\underline{P})+A_1S_2(\underline{P})+r^{-1}\underline{P}\end{bmatrix}, \tag{65}$$

with

$$S_2(y)=-\frac{\sqrt{2r}}{\delta}y^{\frac{3}{2}-\frac{\beta}{2}}I_{v+1}(z(y)), \tag{66}$$

$$T_2(y)=\frac{\sqrt{2r}}{\delta}y^{\frac{3}{2}-\frac{\beta}{2}}K_{v+1}(z(y)), \tag{67}$$

and $I(y)$, $K(y)$, $z(y)$, $\overline{X}$, and $\underline{X}$ as defined as in (58), (59), (62), (63), *and* (64), *respectively.*

Proof. The proof of this proposition is similar to the above proofs. ■

3.3. The mean-reverting CEV process

By a class of mean-reverting CEV processes we mean the volatility modeling specification considered in much of the literature on stochastic volatility models, e.g. Kahl and Jäckel (2006), Andersen and Piterbarg (2007), and Lord et al. (2010), that is:

Definition 3. *The generalized class of mean-reverting CEV processes can be nested into the general framework described by equations* (1) *to* (7) *through the following restrictions: $\mu(P)=\kappa(\theta-P)$, $\sigma(P)=\delta P^{\gamma}$, and $\lambda^*(P)=\lambda\rho P^{1-\gamma}$, where γ represents the elasticity parameter, κ, θ, and δ denote, respectively, the (constant) speed of reversion, the (constant) long-run mean price level, and the (constant) scale parameter fixing the initial instantaneous* *volatility at time* $t = 0$, $\sigma_0 = \sigma(P_0) = \delta P_0^{\gamma}$, *and* $\lambda = (\mathbb{E}[R_m] - r)/\mathbb{D}[R_m]$ *is the market price of risk (with* $\mathbb{E}[R_m]$ *and* $\mathbb{D}[R_m]$ *being, respectively, the expected return and standard deviation of the market portfolio), and* ρ *is the correlation between the output price* P *and the market portfolio, i.e.* $dW^{\mathbb{P}} dW_m^{\mathbb{P}} = \rho dt$. *Both* λ *and* ρ *are assumed constant.*

3.3.1. The inhomogeneous geometric Brownian motion process

The inhomogeneous geometric Brownian motion process (hereafter IGBM) (also known as the geometric mean reversion process or the geometric Ornstein-Uhlenbeck process)[6] is obtained with $\gamma = 1$.

This modeling assumption has been been used in many different contexts, e.g. by Brennan and Schwartz (1980) for analyzing convertible bonds, Insley (2002) to model the optimal tree harvesting decision, Sarkar (2003) to study the effect of mean reversion on investment under cost uncertainty, Abadie and Chamorro (2008) to analyze the choice between an inflexible and a flexible technology for producing electricity, and Tsekrekos (2010) to study the effect of mean reversion on entry and exit decisions under output price uncertainty.

Proposition 5. *Under the restrictions stated in Definition 3, with* $\gamma = 1$, *the optimal solution* $\boldsymbol{X} = [\overline{P}, \underline{P}, A_1, B_0]'$ *is uniquely determined by solving the system* $\boldsymbol{F(X) = 0}$, *where*

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -B_0 U(\overline{P}) + A_1 M(\overline{P}) + \varphi \overline{P} - \overline{X} \\ -B_0 U(\underline{P}) + A_1 M(\underline{P}) + \varphi \underline{P} - \underline{X} \\ -B_0 V(\overline{P}) + A_1 W(\overline{P}) + \varphi \overline{P} \\ -B_0 V(\underline{P}) + A_1 W(\underline{P}) + \varphi \underline{P} \end{bmatrix}, \tag{68}$$

with

$$M(y) = y^{-a_1} M(a_1, b_1, z(y)), \tag{69}$$

$$U(y) = y^{-a_2} U(a_2, b_2, z(y)), \tag{70}$$

[6] Note that, with $\gamma = 1$, the variance rate grows with P, so that the variance is zero if P is zero. This is clearly a more appealing feature than the one associated to the simple Ornstein-Uhlenbeck process in which the variance rate is σdz. In this latter case, as the output price becomes small, the constant volatility could cause prices to became negative, which is not economically reasonable for a practitioner.

$$W(y) = a_1 \left[M(y) + \frac{z(y)}{b_1} y^{-a_1} M(a_1+1, b_1+1, z(y)) \right], \tag{71}$$

$$V(y) = a_2 \left[U(y) - z(y) y^{-a_2} U(a_2+1, b_2+1, z(y)) \right], \tag{72}$$

$$z(y) = \frac{2k\theta}{\delta^2 y}, \tag{73}$$

$$a_1 = -\frac{2(k+\lambda\rho\delta) + \delta^2 - \sqrt{8r\delta^2 + (-2k - 2\lambda\rho\delta - \delta^2)^2}}{2\delta^2}, \tag{74}$$

$$a_2 = -\frac{2(k+\lambda\rho\delta) + \delta^2 + \sqrt{8r\delta^2 + (-2k - 2\lambda\rho\delta - \delta^2)^2}}{2\delta^2}, \tag{75}$$

$$b_1 = 2 + 2a_1 + \frac{2(k+\lambda\rho\delta)}{\delta^2}, \tag{76}$$

$$b_2 = 2 + 2a_2 + \frac{2(k+\lambda\rho\delta)}{\delta^2}, \tag{77}$$

$$\varphi = (r + k + \lambda\rho\delta)^{-1}, \tag{78}$$

$$\overline{X} = -\frac{k\theta}{r(k+\lambda\rho\delta)} + \frac{k\theta}{(r+k+\lambda\rho\delta)(k+\lambda\rho\delta)} + \frac{C}{r} + \overline{K}, \tag{79}$$

$$\underline{X} = -\frac{k\theta}{r(k+\lambda\rho\delta)} + \frac{k\theta}{(r+k+\lambda\rho\delta)(k+\lambda\rho\delta)} + \frac{C}{r} + \underline{K}. \tag{80}$$

Proof. To proof of this proposition is similar to the proof of the above propositions. ■

3.3.2. The mean-reverting square-root process

The mean-reverting square-root process, also known as the Cox-Ingersoll-Ross process (hereafter CIR process), due to Cox et al. (1985) is obtained with $\gamma = 1/2$. This process has been widely used to model volatility, interest rates, and other financial instruments.

In the context of real options, this model assumption has been used by Dias and Shackleton (2011) to study the investment hysteresis problem under stochastic interest rates, while Alvarez (2011) use it to model optimal capital accumulation under price uncertainty and cost reversibility of investment.

Proposition 6. *Under the restrictions stated in Definition 3, with $\gamma = 1/2$, the optimal solution $\boldsymbol{X} = [\overline{P}, \underline{P}, A_0, B_1]'$ is uniquely determined by solving the system $\boldsymbol{F(X)} = \boldsymbol{0}$, where*

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -A_0 M(\overline{P}) + B_1 U(\overline{P}) + \varphi \overline{P} - \overline{X} \\ -A_0 M(\underline{P}) + B_1 U(\underline{P}) + \varphi \underline{P} - \underline{X} \\ -A_0 W(\overline{P}) + B_1 V(\overline{P}) + \varphi \overline{P} \\ -A_0 W(\underline{P}) + B_1 V(\underline{P}) + \varphi \underline{P} \end{bmatrix}, \tag{81}$$

with

$$M(y) = M(a_1, b_1, z(y)), \tag{82}$$

$$U(y) = y^{\xi_2} U(a_2, b_2, z(y)), \tag{83}$$

$$W(y) = y \frac{r}{k\theta} M(a_1 + 1, b_1 + 1, z(y)), \tag{84}$$

$$V(y) = \left(\xi_2 U(y) - a_2 z(y) y^{\xi_2} U(a_2 + 1, b_2 + 1, z(y)) \right), \tag{85}$$

$$z(y) = \frac{2(k + \lambda\rho\delta)}{\delta^2} y, \tag{86}$$

$$a_1 = \frac{r}{k + \lambda\rho\delta}, \tag{87}$$

$$a_2 = \xi_2 + \frac{r}{k + \lambda\rho\delta}, \tag{88}$$

$$b_1 = \frac{2k\theta}{\delta^2}, \tag{89}$$

$$b_2 = 2\xi_2 + \frac{2k\theta}{\delta^2}, \tag{90}$$

$$\xi_2 = 1 - \frac{2k\theta}{\delta^2}, \tag{91}$$

and φ, $\overline{X}$, and $\underline{X}$ as defined as in (78), (79), *and* (80), *respectively.*

Proof. To proof of this proposition is similar to the proof of the above propositions. ∎

3.3.3. The Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process (hereafter OU process) is obtained with $\gamma = 0$.

Despite its apparently less desirable feature of allowing paths with negative prices, the OU process is often used in many capital budgeting decisions given its analytic tractability and its ability to fit historical and futures price data (see, for instance, Smith and McCardle (1999) for a specific application in evaluating investments in the oil and gas industry).

Proposition 7. *Under the restrictions stated in Definition 3, with $\gamma = 0$, the optimal solution $\boldsymbol{X} = [\overline{P}, \underline{P}, A_0, B_1]'$ is uniquely determined by solving the system $\boldsymbol{F(X) = 0}$, where*

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} -A_0 M(\overline{P}) + B_1 U(\overline{P}) + \varphi\overline{P} - \overline{X} \\ -A_0 M(\underline{P}) + B_1 U(\underline{P}) + \varphi\underline{P} - \underline{X} \\ -A_0 W(\overline{P}) + B_1 V(\overline{P}) + \varphi\overline{P} \\ -A_0 W(\underline{P}) + B_1 V(\underline{P}) + \varphi\underline{P} \end{bmatrix}, \tag{92}$$

with

$$M(y) = M(a, b, z(y)), \tag{93}$$

$$U(y) = U(a, b, z(y)), \tag{94}$$

$$W(y) = -\frac{2r(k\theta - (k + \lambda\rho\delta)y)}{\delta^2(k + \lambda\rho\delta)}\, y\, M(a+1, b+1, z(y)), \tag{95}$$

$$V(y) = \frac{r(k\theta - (k + \lambda\rho\delta)y)}{\delta^2(k + \lambda\rho\delta)}\, y\, U(a+1, b+1, z(y)), \tag{96}$$

$$z(y) = \frac{\left(k\theta - (k + \lambda\rho\delta)y\right)^2}{\delta^2(\kappa + \lambda\rho\delta)}, \tag{97}$$

$$a = \frac{r}{2(k + \lambda\rho\delta)}, \tag{98}$$

$$b = \frac{1}{2}, \tag{99}$$

and φ, $\overline{X}$, and $\underline{X}$as defined as in (78), (79), *and* (80), *respectively.*

Proof. To proof of this proposition is similar to the proof of the above propositions. ■

4. Analysis of optimal entry-exit policy

In this section we analyze the optimal entry-exit policy assuming that the underlying output price dynamics follows the aforementioned generalized one-dimensional diffusion subject to the restrictions stated in definitions 1 to 3. Panels (a) and (b) of Figure 1 plot entry and exit thresholds prices, $\overline{P}$ and $\underline{P}$, respectively, as a function of lump-sum entry and exit costs $\overline{K} = -\underline{K}$ for different parameter values. In our modeling framework, this means that $\alpha = -1$. The range $(\overline{P}, \underline{P})$ is the hysteretic band of the problem since idle firms do not invest and operating firms do not abandon the activity within this intermediate level of output prices. Panel (a) is for different β values of the CEV process, namely, $\beta \in \{-4, -2, 0, 1, 2, 3\}$ ($\beta = 2$ corresponds to the GBM assumption), where we have used the following parameter values: $C = 2$, $r = 0.04$, $\sigma_0 = 0.15$, $\mu = 0.08$, $\lambda = 0.4$, $\rho = 1$, and $P_0 = 1$, and panel (b) is for different γ values of the mean-reverting CEV processes, namely, $\gamma = 0$ (OU process), $\gamma = 1/2$ (CIR process) and $\gamma = 1$ (IGBM process), where we have used the following parameter values: $C = 2$, $r = 0.04$, $\sigma_0 = 0.15$, $k = 0.05$, $\theta = 1$, $\lambda = 0.4$, $\rho = 1$, and $P_0 = 1$. As we can see from this figure, the hysteresis emerges only when entry and exit costs are present, otherwise the entry and exit thresholds $\overline{P}, \underline{P}$ drop to the level of the variable cost C. From panels (a) and (b) of Figure 1, we also can see that the hysteresis increases with the diffusion coefficient (i.e, with the parameter β in the CEV and with the parameter γ in the mean-reverting CEV processes). This results are consistent with the findings of Dias and Nunes (2011).

Panels (c) and (d) of the Figure 1 plot the optimal decisions thresholds scaled by the corresponding Marshallian triggers as functions of volatility. Panel (c) is for different β values of the CEV process, namely, $\beta \in \{-4, -2, 0, 1, 2, 3\}$ ($\beta = 2$ corresponds to the GBM assumption), where we have used the following parameter values:$\overline{K} = 3$, $\underline{K} = -2$, $C = 2$, $r = 0.04$, $\mu = 0.08$, $\lambda = 0.4$, $\rho = 1$ and $P_0 = 1$, and panel (b) is for different γ values of the mean-reverting CEV processes, namely, $\gamma \in \{0, 1/2, 1\}$ (as well as the case of GBM with no drift), where we have used the following parameter values: $\overline{K} = 3$, $\underline{K} = -2$, $C = 2$, $r = 0.04$, $k = 0.05$, $\theta = 1$, $\lambda = 0.4$, $\rho = 1$, and $P_0 = 1$. Both panels show that the entry and exit thresholds under the CEV and the mean-reverting CEV processes converge to the Marshallian triggers, $\overline{W}$ and $\underline{W}$, respectively, when $\delta \to 0$, so $\overline{P}/\overline{W} \to 1$ and $\underline{P}/\underline{W} \to 1$. From these panels, we can say that there is a clear trend for a wider range of inaction as the volatility coefficient rises.

An idle firm will enter in the market if the output price rises to high values, but it owns an option to exit later if the output prices fall to a sufficient low level and return to the idle state. Once the project is abandoned, the firms owns an option to reinvest again if the output prices reverse to high levels again. Thus, it is important to evaluate the no-action region, this is, the hysteretic band. Figures 2 and 3 highlight the value of an idle firm, $V_0(P)$, and the value of an active firm, $V_1(P)$, both as functions of the output price P, for the CEV process and mean-reverting CEV process, respectively. Also shown are the entry and exit thresholds prices, $\overline{P}$ and $\underline{P}$. Since the option to invest is exercised as soon as P reaches $\overline{P}$, the option value does not exist for values of P above $\overline{P}$. Similarly, since the abandonment is exercised as soon as P falls to $\underline{P}$, the option value does not exist for values of P below $\underline{P}$. Note that, for all β values, at $P = \underline{P}$, $V_0(P)$ exceeds $V_1(P)$ by the abandonment cost $-\underline{K}$, since at that price is optimal to exercise the abandonment option, giving up $-\underline{K} + V_1$ and receiving V_0. Likewise, at $P = \overline{P}$ it is optimal to invest, so $V_1 = V_0 + \overline{K}$.

In Figure 2 we have used the following parameter values: $\overline{K} = 3$, $\underline{K} = -2$, $C = 2$, $r = 0.04$, $\sigma_0 = 0.15$, $\mu = 0.08$, $\lambda = 0.4$, $\rho = 1$, and $P_0 = 1$ for $\beta \in \{-4, -2, 0, 1, 2, 3\}$. Considering these parameter values, and for $\beta = -4$, the entry and exit thresholds prices are, respectively, $\overline{P} = 2.133$ and $\underline{P} = 1.422$, which originates a range of inaction with width 0.711. Under the GBM process, i.e. $\beta = 2$, the entry and exit trigger points are respectively $\overline{P} = 2.782$ and $\underline{P} = 1.319$, which gives a larger range of inaction of 1.463. With $\beta = 3$, the range of inaction increases to 2.023, where entry and exit thresholds are given by $\overline{P} = 3.34$ and $\underline{P} = 1.312$, respectively. It is clear that the hysteretic band increases with the parameter β in the CEV process, keeping all else equal.

In Figure 3 we have used the following parameter values borrowed from Tsekrekos (2010): $\overline{K} = 3$, $\underline{K} = -2$, $C = 2$, $k = 0.05$, $\theta = 1$, $\lambda = 0.4$, $\sigma_0 = 0.15$, $\rho = 1$, $P_0 = 1$, and $r = 0.04$ for $\gamma = 0, 1/2, 1$. We also plot the GBM process with no drift for reference. We conclude that hysteresis increases with the volatility parameter of the mean-reverting processes, from de 1.809 in the OU process, to 1.892 in the CIR process, and to 2.136 in the IGBM process.

Figures 2 and 3 allow us to conclude that the value of an idle firm and the value of an active firm increase when the β and γ parameters rise.

In summary, we can draw the following conclusions from our analysis, for both CEV and mean-reverting CEV processes:

i) The entry threshold price rise and the exit threshold price falls as the parameters β and γ rises, keeping all remainder parameters equal. Thus, the hysteretic band will be wider;

ii) When equal entry and abandonment cost, $\overline{K} = -\underline{K}$, increase, the entry trigger increases while the exit trigger decreases, leading to a higher inactive region. This holds for all β and γ values.

iii) Keeping all remainder parameter values constants, the hysteric band increases with the volatility parameter.

5. First passage time distributions for entry and exit thresholds

In this section we will compute and analyze the *ex ante* probability of entry for an inactive firm, and the *ex ante* probability of exit for an active firm within an specified horizon. Following Tsekrekos (2010), we will consider that the *ex ante* probability that a single idle firm will enter in an industry/market during the time horizon T will be a measure of the fraction of idle firms, under competitive equilibrium, that will enter the market during this time horizon. The same line of reasoning applies for the *ex ante* probability that a single active firm will exit during time T.

For any optimal policy pair $(\overline{P}, \underline{P})$, the *ex ante* probability that an inactive firm will enter during time T is equal to $P(\tau_U \leq T)$, where $\tau_U := inf\{t \geq t_0 : P_t = \overline{P}\}$ is the first hitting time of the underlying process to the investment threshold $\overline{P}$. Conversely, the *ex ante* probability that an active firm will exit the market during the time horizon T is equal to $P(\tau_L \leq T)$, with $\tau_L := inf\{t \geq t_0 : P_t = \underline{P}\}$ is the first hitting time of the underlying process to the divestment threshold $\underline{P}$.

Considering the particular case of the CEV model, $\beta = 2$ (GBM model), the *ex ante* probabilities of entry and exit can be computed in closed-form, using the following expressions as given by Jeanblanc et al. (2009, expressions 3.3.2 and 3.3.3), respectively,

$$P(\tau_U \leq T) = N\left(\frac{\ln(\frac{P_0^*}{\overline{P}}) + (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) + \left(\frac{\overline{P}}{P_0^*}\right)^{\frac{2\mu}{\sigma^2}-1} N\left(\frac{\ln(\frac{P_0^*}{\overline{P}}) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right), \quad (100)$$

$$P(\tau_L \leq T) = N\left(\frac{\ln(\frac{\underline{P}}{P_0^*}) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) + \left(\frac{\underline{P}}{P_0^*}\right)^{\frac{2\mu}{\sigma^2}-1} N\left(\frac{\ln(\frac{\underline{P}}{P_0^*}) + (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right), \quad (101)$$

where $N(.)$ is the normal standard cumulative distribution function, and $P_0^* = \frac{\overline{P}+\underline{P}}{2}$ is the current level of the price process. Expression (101) corrects the small typo in Tsekrekos (2010, Equation 27).

For the CEV process, with $\beta \neq 2$, and for the mean-reverting CEV process there are no simple closed-form solutions as for the GBM process. Thus, in order to simulate these probabilities, it will be necessary to compute first an approximate solution of the CEV and mean-reverting CEV processes prices.

Table 1 shows the *ex ante* entry and *ex ante* exit probabilities for the CEV diffusion model using the parameter values $\beta \in \{-4, -2, 0, 1, 2, 3\}$, $\sigma_0 \in \{0.10, 0.15, 0.20\}$, $r = 0.04$, and $T = 10$ years. The parameter μ will be adjusted so that the difference $(\mu - \lambda\rho\delta)$ remains constant and equal to 2 (so, for $\sigma_0 = 0.10, 0.15$, and 0.20 we will have $\mu = 0.06, 0.08$, and 0.10, respectively). To compute these probabilities we have used the Euler scheme, described in **?**. Details on how these probabilities were computed are provided in the Appendix.

Table 1 show that, for $\beta < 2$, there is a direct relation between the volatility parameter and the *ex ante* probability of entry, while for $\beta > 2$, there is an inverse relation. For the case of the exit probability, and for all β values, the probability increases with volatility. We can also conclude that there appears to be evidence that the value of entry and exit probabilities decreases when β increases, keeping the remainder parameter values constants.

Volatility	$\beta = 3$	$\beta = 2$	$\beta = 1$	$\beta = 0$	$\beta = -2$	$\beta = -4$
Panel A: Probability of entry						
$\sigma_0 = 0.10$	0.5025	0.5888	0.6340	0.6816	0.7446	0.7976
$\sigma_0 = 0.15$	0.4896	0.5818	0.6354	0.6918	0.8063	0.8862
$\sigma_0 = 0.20$	0.4880	0.5820	0.6372	0.7056	0.8505	0.9147
Panel B: Probability of exit						
$\sigma_0 = 0.10$	0.0597	0.1602	0.2300	0.2760	0.3134	0.3174
$\sigma_0 = 0.15$	0.1263	0.2942	0.3758	0.4298	0.4626	0.4637
$\sigma_0 = 0.20$	0.1765	0.3945	0.4720	0.5082	0.5407	0.5343

This table values the probabilities of entry and exit in the market during the time horizon T under the CEV diffusion model. In each simulation, $70,000$ paths and $1,000$ time steps are used. σ_0 is the volatility of the market price, and μ is the growth rate of the market price, which is adjusted for each σ_0 value ($\mu \in \{0.06, 0.08, 0.10\}$, respectively). In all cases $P_0^* = (\overline{P} + \underline{P})/2$, the time horizon is $T = 10$ years, the variable flow cost is $C = 2$, $r = 0.04$ is the riskless interest rate, $\lambda = 0.4$ is the market price of the risk, $\rho = 1$ is the correlation between output price and the market portfolio, and the entry and exit sunk costs are $\overline{K} = 3$ and $\underline{K} = -2$, respectively.

To compute an approximate solution to the IGBM and CIR processes we will use the *Pathwise Adapted Linearization* and the *Pathwise Adapted Linearization Quadratic* schemes, both proposed by Kahl and Jäckel (2006), and explained in the Appendix. To compute an approximate solution to the OU process, we will simulate the output prices using the following exact discretization:

$$P_{t+1} = e^{-k\Delta_t} P_{t_i} + \theta(1 - e^{-k\Delta_t}) + \delta\sqrt{\frac{(1 - e^{-2k\Delta_t})}{2k}} Z_{t-i+1}, \tag{102}$$

where Z_i are independent $N(0,1)$ variables, and $\Delta_t = t_{i+1} - t_i$.

Tables 5 and 5 give us the *ex ante* probabilities of entry and exit, respectively, for the mean-reverting CEV processes, namely, for the OU, CIR, and IGBM processes.

Our first remark is concerned with the effect of the volatility parameter, σ_0, on the *ex ante* probabilities. As reported in Table 5, the probability of entry increases with σ_0 in all cases. Similarly, we reach the same conclusion for the exit probability, except when the price process reverts to low levels ($\theta = 0.4$) with higher speed ($k = 0.10$). These conclusions are similar to the ones reaches by Tsekrekos (2010) for the IGMB process, and by Sarkar (2000) for the GBM process. So, we can also conclude that the non-monotonic relationship, as reported by the authors referenced above, is present in the all mean-reverting CEV processes analyzed.

We analyze also the effect of the log-run output price level, θ. By the results reported, we can argue that the increase of the θ parameter has a positive effect on the entry decisions and a negative effect on the exit decisions.

Now, we analyze the effect of the mean reverting speed parameter, k, on the investment and disinvestment decisions. There is some evidence of a negative effect on the entry probability, except when the price of the output process revert to hight level (when $\rho = 0$, and for $\rho = 1$ with low volatility).

Finally, we report our conclusions of the effect of the γ parameter. From Table 5 we can conclude that when γ increases, the probability of entry also increases, except when the correlation between the equilibrium output price and the market portfolio is null, $\rho = 0$, the mean reversion speed is hight, $k = 0.10$, and the volatility is hight, $\sigma_0 = 0.15, 0.25$. From Table 5, we may conclude that the probability of exit, for $\rho = 1$, increases when γ increases, except when volatility is low, $\sigma_0 = 0.10$, and the price process reverts to hight level, $\theta = 1.4$. However, there is some evidence of decrease of the exit probability, except when the price process reverts to low levels, $\theta = 0.4$.

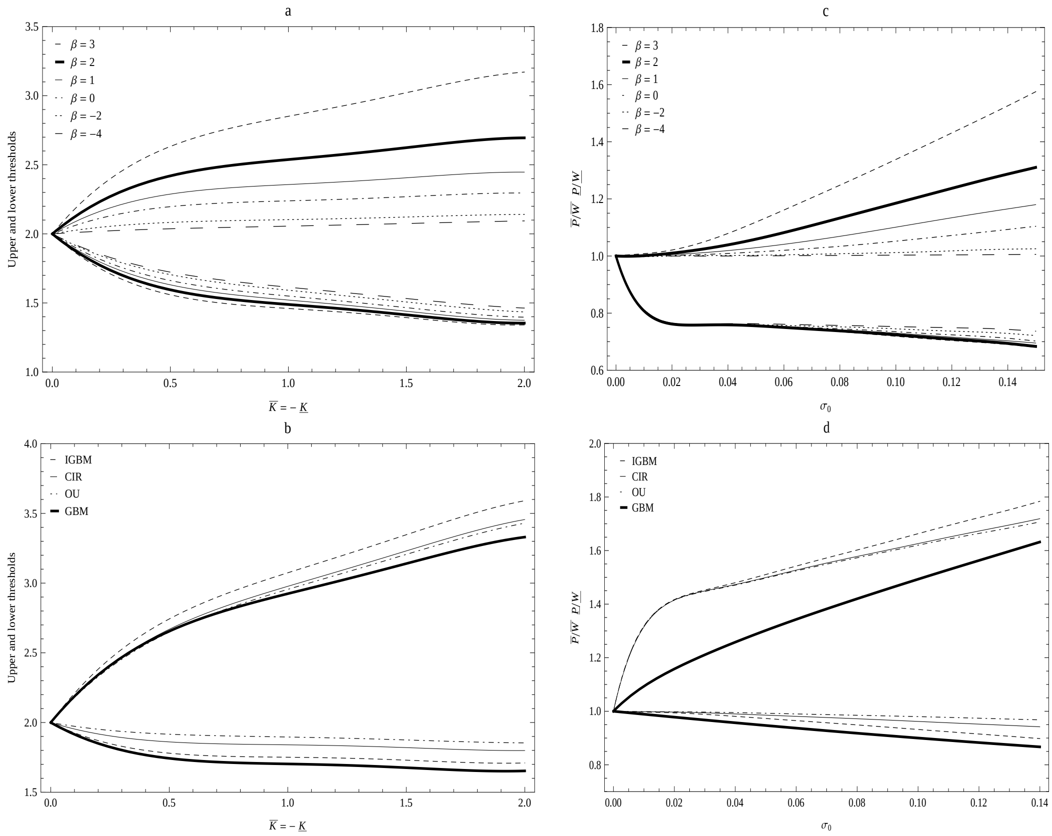

Figure 1: Panels (a) and (b): the optimal entry and exit triggers, $\overline{P}$ and $\underline{P}$, under the CEV and the mean reverting CEV processes, as function of entry and exit cost, in the particular case $\overline{K} = -\underline{K}$. Panels (c) and (d): the ratios of the optimal entry and exit triggers $\overline{P}$, $\underline{P}$ under the CEV and the mean-reverting processes, respectively, over the corresponding thresholds, $\overline{W}$, $\underline{W}$, as function of volatility, σ_0. The rest parameters used are $C = 2$, $r = 0.04$, $\lambda = 0.4$, $\rho = 1$, $P_0 = 1$, and in panel (a) $\beta \in \{-4, -2, 0, 1, 2, 3\}$, $\sigma_0 = 0.15$, $\mu = 0.08$; panel (b) $\gamma \in \{0, 1/2, 1\}$, $\sigma_0 = 0.15$, $k = 0.05$, $\theta = 1$, and the GBM with no drift; panel (c) $\overline{K} = 3$, $\underline{K} = -2$, $\beta \in \{-4, -2, 0, 1, 2, 3\}$, $\mu = 0.08$; and in panel (d) $\overline{K} = 3$, $\underline{K} = -2$, $\gamma \in \{0, 1/2, 1\}$, $k = 0.15$, $\theta = 1$, and the GBM with no drift.

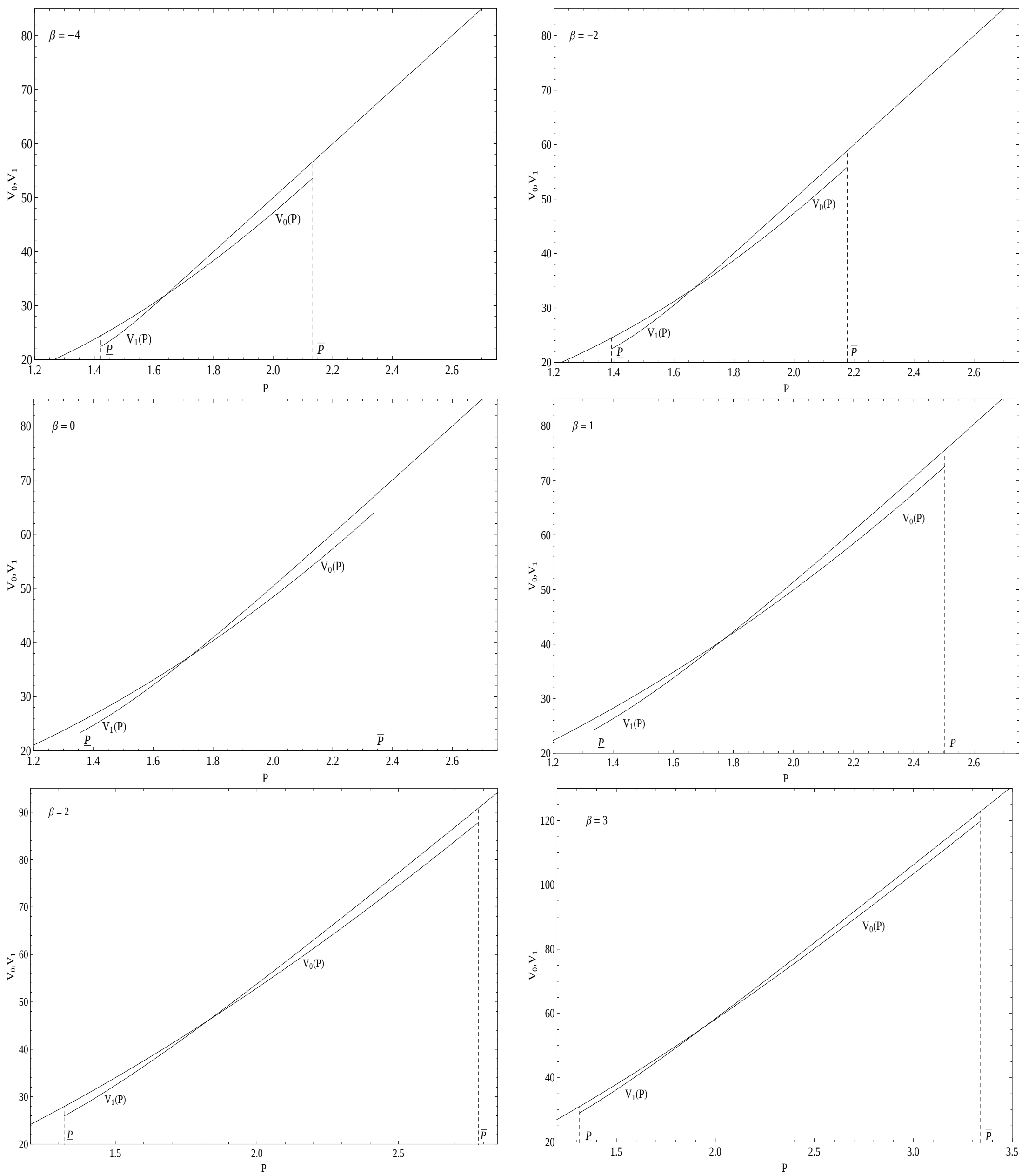

Figure 2: The value of the firm when idle, $V_0(P)$, and active, $V_1(P)$, as a function of the output price P, when the output price follows a CEV process. At the entry and exit thresholds, $\overline{P}$, $\underline{P}$, the firm optimally switches between the idle and active states by sinking the entry and exit costs, $\overline{K}$, $\underline{K}$. The rest of parameters used are: $\overline{K} = 3$, $\underline{K} = -2$, $C = 2$, $r = 0.04$, $\sigma_0 = 0.15$, $\mu = 0.08$, $\lambda = 0.4$, $\rho = 1$, and $P_0 = 1$.

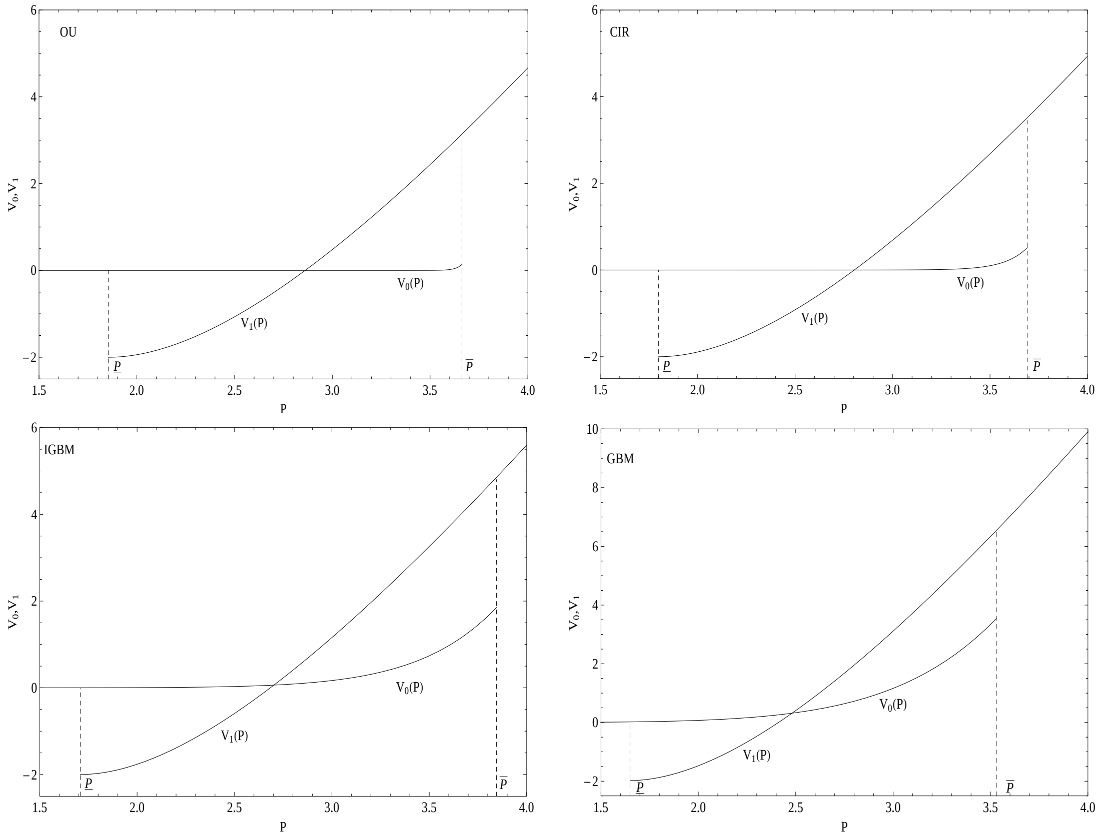

Figure 3: The value of the firm when idle, $V_0(P)$, and active, $V_1(P)$, as a function of the output price P, when the output price follows a CEV mean-reverting process. At the entry and exit thresholds, $\overline{P}$, $\underline{P}$, the firm optimally switches between the idle and active states by sinking the entry and exit costs, $\overline{K}$, $\underline{K}$. The rest of parameters used are: $\overline{K} = 3$, $\underline{K} = -2$, $C = 2$, $r = 0.04$, $k = 0.05$, $\theta = 1$, $\lambda = 0.4$, $\rho =$, $\sigma_0 = 0.15$, and $P_0 = 1$.

	$\theta = 0.40$			$\theta = 1.00$			$\theta = 1.40$		
	$\sigma_0 = 0.10$	$\sigma_0 = 0.15$	$\sigma_0 = 0.25$	$\sigma_0 = 0.10$	$\sigma_0 = 0.15$	$\sigma_0 = 0.25$	$\sigma_0 = 0.10$	$\sigma_0 = 0.15$	$\sigma_0 = 0.25$
Panel A: Probability of entry with $\rho = 1$									
OU process									
$k = 0.05$	0.0001	0.0045	0.0478	0.0100	0.0424	0.1207	0.1021	0.1344	0.2019
$k = 0.10$	0.0000	0.0000	0.0014	0.0000	0.0020	0.0249	0.1568	0.0949	0.1212
CIR process									
$k = 0.05$	0.0045	0.0395	0.1607	0.0448	0.1179	0.2408	0.1614	0.2192	0.3125
$k = 0.10$	0.0000	0.0012	0.0404	0.0026	0.0273	0.1241	0.2002	0.1749	0.2449
IGBM process									
$k = 0.05$	0.0295	0.1194	0.2745	0.1012	0.2015	0.3333	0.2195	0.2905	0.3816
$k = 0.10$	0.0010	0.0274	0.1544	0.0233	0.1002	0.2425	0.2359	0.2482	0.3338
Panel B: Probability of entry with $\rho = 0$									
OU process									
$k = 0.05$	0.0050	0.0550	0.2007	0.1233	0.2153	0.3384	0.3284	0.3771	0.4417
$k = 0.10$	0.0000	0.0005	0.0443	0.0371	0.1240	0.2657	0.4886	0.5005	0.5111
CIR process									
$k = 0.05$	0.0194	0.0982	0.2506	0.1487	0.2433	0.3575	0.3395	0.3888	0.4455
$k = 0.10$	0.0000	0.0079	0.1059	0.0578	0.1534	0.2911	0.4764	0.4902	0.5004
IGBM process									
$k = 0.05$	0.0488	0.1531	0.2966	0.1729	0.2724	0.3750	0.3383	0.3897	0.4437
$k = 0.10$	0.0022	0.0401	0.1792	0.0826	0.1871	0.3148	0.4524	0.4616	0.4741

This table values the probabilities of entry in the market during a time horizon T for the mean-reverting CEV. In each simulation, $100,000$ paths and $1,000$ time steps are used. θ is the long-run output price level, k is the speed of mean reversion, σ is the volatility of the output price process, and ρ is the correlation between the equilibrium output price and the market portfolio. In all cases, $P_0^* = (\overline{P} + \underline{P})/2$, the time horizon is $T = 10$ years, the market price of risk is $\lambda = 0.4$, the variable flow cost is $C = 1$, and the entry and exit sunk costs are $\overline{K} = 3$ and $\underline{K} = -2$, respectively.

	$\theta = 0.40$			$\theta = 1.00$			$\theta = 1.40$		
	$\sigma_0 = 0.10$	$\sigma_0 = 0.15$	$\sigma_0 = 0.25$	$\sigma_0 = 0.10$	$\sigma_0 = 0.15$	$\sigma_0 = 0.25$	$\sigma_0 = 0.10$	$\sigma_0 = 0.15$	$\sigma_0 = 0.25$
Panel A: Probability of exit with $\rho = 1$									
OU process									
$k = 0.05$	0.3074	0.3562	0.4140	0.1098	0.1916	0.3012	0.0414	0.1133	0.2333
$k = 0.10$	0.6487	0.6080	0.5898	0.1018	0.1952	0.3207	0.0019	0.0431	0.1731
CIR process									
$k = 0.05$	0.3391	0.3983	0.4751	0.1264	0.2259	0.3623	0.0457	0.1340	0.2876
$k = 0.10$	0.6568	0.6299	0.6355	0.1225	0.2341	0.3887	0.0007	0.0480	0.2202
IGBM process									
$k = 0.05$	0.3621	0.4306	0.5340	0.1455	0.2585	0.4228	0.0491	0.1535	0.3442
$k = 0.10$	0.6604	0.6423	0.6711	0.1431	0.2723	0.4549	0.0002	0.0532	0.2709
Panel B: Probability of exit with $\rho = 0$									
OU process									
$k = 0.05$	0.4622	0.4741	0.5042	0.1420	0.2319	0.3469	0.0228	0.1042	0.2527
$k = 0.10$	0.7478	0.6910	0.6463	0.0656	0.1566	0.2628	0.0000	0.0061	0.1028
CIR process									
$k = 0.05$	0.4709	0.4873	0.5286	0.1293	0.2245	0.3615	0.0072	0.0685	0.2378
$k = 0.10$	0.7517	0.7082	0.6755	0.0533	0.1446	0.3048	0.0000	0.0000	0.0413
IGBM process									
$k = 0.05$	0.4677	0.4933	0.5497	0.1152	0.2134	0.3750	0.0012	0.0406	0.2249
$k = 0.10$	0.7448	0.7046	0.6936	0.0421	0.1350	0.3173	0.0000	0.0000	0.0048

This table values the probabilities of entry in the market during a time horizon T for the mean-reverting CEV. In each simulation, $100,000$ paths and $1,000$ time steps are used. θ is the long-run output price level, k is the speed of mean reversion, σ is the volatility of the output price process, and ρ is the correlation between the equilibrium output price and the market portfolio. In all cases, $P_0^* = (\overline{P} + \underline{P})/2$, the time horizon is $T = 10$ years, the market price of risk is $\lambda = 0.4$, the variable flow cost is $C = 1$, and the entry and exit sunk costs are $\overline{K} = 3$ and $\underline{K} = -2$, respectively.

6. Conclusions

In this article, we consider the optimal entry and exit policy of a firm in the presence of output price uncertainty and subject to costly reversibility of investment under a generalized class of one-dimensional diffusions. We derive explicit solutions for the value functions for options of reversible investments under CEV and mean-reverting CEV processes.

We compare the different stochastic processes studied by doing an analysis of optimal entry-exit policy. This analysis includes both numerical and graphical illustrations, where we have concluded that the hysteretic band increases when: i) β and γ parameters increase; ii) when both investment and divestment equal costs increase, and iii) the volatility parameter increases.

We have also computed the *ex ante* probabilities of entry and exit and try to show that the choice of the stochastic process for the output price has a significant impact on investment and divestment decisions.

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