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A VARIATIONAL APPROACH TO WAVEFORM DESIGN FOR SYNTHETIC-APERTURE IMAGING

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Abstract. We derive an optimal transmit waveform for filtered backprojection-based synthetic-aperture imaging. The waveform is optimal in terms of minimising the mean square error (MSE) in the resulting image. Our optimization is performed in two steps: First, we consider the minimum-mean-square-error (MMSE) for an arbitrary but fixed waveform, and derive the corresponding filter for the filtered backprojection reconstruction. Second, the MMSE is further reduced by finding an optimal transmit waveform. The transmit waveform is derived for stochastic models of the scattering objects of interest (targets), other scattering objects (clutter), and additive noise. We express the waveform in terms of spatial spectra for the random fields associated with target and clutter, and the spectrum for the noise process. This approach results in a constraint that involves only the amplitude of the Fourier transform of the transmit waveform. Therefore, considerable flexibility is left for incorporating additional requirements, such as minimal variation of transmit amplitude and phase-coding.

1. Introduction

There are two main waveform design approaches in the radar literature, namely the ambiguity and variational-based approaches. In the first approach, a range-Doppler echo model is used with matched-filter processing. The waveforms are designed and combined in order to create an approximate Dirac-delta ambiguity function [1, 25, 7, 19, 20, 5, 11, 10, 13, 4]. In the second approach, the scene is assumed to be static and therefore the range-only echo model is considered. As in the first approach, matched filtering is used as a foundation for joint design of both receive processing and waveforms for target detection, identification and classification [9, 2, 18, 8, 21]. In [2, 18, 8] target detection is considered and the waveforms are designed by maximizing the signal-to-noise or signal-to-interference ratios. In [2], a mutual information criterion is also used to design optimal waveforms for target classification. In both of the aforementioned approaches a monostatic radar system is considered.

In this paper we consider a synthetic-aperture radar (SAR) system over a static scene and present a variational approach to the design of both waveforms and an image reconstruction method for high range-resolution imaging in the presence of clutter and noise. Our waveform design criterion is based on minimization of the

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MSE between the reconstructed image and scatterers of interest. While our method falls into the category of variational approaches, to our knowledge, it is the first attempt to design waveforms for synthetic-aperture imaging.

Our study commences with the physics-based model which takes into account the antenna beam pattern, transmitted waveform and geometric spreading factors. We assume that the scene is composed of two parts, namely the scatterers of interest (*target*) and unwanted scatterers (*clutter*). Furthermore we assume the measured backscattered waves are corrupted by additive noise.

In [27] we showed how statistical information about the target, noise and clutter can be combined with microlocal analysis in order to develop filtered-backprojection-type, edge-preserving, target-enhancing, clutter-and-noise-suppressing reconstruction algorithms for radar imaging. Following [27], we derive the filtered-backprojection-type image formation process that minimizes the MSE between the reconstructed image and the original target for a given waveform. We extend this approach by designing waveforms that further reduce the MSE.

The organization of the paper is as follows: In Section 2, we introduce the mathematical model and relevant notation. In Section 3, we develop the generalized filtered backprojection algorithm for image formation and derive the corresponding reconstruction filter that minimizes the MSE in the presence of clutter and noise. In Section 4 we optimize the result with respect to the transmitted waveforms, in order to further minimize the MSE. We conclude our discussion and summarize our results in Section 5.

2. Forward model

In this section we present our model for the measured backscatter signal resulting from the transmission of a given waveform. First, we will describe the deterministic case. Subsequently, we will present the signal model including additive measurement noise, and employ a stochastic model for the scatterers.

We model the antenna as a time-varying current density over an aperture. This current density creates an electromagnetic field, denoted by u^{in}, that emanates from the antenna. We assume that u^{in} propagates through dry air, and since dry air is a homogeneous medium, Maxwell's equations for the electromagnetic field decouple into three scalar wave equations (one for each vector component). We will consider each vector component of the electric field separately, which means that we ignore polarizing effects. Therefore u^{in} satisfies

$$(\nabla^2 - c_0^{-2}\partial_t^2)u^{\mathrm{in}}(t,\mathbf{x}) = -j_{\mathrm{tr}}(t,\mathbf{x}), \tag{1}$$

where c_0 is the speed of light in dry air, where $j_{\mathrm{tr}}(t,\mathbf{x})$ is proportional to the effective current density, and where $t \in \mathbb{R}$ is time and $\mathbf{x} \in \mathbb{R}^3$ is spatial location. For the sake of simplicity we will assume that every point on the antenna aperture emits the waveform $p(t)$ with unit energy, *i.e.,*

$$j_{\mathrm{tr}}(t,\mathbf{x}) = p(t)J_{\mathrm{tr}}(\mathbf{x}), \tag{2}$$

and

$$\|p\|^2 = \int |p(t)|^2 \,\mathrm{d}t = 1. \tag{3}$$

From (1), we use the free-space Green's function [22]

$$g_0(t, \mathbf{x}) = \frac{\delta(t - |\mathbf{x}|/c_0)}{4\pi|\mathbf{x}|} = \frac{1}{4\pi|\mathbf{x}|} \int e^{-i2\pi\omega(t-|\mathbf{x}|/c_0)} d\omega \tag{4}$$

to obtain

$$u^{\rm in}(t, \mathbf{x}) = (g_0 * j_{\rm tr})(t, \mathbf{x}) = \int \frac{e^{-i2\pi\omega(t-|\mathbf{x}-\mathbf{y}|/c_0)}}{4\pi|\mathbf{x}-\mathbf{y}|} \tilde{j}_{\rm tr}(\omega, \mathbf{y}) \mathrm{d}\omega \mathrm{d}\mathbf{y}. \tag{5}$$

Here we have defined $\tilde{j}_{\rm tr}(\omega, \mathbf{x})$ and $P(\omega)$ as

$$\tilde{j}_{\rm tr}(\omega, \mathbf{x}) = \int j_{\rm tr}(t, \mathbf{x}) e^{i2\pi t\omega} \mathrm{d}t = P(\omega) J_{\rm tr}(\mathbf{x}), \tag{6}$$

and

$$P(\omega) = \int e^{i2\pi t\omega} p(t)\, \mathrm{d}t, \tag{7}$$

respectively.

Next we assume that the antenna (or sensor array) is small compared to the distance between the antenna and the scatterers. Thus we use the far-field approximation (Fraunhofer approximation) to express the incident field as

$$u^{\rm in}(t, \mathbf{x}, \mathbf{y}) \approx \int \frac{e^{-i2\pi\omega(t-|\mathbf{x}-\mathbf{y}|/c_0)}}{4\pi|\mathbf{x}-\mathbf{y}|} P(\omega, \mathbf{y}) \tilde{J}_{\rm tr}(\omega, \widehat{\mathbf{x}-\mathbf{y}}, \mathbf{y}) \mathrm{d}\omega, \tag{8}$$

where the vector $\mathbf{y}$ denotes the center of the antenna, $\widehat{\mathbf{x}-\mathbf{y}}$ denotes a unit vector in the same direction as $\mathbf{x}-\mathbf{y}$, and

$$\tilde{J}_{\rm tr}(\omega, \widehat{\mathbf{x}-\mathbf{y}}, \mathbf{y}) = e^{i2\pi\omega/c_0(\widehat{\mathbf{x}-\mathbf{y}})\cdot\mathbf{y}} \int e^{-i2\pi\omega/c_0(\widehat{\mathbf{x}-\mathbf{y}})\cdot\mathbf{v}} J_{\rm tr}(\mathbf{v}) \mathrm{d}\mathbf{v}. \tag{9}$$

Note that in (8) we have introduced a $\mathbf{y}$ dependence on P which was not present in (7). This facilitates the use of different waveforms at different antenna locations.

2.1. A model for the scattered field. When we place our antenna in an environment with scatterers, the resulting field will differ from the incident field. For our scalar model, the resulting field will be described by

$$(\nabla^2 - c^{-2}(\mathbf{x})\partial_t^2) u(t, \mathbf{x}) = -j_{\rm tr}(t, \mathbf{x}). \tag{10}$$

where $c(\mathbf{x})$ is the speed of light in the medium. We consider the field u to be a sum of two parts; the incident field $u^{\rm in}$ and the scattered field $u^{\rm sc}$. Therefore, substituting

$$u = u^{\rm in} + u^{\rm sc} \tag{11}$$

in (10) and using (1), we obtain

$$(\nabla^2 - c_0^{-2}\partial_t^2) u^{\rm sc}(t, \mathbf{x}) = V(\mathbf{x}) \partial_t^2 u(t, \mathbf{x}), \tag{12}$$

where the *reflectivity density function* V contains all the relevant information about how the scattering medium differs from free space. The explicit relation between V and the propagation speed is

$$V(\mathbf{x}) = \frac{1}{c^2(\mathbf{x})} - \frac{1}{c_0^2}. \tag{13}$$

A location $\mathbf{x}$ which is occupied by a target or by clutter thus corresponds to locations where $V(\mathbf{x}) \neq 0$. Discontinuities and other singularities of V correspond to edges and other structures which produce significant scattering from the object. It is therefore these singularities that we would like to reconstruct as our image.

We employ the *Born* or *single-scattering* approximation [15, 16] to (12). In other words we replace the full field u on the right side of (12) by the incident field u^{in}. Solving the resulting differential equation leads to

$$u^{\text{sc}}(t, \mathbf{x}) \approx -\int g_0(t-\tau, \mathbf{x}-\mathbf{z}) V(\mathbf{z}) \partial_\tau^2 u^{\text{in}}(\tau, \mathbf{z}) \mathrm{d}\tau \mathrm{d}\mathbf{z}$$
$$= -\int \frac{V(\mathbf{z})}{4\pi|\mathbf{x}-\mathbf{z}|} \partial_t^2 u^{\text{in}}(t - |\mathbf{x}-\mathbf{z}|/c_0, \mathbf{z}) \mathrm{d}\mathbf{z}. \tag{14}$$

For the incident field (8), (14) becomes

$$u^{\text{sc}}(t, \mathbf{x}, \mathbf{y}) \approx \int \frac{e^{-i2\pi\omega(t-(|\mathbf{x}-\mathbf{z}|+|\mathbf{z}-\mathbf{y}|)/c_0)}}{(4\pi)^2|\mathbf{x}-\mathbf{z}||\mathbf{z}-\mathbf{y}|} \times (2\pi\omega)^2 P(\omega, \mathbf{y}) \tilde{J}_{\text{tr}}(\omega, \widehat{\mathbf{z}-\mathbf{y}}, \mathbf{y}) V(\mathbf{z}) \mathrm{d}\omega \mathrm{d}\mathbf{z}. \tag{15}$$

2.2. A MODEL FOR THE IDEAL RECEIVED SIGNAL. The field is measured by the receiving antenna; the reception process results in a beam pattern for reception $\tilde{J}_{\text{rc}}(\omega, \widehat{\mathbf{z}-\mathbf{x}}, \mathbf{x})$. Thus a model for the signal received at $\mathbf{x}$ from a source at $\mathbf{y}$ is

$$u^{\text{sc}}(t, \mathbf{x}, \mathbf{y}) \approx \int \frac{e^{-i2\pi\omega(t-(|\mathbf{x}-\mathbf{z}|+|\mathbf{z}-\mathbf{y}|)/c_0)}}{(4\pi)^2|\mathbf{x}-\mathbf{z}||\mathbf{z}-\mathbf{y}|} (2\pi\omega)^2 \times P(\omega, \mathbf{y}) \tilde{J}_{\text{tr}}(\omega, \widehat{\mathbf{z}-\mathbf{y}}, \mathbf{y}) \tilde{J}_{\text{rc}}(\omega, \widehat{\mathbf{z}-\mathbf{x}}, \mathbf{x}) V(\mathbf{z}) \mathrm{d}\omega \mathrm{d}\mathbf{z}. \tag{16}$$

For common SAR applications, the transmitter and the receiver are colocated. Hence we will set $\mathbf{x} = \mathbf{y}$. The ideal received signal then becomes

$$\tilde{u}^{\text{sc}}(t, \mathbf{y}) \approx \int e^{-i2\pi\omega(t-2|\mathbf{y}-\mathbf{z}|/c_0)} P(\omega, \mathbf{y}) A(\mathbf{z}, \mathbf{y}, \omega) V(\mathbf{z}) \mathrm{d}\omega \mathrm{d}\mathbf{z}, \tag{17}$$

where

$$A(\mathbf{z}, \mathbf{y}, \omega) = \frac{(2\pi\omega)^2 \tilde{J}_{\text{tr}}(\omega, \widehat{\mathbf{z}-\mathbf{y}}, \mathbf{y}) \tilde{J}_{\text{rc}}(\omega, \widehat{\mathbf{z}-\mathbf{y}}, \mathbf{y})}{(4\pi)^2|\mathbf{y}-\mathbf{z}|^2}. \tag{18}$$

So far all spatial variables have been vectors in $\mathbb{R}^3$. We now assume the earth's surface is located at the position given by $\mathbf{z} = \boldsymbol{\psi}(\boldsymbol{x})$, where the function $\boldsymbol{\psi} : \mathbb{R}^2 \to \mathbb{R}^3$ is known. Throughout this document we will use a bold roman font and a bold italic variation to distinguish between vectors in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively: $\mathbf{x} \in \mathbb{R}^3$ and $\boldsymbol{x} \in \mathbb{R}^2$. Because electromagnetic waves are rapidly attenuated in the earth, we assume that the scattering takes place in a thin region near the surface; hence we also assume that the perturbation in wave speed c is of the form

$$V(\mathbf{z}) = c^{-2}(\mathbf{z}) - c_0^{-2} = V_G(\boldsymbol{z}) \delta(\boldsymbol{z} - \boldsymbol{\psi}(\boldsymbol{z})). \tag{19}$$

Thus, we are going to image V_G, which will be referred to as the *ground reflectivity function*. In this situation, we replace the 3D volume integral in (17) by a 2D surface integral. The received field at sensor location $\mathbf{y}$ and time t can therefore be approximated by the expression [17]

$$\tilde{u}^{\text{sc}}(t, \mathbf{y}) = \iint e^{-i2\pi\omega(t-2|\boldsymbol{\psi}(\boldsymbol{x})-\mathbf{y}|/c_0)} P(\omega, \mathbf{y}) A(\boldsymbol{x}, \mathbf{y}, \omega) V_G(\boldsymbol{x}) \mathrm{d}\omega \mathrm{d}\boldsymbol{x}, \tag{20}$$

where ω denotes the angular frequency. Here $\boldsymbol{x} \in \mathbb{R}^2$, $\mathrm{d}\boldsymbol{x}$ is the 2D surface measure, and $A(\boldsymbol{x}, \mathbf{y}, \omega)$ is shorthand for $A(\boldsymbol{\psi}(\boldsymbol{x}), \mathbf{y}, \omega)$.

The idealized inverse problem is to determine V_G from the knowledge of $\tilde{u}^{\text{sc}}$ for $t \in (T_1, T_2)$ and for $\mathbf{y}$ over a curve γ in $\mathbb{R}^3$. We parameterize the curve $\boldsymbol{\gamma}$ as $\boldsymbol{\gamma} := \{ \ \boldsymbol{\gamma}(s) : \ s_{\min} < s < s_{\max} \ \}$. We now make the *start-stop approximation*, *i.e.*, we assume that the scene does not change during the time it takes for a single transmit pulse to illuminate the scene. This allows us to write the data for the idealized inverse problem in terms of the two parameters s ("slow time") and t ("fast time"):

$$\tilde{d}(t,s) = F_P[V_G](t,s), \tag{21}$$

where

$$F_P[V_G](t,s) = \int e^{-i2\pi\omega[t-2|\mathbf{r}_{s,\boldsymbol{x}}|/c_0]} A_P(\boldsymbol{x}, s, \omega) V_G(\boldsymbol{x}) \, \mathrm{d}\omega \mathrm{d}\boldsymbol{x}, \tag{22}$$

where we have used the notation $\mathbf{r}_{s,\boldsymbol{x}} = \boldsymbol{\psi}(\boldsymbol{x}) - \boldsymbol{\gamma}(s)$ and

$$A_P(\boldsymbol{x}, s, \omega) = P(\omega, s) A(\boldsymbol{x}, s, \omega). \tag{23}$$

Here we have used the shorthand notation $P(\omega, s)$ and $A(\boldsymbol{x}, s, \omega)$ for $P(\omega, \boldsymbol{\gamma}(s))$ and $A(\boldsymbol{x}, \boldsymbol{\gamma}(s), \omega)$.

We will assume that the amplitude A_P of (22) satisfies the following criterion:

$$\sup_{(s,\boldsymbol{x}) \in K} \ | \ \partial_\omega^\alpha \partial_s^\beta \partial_{x_1}^{\rho_1} \partial_{x_2}^{\rho_2} A_P(\boldsymbol{x}, s, \omega) \ | \leq C \ (1+\omega^2)^{(2-|\alpha|)/2}, \tag{24}$$

where K is any compact subset of $\mathbb{R} \times \mathbb{R}^2$, and the constant C depends on K, α, β, ρ_1, and ρ_2. This assumption is needed in order for various stationary phase calculations hold, and ensures that the "forward" operator F_P is a Fourier Integral Operator [6, 23, 12]. Assumption (24) is valid, for example, when the transmit waveform is a short pulse and the antenna is sufficiently broadband. We note that A_P can be complex; it can be used to model non-ideal antenna behavior such as phase aberrations and frequency-dependent changes in the beam pattern. With minor modifications it can also be used to model phased arrays.

We will consider the case where the reflectivity function is a random field, and the measurement is contaminated by additive noise. In this regard we will modify the model (21) in two ways. First, we assume that V_G is composed of two parts

$$V_G = T + C, \tag{25}$$

where T corresponds to the target (scatterers of interest) and C to clutter (unwanted scatterers). Second, we include additive noise n, which models thermal fluctuations in the receiver. The data is in this case

$$d(t,s) = F_P[T + C](t,s) + n(t,s). \tag{26}$$

We assume that the noise is a zero-mean second-order stochastic process. Furthermore, we assume that the noise is stationary in the fast time variable t, statistically uncorrelated in the slow time variable s, and that the power spectral density S_n is given by

$$\int e^{i2\pi\omega t} e^{-i2\pi\omega' t'} \mathrm{E}[n(t,s) n(t',s')] \mathrm{d}t \mathrm{d}t' = S_n(\omega, s) \delta(\omega - \omega') \delta(s - s'). \tag{27}$$

Similarly, we assume that the target T and clutter C are realizations of zero-mean second-order random fields, and C and T are statistically independent, *i.e.,* $\mathrm{E}[T(\boldsymbol{x}) C(\boldsymbol{x}')] = 0$. The zero-mean assumption about the target is needed for the current analysis to result in a MMSE reconstruction method. If this assumption is removed we will instead obtain a minimum-variance reconstruction [27].

Let $\mathcal{R}_T$ and $\mathcal{R}_C$ denote the autocorrelation functions of T and C, respectively:

$$\mathrm{E}[T(\boldsymbol{x})T(\boldsymbol{x}')] = \mathcal{R}_T(\boldsymbol{x}, \boldsymbol{x}'), \tag{28}$$

$$\mathrm{E}[C(\boldsymbol{x})C(\boldsymbol{x}')] = \mathcal{R}_C(\boldsymbol{x}, \boldsymbol{x}'). \tag{29}$$

Then the power spectral densities $\tilde{S}_T$ and $\tilde{S}_C$ of T and C respectively, are defined by

$$\mathcal{R}_T(\boldsymbol{x}, \boldsymbol{x}') = \iint e^{-i2\pi\boldsymbol{x}\cdot\boldsymbol{\zeta}} e^{i2\pi\boldsymbol{x}'\cdot\boldsymbol{\zeta}'} \tilde{S}_T(\boldsymbol{\zeta}, \boldsymbol{\zeta}')\mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}' \tag{30}$$

$$\mathcal{R}_C(\boldsymbol{x}, \boldsymbol{x}') = \iint e^{-i2\pi\boldsymbol{x}\cdot\boldsymbol{\zeta}} e^{i2\pi\boldsymbol{x}'\cdot\boldsymbol{\zeta}'} \tilde{S}_C(\boldsymbol{\zeta}, \boldsymbol{\zeta}')\mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}'. \tag{31}$$

3. Image Formation

In this section we present our image formation process. We will form the image $\tilde{T}(\boldsymbol{z})$ by means of a filtered backprojection (FBP) [17, 27] operator B_Q

$$\tilde{T}(\boldsymbol{z}) = B_Q[d](\boldsymbol{z}), \tag{32}$$

where B_Q is defined by

$$B_Q[d](\boldsymbol{z}) := \iint Q(\boldsymbol{z}, s, \omega) e^{i2\pi\omega[t-2|\mathbf{r}_{s,\boldsymbol{z}}|/c_0]} d(t, s)\ \mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}s, \tag{33}$$

with Q being a filter to be determined below. The FBP-type image formation methods are both computationally efficient and simple to implement. Furthermore, FBP results in an image in which visible edges are preserved [17]. FBP type image formation methods are therefore attractive from a practical point of view [24].

In order to determine the filter Q, we will examine the point-spread function (PSF) which is obtained as the resulting integral kernel from inserting (22) into (33). We see that $B_Q(F_P[V_G])(\boldsymbol{z})$ now is

$$\begin{aligned} B_Q(F_P[V_G])(\boldsymbol{z}) = \int & e^{i2\pi\left(\omega[t-2|\mathbf{r}_{s,\boldsymbol{z}}|/c_0] - \omega'[t-2|\mathbf{r}_{s,\boldsymbol{x}}|/c_0]\right)} \\ & \times Q(\boldsymbol{z}, s, \omega) A_P(\boldsymbol{x}, s, \omega') V_G(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\,\mathrm{d}\omega'\mathrm{d}t\,\mathrm{d}\omega\,\mathrm{d}s. \end{aligned} \tag{34}$$

We perform the t and ω' integrations in (34) to obtain

$$B_Q(F_P[V_G])(\boldsymbol{z}) = \int e^{i2\pi 2\omega[|\mathbf{r}_{s,\boldsymbol{x}}| - |\mathbf{r}_{s,\boldsymbol{z}}|]/c_0} Q(\boldsymbol{z}, s, \omega) A_P(\boldsymbol{x}, s, \omega) V_G(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\omega\,\mathrm{d}s. \tag{35}$$

As in [17], we use the identity

$$h(\boldsymbol{x}) - h(\boldsymbol{z}) = (\boldsymbol{x} - \boldsymbol{z}) \cdot \int_0^1 \nabla h(\boldsymbol{z} + \lambda(\boldsymbol{x} - \boldsymbol{z}))\mathrm{d}\lambda \tag{36}$$

with $h(\boldsymbol{x}) = (2\omega/c_0)\ |\mathbf{r}_{s,\boldsymbol{x}}|$ to write the exponent of (36) as

$$\frac{2\omega}{c_0}(|\mathbf{r}_{s,\boldsymbol{x}}| - |\mathbf{r}_{s,\boldsymbol{z}}|) = \frac{2\omega}{c_0}(\boldsymbol{x} - \boldsymbol{z}) \cdot \boldsymbol{\Xi}(s, \boldsymbol{x}, \boldsymbol{z}). \tag{37}$$

For $\boldsymbol{x} = \boldsymbol{z}$ this becomes

$$\boldsymbol{\Xi}(s, \boldsymbol{x}, \boldsymbol{x}) = \nabla h(\boldsymbol{x}) = \widehat{\mathbf{r}_{s,\boldsymbol{x}}} \cdot D\boldsymbol{\psi}(\boldsymbol{x}), \tag{38}$$

where $D\boldsymbol{\psi}$ denotes the Jacobian matrix of derivatives with respect to $\boldsymbol{x}$. Making the Stolt change of variables

$$(s,\omega) \to \boldsymbol{\xi} = \frac{2\omega}{c_0}\boldsymbol{\Xi}(s,\boldsymbol{x},\boldsymbol{z}), \tag{39}$$

we transform (35) into

$$B_Q(F_P[V_G])(\boldsymbol{z}) = \int e^{i2\pi(\boldsymbol{x}-\boldsymbol{z})\cdot\boldsymbol{\xi}} Q(\boldsymbol{z},\boldsymbol{\xi}) A_P(\boldsymbol{x},\boldsymbol{\xi}) \eta(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi}) V_G(\boldsymbol{x})\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\boldsymbol{x}, \tag{40}$$

where

$$\eta(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi}) = |\partial(s,\omega)/\partial\boldsymbol{\xi}| \tag{41}$$

is the corresponding Jacobian determinant.

In [27], to determine Q, we examined the degree to which the image $\tilde{T}$ reproduces the true T, or at least the best approximation to T we could hope to obtain from our limited data. The best approximation to T is determined by the flight path and the frequency band of the radar system. In particular, the best mean square approximation T_Ω to T is

$$\begin{aligned} T_\Omega(\boldsymbol{z}) = I_\Omega T(\boldsymbol{z}) &= \int_{\Omega_{\boldsymbol{z}}} e^{i2\pi(\boldsymbol{x}-\boldsymbol{z})\cdot\boldsymbol{\xi}} T(\boldsymbol{x})\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\boldsymbol{x} \\ &= \int e^{i2\pi(\boldsymbol{x}-\boldsymbol{z})\cdot\boldsymbol{\xi}} I_\Omega(\boldsymbol{z},\boldsymbol{\xi}) T(\boldsymbol{x})\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{42}$$

where $I_\Omega(\boldsymbol{z},\boldsymbol{\xi})$ is the function that is 1 if $\boldsymbol{\xi}$ is in the set $\Omega_{\boldsymbol{z}}$ and zero otherwise. Here the set $\Omega_{\boldsymbol{z}}$ is the set of $\boldsymbol{\xi}$ determined by the flight path and frequency band obtained from the Stolt change of variables (39)

$$\Omega_{\boldsymbol{z}} = \{\boldsymbol{\xi} = 2(\omega/c_0)\widehat{\mathbf{r}_{s,\boldsymbol{z}}} \cdot D\boldsymbol{\psi}(\boldsymbol{z}) \mid s_{\min} \le s \le s_{\max}, \omega_{\min} \le \omega \le \omega_{\max}\}. \tag{43}$$

For further details on this set we refer to [26, 27].

We will now determine the filter Q_P which is optimal in the sense that it yields an image with minimum MSE when compared to T_Ω.

Let

$$\begin{aligned} \mathcal{E}(\boldsymbol{z}) &= \tilde{T}(\boldsymbol{z}) - T_\Omega(\boldsymbol{z}) \\ &= B_Q\left(F_P[V_G]\right)(\boldsymbol{z}) - I_\Omega T(\boldsymbol{z}), \end{aligned} \tag{44}$$

be the error and

$$\mathcal{J}(Q) = \int E|\mathcal{E}(\boldsymbol{z})|^2 d\boldsymbol{z} \tag{45}$$

be the MSE. Then the optimal filter Q_P is defined such that

$$Q_P = \operatorname*{argmin}_Q \mathcal{J}(Q). \tag{46}$$

In order to find Q_P explicitly, first we calculate $\mathcal{J}(Q)$

$$\begin{aligned} \mathcal{J}(Q) &= \int E\left[\left|B_Q\left(F_P[V_G]\right)(\boldsymbol{z}) - I_\Omega T(\boldsymbol{z})\right|^2\right] \mathrm{d}\boldsymbol{z} \\ &= \int E\left[\left|(B_Q F_P - I_\Omega)T(\boldsymbol{z}) + B_Q F_P C(\boldsymbol{z}) + B_Q n(\boldsymbol{z})\right|^2\right] \mathrm{d}\boldsymbol{z} \\ &= \mathcal{J}_T(Q) + \mathcal{J}_C(Q) + \mathcal{J}_n(Q), \end{aligned} \tag{47}$$

where I_Ω is defined as in (42) and

$$\mathcal{J}_T(Q) = \int E\left[|(B_Q F_P - I_\Omega)T(\boldsymbol{z})|^2\right] \mathrm{d}\boldsymbol{z} \tag{48}$$

$$\mathcal{J}_C(Q) = \int E\left[|B_Q F_P C(\boldsymbol{z})|^2\right] \mathrm{d}\boldsymbol{z} \tag{49}$$

$$\mathcal{J}_n(Q) = \int E\left[|B_Q n(\boldsymbol{z})|^2\right] \mathrm{d}\boldsymbol{z}. \tag{50}$$

The cross terms in (47) disappear because we are assuming that the target, clutter and noise are all statistically independent, and clutter and noise have zero mean.

We now make the assumption that target and clutter are stationary random fields, *i.e.,*

$$\tilde{S}_T(\boldsymbol{\xi}, \boldsymbol{\xi}') = S_T(\boldsymbol{\xi})\delta(\boldsymbol{\xi} - \boldsymbol{\xi}') \tag{51}$$

$$\tilde{S}_C(\boldsymbol{\xi}, \boldsymbol{\xi}') = S_C(\boldsymbol{\xi})\delta(\boldsymbol{\xi} - \boldsymbol{\xi}'). \tag{52}$$

In this case the leading order terms of $\mathcal{J}_T(Q)$, $\mathcal{J}_C(Q)$ and $\mathcal{J}_n(Q)$ are given by

$$\mathcal{J}_T(Q) \sim \int |Q(\boldsymbol{z},\boldsymbol{\xi})A_P(\boldsymbol{z},\boldsymbol{\xi})\eta(\boldsymbol{z},\boldsymbol{z},\boldsymbol{\xi}) - I_\Omega(\boldsymbol{z},\boldsymbol{\xi})|^2 S_T(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\boldsymbol{z}, \tag{53}$$

$$\mathcal{J}_C(Q) \sim \int |Q(\boldsymbol{z},\boldsymbol{\xi})A_P(\boldsymbol{z},\boldsymbol{\xi})\eta(\boldsymbol{z},\boldsymbol{z},\boldsymbol{\xi})|^2 S_C(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\boldsymbol{z}, \tag{54}$$

$$\mathcal{J}_n(Q) \sim \int |Q(\boldsymbol{z},\boldsymbol{\xi})|^2 \eta(\boldsymbol{z},\boldsymbol{z},\boldsymbol{\xi})S_n(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\boldsymbol{z}. \tag{55}$$

See Appendix B for details.

In Appendix B we also compute the variational derivative of (47) with respect to Q as

$$\mathcal{J}(Q + \epsilon Q_\epsilon) = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathcal{J}_T(Q + \epsilon Q_\epsilon) + \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathcal{J}_C(Q + \epsilon Q_\epsilon) + \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathcal{J}_n(Q + \epsilon Q_\epsilon) \tag{56}$$

$$\sim 2\mathrm{Re} \iint \overline{Q_\epsilon}\left[\overline{A_P}\eta\left(QA_P\eta[S_T + S_C] - S_T I_\Omega\right) + Q\eta S_n\right] \mathrm{d}\boldsymbol{\xi}\mathrm{d}\boldsymbol{z}. \tag{57}$$

In order for (56) to be zero at Q_P for any Q_ϵ,

$$\overline{A_P}\left(Q_P A_P \eta[S_T + S_C] - S_T I_\Omega\right) + S_n Q_P = 0. \tag{58}$$

Thus we obtain [26]

$$Q_P(\boldsymbol{z},\boldsymbol{\xi}) = \frac{I_\Omega(\boldsymbol{z},\boldsymbol{\xi})S_T(\boldsymbol{\xi})\overline{A_P(\boldsymbol{z},\boldsymbol{\xi})}}{|A_P(\boldsymbol{z},\boldsymbol{\xi})|^2\eta(\boldsymbol{z},\boldsymbol{z},\boldsymbol{\xi})[S_T(\boldsymbol{\xi}) + S_C(\boldsymbol{\xi})] + S_n(\boldsymbol{\xi})}. \tag{59}$$

Substituting (59) back into (53)-(55) and using (47) we get a high-frequency asymptotic expression for the MMSE in the reconstructed image

$$\mathcal{J}(Q_P) \sim \int \frac{[|P(\boldsymbol{\xi})|^2|A(\boldsymbol{z},\boldsymbol{\xi})|^2\eta(\boldsymbol{z},\boldsymbol{z},\boldsymbol{\xi})S_C(\boldsymbol{\xi}) + S_n(\boldsymbol{\xi})]I_\Omega(\boldsymbol{z},\boldsymbol{\xi})S_T(\boldsymbol{\xi})}{|P(\boldsymbol{\xi})|^2|A(\boldsymbol{z},\boldsymbol{\xi})|^2\eta(\boldsymbol{z},\boldsymbol{z},\boldsymbol{\xi})[S_T(\boldsymbol{\xi}) + S_C(\boldsymbol{\xi})] + S_n(\boldsymbol{\xi})}\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\boldsymbol{z}. \tag{60}$$

Here $A_P = P(\boldsymbol{\xi})A(\boldsymbol{z},\boldsymbol{\xi})$ has been written out explicitly.

In the ideal situation, where there is no clutter and no measurement noise, employing Q_P from (59) yields an accurate reconstruction in the sense that the asymptotic expression for MMSE in (60) is zero. This is the filter presented in [17], which provides an unbiased, and hence high-resolution reconstruction of the target. However, in the presence of clutter and noise, the reconstructed image will be contaminated by clutter and filtered backprojected noise. In the case when target has zero mean, (59) gives the filter presented in [27] that minimizes the MSE. The MMSE criterion results in a reconstruction which preserves the visible edges of the target are preserved while optimally suppressing the edges due to clutter and noise. However, it is a biased reconstruction, and achieves this suppression by introducing loss of contrast [27].

4. Waveform Design

In Section 3 we presented a filtered-backprojection-type reconstruction method that minimizes the MSE. By (60), the resulting error depends on the choice of the transmitted waveform. We will now proceed to further minimize the reconstruction error by designing an optimal transmit waveform. To this end we will minimize the MMSE subject to the constraint that the total transmitted energy along the flightpath is a constant M, *i.e.,*

$$\int |P(\omega,s)|^2 \mathrm{d}\omega\,\mathrm{d}s = M. \tag{61}$$

First, we perform an inverse Stolt change of variables $\xi \to (s,\omega)$ and re-write (60)

$$\mathcal{J}(Q_P) \sim \int \frac{[\Lambda(\boldsymbol{z},s,\omega)S_C(\boldsymbol{z},\omega,s)+S_n(\omega,s)]I_\Omega(\boldsymbol{z},\omega,s)S_T(\boldsymbol{z},\omega,s)}{\Lambda(\boldsymbol{z},s,\omega)[S_T(\boldsymbol{z},\omega,s)+S_C(\boldsymbol{z},\omega,s)]+S_n(\omega,s)}\,\frac{\mathrm{d}\omega\,\mathrm{d}s}{\eta(\boldsymbol{z},\omega,s)}\,\mathrm{d}\boldsymbol{z}, \tag{62}$$

where

$$\Lambda(\boldsymbol{z},s,\omega) = |P(\omega,s)|^2|A(\boldsymbol{z},s,\omega)|^2\eta(\boldsymbol{z},\omega,s). \tag{63}$$

In order to determine the waveform which minimize the asymptotic MMSE, we employ the method of Lagrange multipliers. To this end we define a cost functional $J_\lambda(P)$

$$J_\lambda(P) = J(Q_P) + \lambda\left(\int |P(\omega,s)|^2\mathrm{d}\omega\,\mathrm{d}s - M\right), \tag{64}$$

which we then minimize with respect to P and λ. In order to keep the expressions cleaner, we will now mostly drop writing out the arguments of each function.

To simplify the computation of the variational derivative we note that $J_\lambda(P)$ depends only on $|P|^2$. We may therefore equivalently minimize $\hat{J}_\lambda(|P|^2)$ with respect to $|P|^2$

$$\hat{J}_\lambda(|P|^2) = J_\lambda(P). \tag{65}$$

Let $W = |P|^2$. By (65), the square root of the minimizer W of $\hat{J}_\lambda(W)$ is also a minimizer of $J_\lambda(P)$. Taking the variational derivative of $\hat{J}_\lambda(W)$ with respect to W, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\hat{J}_\lambda(W+\epsilon W_\epsilon) = \int W_\epsilon \frac{-|A|^2 S_T^2 S_n}{[W|A|^2\eta[S_T+S_C]+S_n]^2}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\omega\,\mathrm{d}s + \lambda\int W_\epsilon\,\mathrm{d}\omega\,\mathrm{d}s. \tag{66}$$

In order for the right hand side of (66) to be zero for all W_ϵ, we must have

$$\int \frac{|A|^2 S_T^2/S_n}{[W|A|^2\eta[S_T+S_C]/S_n+1]^2}\,\mathrm{d}\boldsymbol{x} = \lambda. \tag{67}$$

Furthermore, since $[W|A|^2\eta[S_T+S_C]/S_n+1]^2 \geq 1$, we conclude from (67) that

$$0 \leq \lambda \leq \int |A|^2 S_T^2 \mathrm{d}\boldsymbol{x}/S_n. \tag{68}$$

To gain some insight, we will consider two special cases; namely, a low-noise and noise-dominated situation.

Low-noise case: If there is no additive noise, we see already from (60) that the waveform is not important since it simply drops from the expression for the MMSE

$$\mathcal{J}_{\text{noisefree}}(Q_P) \sim \int \frac{S_C(\boldsymbol{\xi})S_T(\boldsymbol{\xi})}{S_T(\boldsymbol{\xi})+S_C(\boldsymbol{\xi})} I_\Omega(\boldsymbol{z},\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\boldsymbol{z}. \tag{69}$$

Consider therefore instead the limiting case where the noise level goes to zero. We can approximate (67) by

$$S_n \int \frac{|A|^2 S_T^2}{[W|A|^2\eta[S_T+S_C]]^2}\, \mathrm{d}\boldsymbol{x} = \lambda. \tag{70}$$

This can be solved in terms of W to obtain the solution

$$W \approx \sqrt{\frac{S_n}{\lambda} \int \frac{|A|^2 S_T^2}{[|A|^2\eta[S_T+S_C]]^2}\, \mathrm{d}\boldsymbol{x}}. \tag{71}$$

We see that the Lagrange multiplier λ turns out to be a parameter which is adjusted to satisfy the power constraint. Therefore, λ is determined from

$$M = \int W \mathrm{d}\omega \mathrm{d}s = \int \sqrt{\frac{S_n}{\lambda} \int \frac{|A|^2 S_T^2}{[|A|^2\eta[S_T+S_C]]^2}\, \mathrm{d}\boldsymbol{x}} \mathrm{d}\omega \mathrm{d}s. \tag{72}$$

At frequencies with strong clutter scattering, the denominator of (71) is large, which results in a waveform with low power at the clutter frequencies. On the other hand, at frequencies at which the target scatters strongly, the numerator $|A|^2 S_T^2$ is large, so the waveform has high power at the target frequencies.

Noise-dominant case: For the noise-dominant case we will assume that

$$W|A|^2\eta[S_T+S_C]/S_n << 1. \tag{73}$$

This enables us to use a first-order series expansion of the fraction in (67) to obtain

$$\lambda \approx \int \frac{|A|^2 S_T^2}{S_n}\left[1 - 2W|A|^2\eta\frac{[S_T+S_C]}{S_n}\right] \mathrm{d}\boldsymbol{x}. \tag{74}$$

We therefore obtain an expression for $W(\omega,s)$ in the noise-dominant case

$$W \approx \frac{S_n\left(\int |A|^2 S_T^2\, \mathrm{d}\boldsymbol{x} - \lambda S_n\right)}{2\int |A|^4 \eta S_T^2 [S_T+S_C]\, \mathrm{d}\boldsymbol{x}}. \tag{75}$$

Again, λ must be determined using the total power constraint. Obviously, we need for $\lambda < \int |A|^2 S_T^2 \mathrm{d}\boldsymbol{x}/S_n$ to hold in order to avoid negative values for $W(\omega,s)$, which is positive by definition. This constraint on λ, however, coincides with the one already noted in (68).

General case: The general case of (67) is unfortunately hard to solve analytically. However, we can gain some further insight into the solution by defining the following quantities

$$\phi_T(\boldsymbol{x},\omega,s) = \frac{|A|^2 S_T^2}{\int |A|^2 S_T^2 \mathrm{d}\boldsymbol{x}} \tag{76}$$

$$\phi_n(\omega,s) = \frac{S_n}{\int |A|^2 S_T^2 \mathrm{d}\boldsymbol{x}}. \tag{77}$$

We note that ϕ_T is a unit-mass weight function which emphasizes spatial regions where the antenna beam pattern multiplied by the target spectrum is large, *i.e.,* regions from which we expect strong contributions from the target. On the other hand, ϕ_n represents a noise-to-signal ratio. Using definitions (76) and (77), we can simplify (67) as follows

$$\sqrt{\frac{1}{\lambda\phi_n}\int \frac{\phi_T}{[|A|^2\eta[S_T+S_C]+S_n/W]^2}\,\mathrm{d}\boldsymbol{x}} = \frac{W}{S_n}, \tag{78}$$

with the requirement that $0 \le \lambda \le 1/\phi_n$. Thus we see that the important quantity which we determine is not the amplitude squared, W, but the ratio between the amplitude squared and the noise spectrum, W/S_n. We observe that by choosing $\lambda > 0$ close to zero, the value of $W(\omega,s)/S_n(\omega,s)$ can be made arbitrarily large. This is the unconstrained solution which combats additive noise by transmitting a large amount of energy. As λ increases, the magnitude of W/S_n decreases monotonically. Thus, a solution can be found numerically by solving (78) for an initial value of λ to obtain W_λ. The value of λ should then be gradually increased until the power constraint

$$\int W_\lambda(\omega,s)\mathrm{d}\omega\mathrm{d}s = M \tag{79}$$

is satisfied. This will yield a solution for the amplitude squared of the transmit waveform.

Summary: We recall that the temporal Fourier transform of the optimal waveform should therefore have amplitude $|P| = \sqrt{W}$. Our two explicit cases are therefore

$$\frac{|P(\omega,s)|}{\sqrt{S_n(\omega,s)}} \approx \begin{cases} \sqrt{\frac{1-\lambda\phi_n}{2\int \phi_T |A|^2\eta[S_T+S_C]\,\mathrm{d}\boldsymbol{x}}} & \text{, noise-dominated case} \\ \left(\frac{1}{\lambda\phi_n}\int \frac{\phi_T}{(|A|^2\eta[S_T+S_C])^2}\,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{4}} & \text{, low-noise case} \end{cases}. \tag{80}$$

We see for both that the ratio between transmit waveform amplitude and noise spectrum involves dividing by a spatial weighted average of $f = |A|^2\eta(S_T+S_C)$. In the noise-dominated case, we divide by a weighted arithmetic mean $\int \phi_T f$. In the low-noise case, we divide by a weighted harmonic mean $\left[\int(\phi_T/f^2)\right]^{-1}$ of f^2. In both cases, the weight ϕ_T emphasizes regions where the target scatters strongly.

The harmonic mean of a function will be more influenced by its smallest values than the arithmetic mean. Dividing by the harmonic mean for the low-noise case will thus ensure that more emphasis is placed on spectral components with little clutter than in the noise-dominated case. This difference is further enhanced since in the low-noise case we employ an average of $(|A|^2\eta[S_T+S_C])^2$ instead of $|A|^2\eta[S_T+S_C]$.

We also see that the noise-to-signal quantity $\lambda\phi_n$ will determine the magnitude of P as a function of ω when λ is set to satisfy the power constraint. It is reasonable to expect a similar behaviour also for the general case.

An interesting observation is that only the amplitude, and not the phase, is constrained for the Fourier transform of the optimal waveform. Therefore, we do not determine a single waveform, but rather obtain a class of waveforms which satisfy this condition. We have considerable flexibility to impose additional design criteria within this class, such as constant transmit amplitude. However, this is is beyond the scope of our current investigation, and is left for future work.

5. Concluding remarks

In this paper we presented a waveform design method for synthetic-aperture imaging, where we modeled the objects of interest as random fields. We determine the optimal transmit waveform with respect to the MSE between the object of interest and the reconstructed image subject to the total transmit power constraint over a given flight path.

In [26], for fixed waveform, we derived a filtered-backprojection-type reconstruction method the filter of which was optimally designed to achieve the MMSE. The MMSE, however, depends on the transmitted waveform. In the current work, we designed the optimal waveform that further reduces the MMSE by appropriately chosen waveforms.

Two important cases for the waveform design are computed analytically, namely low-noise and noise-dominated measurements. This is summarized in (80). In both these cases λ is a parameter which is adjusted to obtain a specific total transmit power according to (61); S_n is the spectrum for the additive noise; the functions S_T and S_C are spectral characterizations of the target and clutter distributions; and ϕ_n defined in (77) is a noise-to-signal ratio. Furthermore, the quantities A and η are related to the physical model of the antenna (18) and antenna trajectory (41), and ϕ_T is a weight function ϕ_T defined in (76).

The geometric spreading factors included in the quantity A imply that $|A|$ decays rapidly as a function of the distance between the antenna and the scatterers. Many applications will therefore be close to the noise-dominated case.

An interesting feature of our approach is that only the transmit waveform amplitude is constrained by the MMSE reconstruction criterion. In effect what we are doing is determining an optimal frequency band for the transmit waveform. The results appear somewhat surprising at first glance: they do not avoid frequency bands with high levels of additive noise. In fact, in both the noise-dominant and low-noise situation, the important quantity which is determined is $|P|^2/S_n$, thus to a certain extent emphasizing frequencies with a lot of additive noise. A comment about this is therefore warranted.

First of all we should remember that the measurement noise will not be dependent upon the transmit waveform. It will be present in the measurement regardless of what is transmitted. The only way to filter out measurement noise is to filter the measurements. However, the clutter signal will depend on the transmit waveform. It therefore makes sense to design the waveform in such a way as to stay away from the clutter spectrum. Since our reconstruction criterion implicitly takes image resolution into account, it will be beneficial to perform image reconstruction using a broad bandwidth. The optimal pulse accomplishes this by distributing the transmit power over the entire frequency band in such a way that the clutter signal is suppressed whilst also maintaining an appropriate ratio between target signal and additive noise. We therefore transmit more power on frequencies where there is a lot of noise, and filter out the noise in the receive signal instead. The result can be observed by inserting the waveform for the noise-dominant case of (80) into the expression for the asymptotic MMSE in (62) to obtain

$$(81)\qquad \mathcal{J}(Q_P) \sim \int \frac{\left[\frac{(1-\lambda\phi_n)}{2\int \phi_T |A|^2\eta[S_T+S_C]\,\mathrm{d}\boldsymbol{x}}|A|^2\eta S_C + 1\right] I_\Omega S_T}{\frac{(1-\lambda\phi_n)}{2\int \phi_T |A|^2\eta[S_T+S_C]\,\mathrm{d}\boldsymbol{x}}|A|^2\eta[S_T+S_C]+1} \frac{\mathrm{d}\omega\,\mathrm{d}s}{\eta(\boldsymbol{z},\omega,s)}\,\mathrm{d}\boldsymbol{z}.$$

When compared to (62) for a general waveform, we see that the leading-order terms containing S_n have been cancelled out. The remaining effect of the additive noise is purely from the transmit power constraint in terms of the noise-to-signal-ratio ϕ_n.

The additional flexibility in waveform design provided in the phase can be exploited to introduce additional design criteria. For example, it might be desirable to choose the phase in such a way that the transmit amplitude remains almost constant over the transmit pulse, or employ phase-coding schemes. This will not, however, have any effect on the resulting error measured by our design criterion which is the MMSE. It is therefore not addressed in the current paper.

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Appendix A. The Stationary Phase Theorem

The stationary phase theorem [3, 12, 14] states that if a is a smooth function of compact support on $\mathbb{R}^n$, and ϕ has only non-degenerate critical points, then as $\omega \to \infty$,

$$(82)\qquad \int e^{i2\pi\omega\phi(\mathbf{x})}a(\mathbf{x})d\mathbf{x} = \sum_{\{\boldsymbol{x}^0 : D\phi(\boldsymbol{x}^0)=0\}} \left(\frac{1}{\omega}\right)^{n/2} a(\mathbf{x}^0)\frac{e^{i2\pi\omega\phi(\mathbf{x}^0)}e^{i(\pi/4)\mathrm{sgn}D^2\phi(\boldsymbol{x}^0)}}{\sqrt{|\det D^2\phi(\mathbf{x}^0)|}} + O(\omega^{-n/2-1}).$$

Here $D\phi$ denotes the gradient of ϕ, $D^2\phi$ denotes the Hessian, and sgn denotes the signature of a matrix, *i.e.,* the number of positive eigenvalues minus the number of negative ones.

Appendix B. Computing $\mathcal{J}_T(Q)$, $\mathcal{J}_C(Q)$ and $\mathcal{J}_n(Q)$

Writing $B_Q F_P$ as in (40) and I_Ω as in (42),

$$(83)\qquad \mathcal{J}_T(Q) = \int e^{i2\pi(\boldsymbol{x}-\boldsymbol{z})\cdot\boldsymbol{\xi}}[Q(\boldsymbol{z},\boldsymbol{\xi})A_P(\boldsymbol{x},\boldsymbol{\xi})\eta(\boldsymbol{x},\boldsymbol{z},\boldsymbol{\xi}) - I_\Omega(\boldsymbol{z},\boldsymbol{\xi})]\mathcal{R}_T(\boldsymbol{x},\boldsymbol{x}') \times e^{-i2\pi(\boldsymbol{x}'-\boldsymbol{z})\cdot\boldsymbol{\xi}'}\overline{[Q(\boldsymbol{z},\boldsymbol{\xi}')A_P(\boldsymbol{x}',\boldsymbol{\xi}')\eta(\boldsymbol{x}',\boldsymbol{z},\boldsymbol{\xi}') - I_\Omega(\boldsymbol{z},\boldsymbol{\xi}')]}\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{\xi}'\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{z}.$$

We apply the method of stationary phase (see Appendix A) to the $\boldsymbol{\xi}$ and $\boldsymbol{z}$ integrals of (83). The phase is proportional to $(\boldsymbol{x}-\boldsymbol{z})\cdot\boldsymbol{\xi} - (\boldsymbol{x}'-\boldsymbol{z})\cdot\boldsymbol{\xi}'$, so the critical

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conditions are $\boldsymbol{\xi} = \boldsymbol{\xi}'$ and $\boldsymbol{z} = \boldsymbol{x}'$. Thus the contribution of the leading order term of (83) is

$$\begin{aligned}\mathcal{J}_T(Q) \sim \int & e^{i2\pi(\boldsymbol{x}-\boldsymbol{x}')\cdot\boldsymbol{\xi}}[Q(\boldsymbol{x}',\boldsymbol{\xi})A_P(\boldsymbol{x},\boldsymbol{\xi})\eta(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{\xi}) - I_\Omega(\boldsymbol{x}',\boldsymbol{\xi})]\mathcal{R}_T(\boldsymbol{x},\boldsymbol{x}') \\ & \times \overline{[Q(\boldsymbol{x}',\boldsymbol{\xi})A_P(\boldsymbol{x}',\boldsymbol{\xi})\eta(\boldsymbol{x}',\boldsymbol{x}',\boldsymbol{\xi}) - I_\Omega(\boldsymbol{x}',\boldsymbol{\xi})]}\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}'.\end{aligned} \tag{84}$$

Similarly,

$$\begin{aligned}\mathcal{J}_C(Q) \sim \int & e^{i2\pi(\boldsymbol{x}-\boldsymbol{x}')\cdot\boldsymbol{\xi}}Q(\boldsymbol{x}',\boldsymbol{\xi})A_P(\boldsymbol{x},\boldsymbol{\xi})\eta(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{\xi})\mathcal{R}_C(\boldsymbol{x},\boldsymbol{x}') \\ & \times \overline{Q(\boldsymbol{x}',\boldsymbol{\xi})A_P(\boldsymbol{x}',\boldsymbol{\xi})}\eta(\boldsymbol{x}',\boldsymbol{x}',\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}'.\end{aligned} \tag{85}$$

We substitute (30) and (31) in (84) and (85)

$$\begin{aligned}\mathcal{J}_T(Q) \sim \int & e^{i2\pi(\boldsymbol{x}-\boldsymbol{x}')\cdot\boldsymbol{\xi}}e^{-i2\pi\boldsymbol{x}\cdot\boldsymbol{\zeta}}e^{i2\pi\boldsymbol{x}'\cdot\boldsymbol{\zeta}'} \\ & \times [Q(\boldsymbol{x}',\boldsymbol{\xi})A_P(\boldsymbol{x},\boldsymbol{\xi})\eta(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{\xi}) - I_\Omega(\boldsymbol{x}',\boldsymbol{\xi})]\tilde{S}_T(\boldsymbol{\zeta},\boldsymbol{\zeta}') \\ & \times \overline{[Q(\boldsymbol{x}',\boldsymbol{\xi})A_P(\boldsymbol{x}',\boldsymbol{\xi})\eta(\boldsymbol{x}',\boldsymbol{x}',\boldsymbol{\xi}) - I_\Omega(\boldsymbol{x}',\boldsymbol{\xi})]}\,\mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}'\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}',\end{aligned} \tag{86}$$

$$\begin{aligned}\mathcal{J}_C(Q) \sim \int & e^{i2\pi(\boldsymbol{x}-\boldsymbol{x}')\cdot\boldsymbol{\xi}}e^{-i2\pi\boldsymbol{x}\cdot\boldsymbol{\zeta}}e^{i2\pi\boldsymbol{x}'\cdot\boldsymbol{\zeta}'}Q(\boldsymbol{x}',\boldsymbol{\xi})A_P(\boldsymbol{x},\boldsymbol{\xi})\eta(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{\xi})\tilde{S}_C(\boldsymbol{\zeta},\boldsymbol{\zeta}') \\ & \times \overline{Q(\boldsymbol{x}',\boldsymbol{\xi})A_P(\boldsymbol{x}',\boldsymbol{\xi})\eta(\boldsymbol{x}',\boldsymbol{x}',\boldsymbol{\xi})}\,\mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}'\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{x}',\end{aligned} \tag{87}$$

and then apply the method of stationary phase to the $\boldsymbol{x}'$ and $\boldsymbol{\xi}$ integrals. The phase is proportional to $\boldsymbol{x}\cdot(\boldsymbol{\xi}-\boldsymbol{\zeta}) + \boldsymbol{x}'\cdot(\boldsymbol{\zeta}'-\boldsymbol{\xi})$, so the critical conditions are $\boldsymbol{\xi} = \boldsymbol{\zeta}'$ and $\boldsymbol{x}' = \boldsymbol{x}$. The leading order terms of (48) and (49) are then

$$\begin{aligned}\mathcal{J}_T(Q) \sim \int & e^{-i2\pi\boldsymbol{x}\cdot(\boldsymbol{\zeta}-\boldsymbol{\zeta}')}|Q(\boldsymbol{x},\boldsymbol{\zeta}')A_P(\boldsymbol{x},\boldsymbol{\zeta}')\eta(\boldsymbol{x},\boldsymbol{x},\boldsymbol{\zeta}') - I_\Omega(\boldsymbol{x},\boldsymbol{\xi})|^2 \\ & \times \tilde{S}_T(\boldsymbol{\zeta},\boldsymbol{\zeta}')\mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}'\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{88}$$

$$\mathcal{J}_C(Q) \sim \int e^{-i2\pi\boldsymbol{x}\cdot(\boldsymbol{\zeta}-\boldsymbol{\zeta}')}|Q(\boldsymbol{x},\boldsymbol{\zeta}')A_P(\boldsymbol{x},\boldsymbol{\zeta}')\eta(\boldsymbol{x},\boldsymbol{x},\boldsymbol{\zeta}')|^2\tilde{S}_C(\boldsymbol{\zeta},\boldsymbol{\zeta}')\mathrm{d}\boldsymbol{\zeta}\mathrm{d}\boldsymbol{\zeta}'\,\mathrm{d}\boldsymbol{x}. \tag{89}$$

Using (51) and (52) in (88) and (89) the results in (53) and (54) follows.

Since we assumed that noise is stationary in t and statistically uncorrelated in s, we can insert (27) into (50) to obtain

$$\mathcal{J}_n(Q) = \int S_n(\omega,s)|Q(\boldsymbol{z},s,\omega)|^2\,\mathrm{d}\omega\,\mathrm{d}s\,\mathrm{d}\boldsymbol{z} \tag{90}$$

The Stolt change of variables then leads to the result stated in (55)

From (88), (89) and (90), we compute the variational derivatives of $\mathcal{J}_T$, $\mathcal{J}_C$ and $\mathcal{J}_n$ to be

(91)
$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \mathcal{J}_T(Q+\epsilon Q_\epsilon) \sim 2\mathrm{Re}\int e^{-i2\pi\boldsymbol{x}\cdot(\boldsymbol{\zeta}-\boldsymbol{\zeta}')}\overline{Q_\epsilon A_P \eta}\tilde{S}_T(\boldsymbol{\zeta},\boldsymbol{\zeta}')(QA_P\eta - I_\Omega)\,\mathrm{d}\boldsymbol{\zeta}\,\mathrm{d}\boldsymbol{\zeta}'\,\mathrm{d}\boldsymbol{x}$$

(92)
$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \mathcal{J}_C(Q+\epsilon Q_\epsilon) \sim 2\mathrm{Re}\int e^{-i2\pi\boldsymbol{x}\cdot(\boldsymbol{\zeta}-\boldsymbol{\zeta}')}\overline{Q_\epsilon A_P \eta}\tilde{S}_C(\boldsymbol{\zeta},\boldsymbol{\zeta}')QA_P\eta\,\mathrm{d}\boldsymbol{\zeta}\,\mathrm{d}\boldsymbol{\zeta}'\,\mathrm{d}\boldsymbol{x}$$

(93)
$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \mathcal{J}_n(Q+\epsilon Q_\epsilon) \sim 2\mathrm{Re}\int \overline{Q_\epsilon}Q\eta S_n\,\mathrm{d}\boldsymbol{\zeta}\,\mathrm{d}\boldsymbol{x}$$

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