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Robust Full-Order and Reduced-Order Observers for a Class of Uncertain Switched Systems

This paper deals with the problem of robust state estimation for a class of switched linear systems with unknown inputs under average dwell time (ADT) switching, where the switching of the observers is synchronous with that of the estimated system. First, based on the feasibility of an optimization problem with linear matrix inequality (LMI) constraint, a robust sliding-mode switched observer is developed such that the asymptotic state reconstruction is guaranteed even if the switched system is with unknown inputs. Second, a reduced-order switched system which avoids the influence of unknown inputs is constructed by the technique of state transformation, and a reduced-order switched observer is proposed to estimate the continuous states of the original switched system. Next, the conditions under which a full-order switched observer exists also guarantee the existence of a reduced-order switched observer. The convergence of the state estimate is proved to be exponential by appropriate Lyapunov analysis. Finally, the simulation results confirm the predicted performance and applicability by a simplified three-tank system. [DOI: 10.1115/1.4032067]

1 Introduction

In recent years, there has been an increasing interest in the state observation for switched systems due to their significance both in theory and applications [1–5]. A switched system is a dynamical system that consists of both a family of subsystems which are usually described by a collection of indexed differential or difference equations and a switching rule that orchestrates switching between these subsystems in Ref. [6]. Most of the attention has been focused on the problems of stability [6–19] and control [20–27] issues with extensive and satisfactory results for linear or nonlinear switched system. Meanwhile, some efforts have also been put on the state estimation of linear [1,2,28–37] or nonlinear [3–5,38,39] switched systems. For continuous-time and discrete-time linear switching systems, Alessandri and Coletta proposed a switched version of the conventional Luenberger observer, and the problem of finding a common Lyapunov function (CLF) for the switching dynamics of the error system is addressed via LMI technique in Ref. [28], where a common quadratic Lyapunov function is used to guarantee the stability of switched system regardless of the mode switches in the system. By using multiple Lyapunov functions (MLF), where one for each observer mode, Pettersson deals with the estimation issue of the continuous states for a class of switched system, and the observer states are reset in order to guarantee the boundedness of the observation error in Ref. [29]. Based on algebraic tools and distribution theory, Tian et al. discussed a method for the finite time estimation of the switching times of linear switched systems in Ref. [30]. Bejarano et al. addressed the state observation problem for a class of switched linear systems with unknown inputs by using high-order sliding-mode observer in Ref. [31]. The positive observers for switched positive linear systems with time-varying delays are also concerned in Ref. [32]. The problems of state estimation are investigated for switched linear systems with ADT switching in both continuous-time and discrete-time contexts in Ref. [33]. An observer design method for switched linear systems is presented based on the idea of accumulating the information from the individual subsystems in Ref. [34]. The issue of H_∞ filtering is discussed for a class of switched linear systems in discrete-time domain in Ref. [35]. For nonlinear switched systems, Barbot et al. dealt with nonlinear observer synthesis for a particular class of autonomous switching systems with jumps by super twisting algorithm in Ref. [38]. References [35] and [38] deal with the stochastic systems. The state estimation problem is investigated for a class of stochastic linear switching-output systems by Germani et al. in Ref. [36].

In the existing literature, some works consider the issues of the state estimation for linear or nonlinear switched systems without unknown inputs [5,28–30,33–35,38] and stochastic switched systems [36]. The work of Alessandri and Coletta [28] adopts the CLF approach to design state observer, and Bejarano and Pisano tackled the issues of reduced-order observer design for some classes of switched linear systems with unknown inputs by using CLF methods in Ref. [2]. However, the existing results are conservative since the estimation error might converge or be bounded without the existence of a CLF. On the other hand, many state estimation methods have been proposed for switched systems without unknown inputs. For example, Hernandez and García proposed an alternative approach to the state observation problem for switched Lipschitz continuous systems without unknown inputs in Ref. [5]. Similarly, the unknown inputs are also not considered in Refs. [28–30,33–35,38], when they discuss state estimation issues. So, how to deal with the state estimation problem of switched system with unknown inputs is still not trivial, and which is one of the research motivations of this paper. Besides, Germani et al. [36] considered the state estimation for stochastic linear switching-output system which is different from the present paper. In this paper, the state estimation problem for a class of switched linear systems with unknown inputs is tackled by incorporating

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the ADT switching to investigate the state estimation and using MLF and sliding-mode control techniques. The difference between this paper and other existing ones can be summarized as follows: (1) A class of switched system with unknown inputs is considered in this paper, while the systems discussed in Refs. [5,16–18,21–23] are without unknown inputs and (2) the techniques of MLF, sliding-mode control, and LMI-optimization, which are different from Refs. [2,26] where CLF approach is used, are adopted to deal with the state estimation issues of switched system, so the feasibility of the methods proposed in this paper is general since the MLF approach is relatively less conservative than the CLF one. Besides, it is also pointed out that the conditions under which a full-order switched observer exists also guarantee the existence of a reduced-order switched observer, and the global exponential stability conditions of observer error dynamics are derived based on the feasibility of an optimization problem with LMI constraint.

The remainder of the paper is organized as follows: Section 2 presents general model and preliminaries. In Sec. 3, by the use of both MLF and ADT approaches, a robust sliding-mode switched observer and a reduced-order observer are proposed to estimate the states of original switched system. In Sec. 4, an example is given to show the performance of the proposed methods. Some conclusions are summarized in Sec. 5.

The notations used throughout the paper are fair standard. R^n denotes the n-dimensional Euclidean space, and $R^{n\times m}$ is the set of real $n\times m$ matrices. For matrix $X\in R^{n\times n}$, $X>0$ ($X<0$) means that X is real symmetric positive definite (negative definite). The notation $X>Y$, where X and Y are symmetric matrices, means that $X-Y$ is positive definite. $\mathrm{N}_{\mathrm{set}}$ is the set of natural number.

2 General Model and Preliminaries

Let us consider a class of linear switched systems with unknown inputs given by

$$\begin{cases} \dot{x}(t) = A_{\rho(t)}x(t) + B_{\rho(t)}u(t) + D_{\rho(t)}\eta(t) \\ y(t) = C_{\rho(t)}x(t) \end{cases} \tag{1}$$

where $x\in R^n$ is the state vector; $u\in R^m$ is the known input vector; and $y\in R^p$ and $\eta\in R^q$ stand for the system output and unknown input vectors, respectively. $\rho(\cdot): R^+\to\Lambda$ is a piecewise constant function of time, called a switching signal, which takes its values in the finite set $\Lambda=\{1,2,\ldots,M\}$. The so-called discrete state $\rho(t)=k\in\Lambda$ determines which actual subsystem dynamic is activated among the possible M operating modes corresponding to a specific instance of matrices A_k, B_k, C_k, and D_k. Also, for a switching sequence $t_0<t_1<\cdots<t_i<t_{i+1}<\cdots$, we say the kth subsystem is active when $\rho(t)=k$ and $t\in[t_i,t_{i+1})$, where $t_0=0$ is the initial time of switched system, and t_i^+ and t_{i+1}^- denote the starting and ending times of the kth mode, respectively, for the ith switching ($i\in\mathrm{N}_{\mathrm{set}}$), i.e., $\rho(t_i^+)=\rho(t_{i+1}^-)=k$. It is assumed that the dimension of output matrix C_k and the distribution matrix D_k satisfy $p\geq q$ for any $k\in\Lambda$.

Assumption 1. *The following rank conditions*, $\forall k\in\Lambda$:

$$\mathrm{rank}(C_kD_k)=\mathrm{rank}D_k \tag{2}$$

and

$$\mathrm{rank}\begin{bmatrix} sI-A_k & D_k \\ C_k & 0 \end{bmatrix} = n+\mathrm{rank}D_k \tag{3}$$

hold for every complex number s with non-negative real part.

Assumption 2. *The unknown input vector function $\eta(\cdot)$ is bounded in norm, i.e., there exists a positive constant $\hbar$ such that $\|\eta(t)\|\leq\hbar$ holds for all $t\geq 0$.*

Lemma 1. *Assumption 1 means that condition (2) together with condition (3) holds if and only if for symmetric positive definite matrices $Q_k\in R^{n\times n}$, there exist matrices $L_k\in R^{n\times p}$ and $G_k\in R^{q\times p}$, such that* [40]

$$\begin{cases} (A_k-L_kC_k)^{\mathrm{T}}P_k + P_k(A_k-L_kC_k) = -Q_k \\ D_k^{\mathrm{T}}P_k = G_kC_k \end{cases} \tag{4}$$

holds for symmetric positive definite matrices $P_k\in R^{n\times n}$, where $k\in\Lambda=\{1,2,\ldots,M\}$.

A way of computing to find the matrices L_k, G_k, P_k, and Q_k is given by LMI approach in detailed, which is described as the following optimization problem with feasible solution [40]:

$$\begin{cases} \min\delta \\ P_k>I \\ P_kA_k+\Gamma_kC_k+(P_kA_k+\Gamma_kC_k)^{\mathrm{T}}<0 \\ \begin{bmatrix} \delta I & D_k^{\mathrm{T}}P_k-G_kC_k \\ (D_k^{\mathrm{T}}P_k-G_kC_k)^{\mathrm{T}} & \delta I \end{bmatrix}>0 \end{cases} \tag{5}$$

and matrices $Q_k=-\left[(A_k-L_kC_k)^{\mathrm{T}}P_k+P_k(A_k-L_kC_k)\right]$, $L_k=-P_k^{-1}\Gamma_k$.

It is well know that the filters or observers are not always found for switched systems to achieve desired performances under arbitrary switching since the switched system is with the discontinuity feature at the switching instants [33]. For ensuring system performances, it is necessary to design appropriate switching laws, and the ADT switching which leads to a stability condition involving an ADT constraint on the switching sequence $t_i(\forall i=1,2,\ldots)$ is an efficient one. So, let us first recall the concept of ADT by giving the following definition.

Definition 1. *Let $N_{\rho(t)}(\tau_1,\tau_2)$ be the number of discontinuities of the switching signal $\rho(t)$ on the interval (τ_2,τ_1)* [33]. *We say that $\rho(t)$ has an ADT τ_a if there exist two bounded positive numbers N_0 and τ_a, such that*

$$N_{\rho(t)}(\tau_1,\tau_2)\leq N_0+\frac{\tau_1-\tau_2}{\tau_a}$$

holds for the given $\tau_1>\tau_2\geq 0$.

Remark 1. In this paper, the discrete mode (or switching signal) of switched systems is assumed as known. Under the case of unknown switching signal, how to identify the current discrete mode has been discussed in the literature, such as Ref. [30] and our previous work [37]. Besides, the switching of the designed observers is synchronous with that of switched system modes.

3 Observer Design for State Estimation

The purpose of this section is to study that under which specific conditions the full-order and reduced-order switched observers can be designed to determine an estimation $\hat{x}$ of state x. We particularize the class of switched systems considered in Sec. 2, and the methods that will be presented here use sliding-mode technique and reduced-order approach to design full-order and reduced-order observers such that the aim of state estimation is reached. First, let us give the design procedure of full-order switched observer based on the ADT switching.

3.1 Full-Order Observer Design. Consider the following robust sliding-mode switched observer:

$$\begin{cases} \dot{\hat{x}}(t) = A_{\rho(t)}\hat{x}(t) + B_{\rho(t)}u(t) + L_{\rho(t)}(y(t)-\hat{y}(t)) + \alpha_{\rho(t)}(t) \\ \hat{y}(t) = C_{\rho(t)}\hat{x}(t) \end{cases} \tag{6}$$

with sliding-mode control law

$$\alpha_{\rho(t)}(t) = \lambda\frac{D_kG_k(y(t)-\hat{y}(t))}{\|G_k(y(t)-\hat{y}(t))\|} \tag{7}$$

for $\forall\rho(t)=k\in\Lambda$, where $\hat{\lambda}$ is a large enough such that $\hat{\lambda}$ holds.

In order to achieve an asymptotically stable error dynamics, we need to obtain the observer gain matrices $L_1, L_2, \ldots, L_N$ which will be derived from Eq. (4) and define the observation error $\tilde{x}=x-\hat{x}$. When the switching of the observer is synchronous with that of the system modes, the error dynamic system can be described from observer (6) and system (1) as

$$\dot{\tilde{x}}(t)=\left(A_{\rho(t)}-L_{\rho(t)}C_{\rho(t)}\right)\tilde{x}(t)+D_{\rho(t)}\eta(t)-\alpha_{\rho(t)}(t) \tag{8}$$

THEOREM 1. *Consider system (1) fulfilling Assumptions 1 and 2 and observer (6) with sliding-mode control law (7). Then, the asymptotic estimation of the continuous state vector x is obtained provided that for given scalars $\mu_2>\mu_1>0$, there exists a symmetric positive definite matrix $S\in R^{n\times n}$, such that*

$$\mu_1 S<P_k<\mu_2 S \tag{9}$$

holds, and τ_a is sufficiently large according to

$$\tau_a>\frac{\ln(\kappa_1)}{\kappa_2} \tag{10}$$

where $\kappa_1=(\mu_2/\mu_1)$ and $\kappa_2=\inf_{\ell\in\Lambda}\left(\lambda_{\min}(Q_\ell)/\lambda_{\max}(P_\ell)\right)$.

Proof. If symmetric positive definite matrix P_k, for any $k=\rho(t)$, satisfies Eq. (4), then we choose

$$V_k(\tilde{x}(t))=\tilde{x}^{\mathrm{T}}(t)P_k\tilde{x}(t) \tag{11}$$

as a Lyapunov function candidate. So, the time derivative of the Lyapunov function along the trajectories of the switched error system (8) is

$$\begin{aligned}\dot{V}_k(\tilde{x}(t))&=\left[(A_k-L_kC_k)\tilde{x}(t)+D_k\eta(t)-\alpha_k(t)\right]^{\mathrm{T}}P_k\tilde{x}(t)\\&\quad+\tilde{x}^{\mathrm{T}}(t)P_k\left[(A_k-L_kC_k)\tilde{x}(t)+D_k\eta(t)-\alpha_k(t)\right]\\&=\tilde{x}^{\mathrm{T}}(t)\Big[(A_k-L_kC_k)^{\mathrm{T}}P_k+P_k(A_k-L_kC_k)\Big]\tilde{x}(t)\\&\quad+2\tilde{x}^{\mathrm{T}}(t)P_kD_k\eta(t)-2\tilde{x}^{\mathrm{T}}(t)P_k\alpha_k(t)\\&=-\tilde{x}^{\mathrm{T}}(t)Q_k\tilde{x}(t)+2\tilde{x}^{\mathrm{T}}(t)P_kD_k\eta(t)-2\tilde{x}^{\mathrm{T}}(t)P_k\alpha_k(t)\end{aligned}$$

Since

$$\begin{aligned}2\tilde{x}^{\mathrm{T}}(t)P_kD_k\eta(t)&\le 2|\tilde{x}^{\mathrm{T}}(t)P_kD_k\eta(t)|=2|\left(D_k^{\mathrm{T}}P_k\tilde{x}(t)\right)^{\mathrm{T}}\eta(t)|\\&=2|(G_kC_k\tilde{x}(t))^{\mathrm{T}}\eta(t)|\le 2\|\eta(t)\|\|G_kC_k\tilde{x}(t)\|\\&\le 2\hbar\|G_kC_k\tilde{x}(t)\|\end{aligned}$$

and

$$\begin{aligned}-2\tilde{x}^{\mathrm{T}}(t)P_k\alpha_k(t)&=-2\hat{\lambda}\frac{\tilde{x}^{\mathrm{T}}(t)(G_kC_k)^{\mathrm{T}}G_k(y(t)-\hat{y}(t))}{\|G_k(y(t)-\hat{y}(t))\|}\\&=-2\hat{\lambda}\frac{\tilde{x}^{\mathrm{T}}(t)(G_kC_k)^{\mathrm{T}}G_kC_k\tilde{x}(t)}{\|G_kC_k\tilde{x}(t)\|}=-2\hat{\lambda}\|G_kC_k\tilde{x}(t)\|\end{aligned}$$

we can obtain

$$\begin{aligned}\dot{V}_k(\tilde{x}(t))&=-\tilde{x}^{\mathrm{T}}(t)Q_k\tilde{x}(t)+2\tilde{x}^{\mathrm{T}}(t)P_kD_k\eta(t)-2\tilde{x}^{\mathrm{T}}(t)P_k\alpha_k(t)\\&\le-\tilde{x}^{\mathrm{T}}(t)Q_k\tilde{x}(t)+2\hbar\|G_kC_k\tilde{x}(t)\|-2\hat{\lambda}\|G_kC_k\tilde{x}(t)\|\\&\le-\tilde{x}^{\mathrm{T}}(t)Q_k\tilde{x}(t)\end{aligned}$$

Thus, one can further rewrite the above inequality as follows:

$$\begin{aligned}\dot{V}_k(\tilde{x}(t))&\le-\tilde{x}^{\mathrm{T}}(t)Q_k\tilde{x}(t)\le-\lambda_{\min}(Q_k)\tilde{x}^{\mathrm{T}}(t)\tilde{x}(t)\\&\le-\frac{\lambda_{\min}(Q_k)}{\lambda_{\max}(P_k)}V_k(\tilde{x}(t))\\&\le-\inf_{\ell\in\Lambda}\left(\frac{\lambda_{\min}(Q_\ell)}{\lambda_{\max}(P_\ell)}\right)V_k(\tilde{x}(t))=-\kappa_2V_k(\tilde{x}(t))\end{aligned} \tag{12}$$

where $\kappa_2=\inf_{\ell\in\Lambda}\left(\lambda_{\min}(Q_\ell)/\lambda_{\max}(P_\ell)\right)$. By integrating this for any $t\in[t_i,t_{i+1})$ and $\rho(t_i)=k$, one can obtain from Eq. (12) that

$$V_{\rho(t_i)}(\tilde{x}(t))\le e^{-\kappa_2(t-t_i)}V_{\rho(t_i)}(\tilde{x}(t_i)) \tag{13}$$

which is equivalent to

$$V_{\rho(t_i)}(\tilde{x}(t))\le e^{-\kappa_2(t-t_i)}\frac{V_{\rho(t)}(\tilde{x}(t_i))}{V_{\rho(t_i^-)}\left(\tilde{x}(t_i^-)\right)}V_{\rho(t_i^-)}\left(\tilde{x}(t_i^-)\right) \tag{14}$$

where $V_{\rho(t_i^-)}\left(\tilde{x}(t_i^-)\right)=\lim_{t\to t_i^-}V_{\rho(t)}(\tilde{x}(t))$.

We can obtain from Eq. (9) that $\forall(k,k')\in\Lambda\times\Lambda, k\neq k'$

$$P_k<\mu_2S, P_{k'}>\mu_1S \tag{15}$$

Based on Eqs. (11) and (15), one can obtain the following result:

$$\frac{V_{\rho(t)}(\tilde{x}(t_i))}{V_{\rho(t_i^-)}\left(\tilde{x}(t_i^-)\right)}=\frac{\tilde{x}^{\mathrm{T}}(t_i)P_k\tilde{x}(t_i)}{\tilde{x}^{\mathrm{T}}(t_i^-)P_{k'}\tilde{x}(t_i^-)}<\frac{\mu_2\tilde{x}^{\mathrm{T}}(t_i)S\tilde{x}(t_i)}{\mu_1\tilde{x}^{\mathrm{T}}(t_i^-)S\tilde{x}(t_i^-)}=\frac{\mu_2}{\mu_1}$$

So, Eq. (14) is equivalent to

$$V_{\rho(t_i)}(\tilde{x}(t))<\frac{\mu_2}{\mu_1}e^{-\kappa_2(t-t_i)}V_{\rho(t_i^-)}\left(\tilde{x}(t_i^-)\right)$$

which means that

$$V_{\rho(t_i)}(\tilde{x}(t_i))<\kappa_1V_{\rho(t_i^-)}\left(\tilde{x}(t_i^-)\right)=\kappa_1\tilde{x}^{\mathrm{T}}(t_i^-)P_{\rho(t_i^-)}\tilde{x}(t_i^-) \tag{16}$$

Thus, for any $t\in[t_i,t_{i+1})$, we can obtain by iterating Eqs. (13) and (16) from $i=0$ to $i=N_{\rho(t)}(t,0)$ that

$$\begin{aligned}V_{\rho(t)}(\tilde{x}(t))&\le e^{-\kappa_2(t-t_i)}V_{\rho(t_i)}(\tilde{x}(t_i))<\kappa_1e^{-\kappa_2(t-t_i)}\tilde{x}^{\mathrm{T}}(t_i^-)P_{\rho(t_i^-)}\tilde{x}(t_i^-)\\&=\kappa_1e^{-\kappa_2(t-t_i)}V_{\rho(t_i^-)}\left(\tilde{x}(t_i^-)\right)\\&=\kappa_1e^{-\kappa_2(t-t_i)}V_{\rho(t_{i-1})}\left(\tilde{x}(t_i^-)\right)\\&\le\kappa_1e^{-\kappa_2(t-t_i)}e^{-\kappa_2(t_i-t_{i-1})}V_{\rho(t_{i-1})}(\tilde{x}(t_{i-1}))\\&<\kappa_1^2e^{-\kappa_2(t-t_i)}e^{-\kappa_2(t_i-t_{i-1})}\tilde{x}^{\mathrm{T}}(t_{i-1}^-)P_{\rho(t_{i-1}^-)}\tilde{x}(t_{i-1}^-)\\&=\kappa_1^2e^{-\kappa_2(t-t_i)}e^{-\kappa_2(t_i-t_{i-1})}V_{\rho(t_{i-1}^-)}\left(\tilde{x}(t_{i-1}^-)\right)\\&=\kappa_1^2e^{-\kappa_2(t-t_i)}e^{-\kappa_2(t_i-t_{i-1})}V_{\rho(t_{i-2})}\left(\tilde{x}(t_{i-1}^-)\right)\\&\le\kappa_1^2e^{-\kappa_2(t-t_i)}e^{-\kappa_2(t_i-t_{i-1})}e^{-\kappa_2(t_{i-1}-t_{i-2})}V_{\rho(t_{i-2})}(\tilde{x}(t_{i-2}))\\&\;\;\vdots\\&=\kappa_1^{N_{\rho(t)}(t,0)}e^{-\kappa_2(t-t_1)}V_{\rho(t_0)}\left(\tilde{x}(t_1^-)\right)\\&\le\kappa_1^{N_{\rho(t)}(t,0)}e^{-\kappa_2(t-t_1)}e^{-\kappa_2(t_1-t_0)}V_{\rho(t_0)}(\tilde{x}(t_0))\\&=\kappa_1^{N_{\rho(t)}(t,0)}e^{-\kappa_2(t-t_0)}V_{\rho(t_0)}(\tilde{x}(t_0))\end{aligned}$$

Since $N_{\rho(t)}(t,0)\le N_0+\left((t-0)/\tau_a\right)=N_0+(t/\tau_a)$ and $\kappa_1=(\mu_2/\mu_1)>1$, we can derive that

$$
\begin{aligned}
V_{\rho(t)}(\tilde{x}(t)) &< \kappa_1^{N_0+\frac{t}{\tau_a}} e^{-\kappa_2 t} V_{\rho(t_0)}(\tilde{x}(t_0)) = \kappa_1^{N_0} \kappa_1^{\frac{t}{\tau_a}} e^{-\kappa_2 t} V_{\rho(t_0)}(\tilde{x}(t_0)) \\
&= \kappa_1^{N_0} e^{\ln \kappa_1^{\frac{t}{\tau_a}}} e^{-\kappa_2 t} V_{\rho(t_0)}(\tilde{x}(t_0)) \\
&= \kappa_1^{N_0} e^{\frac{t}{\tau_a}\ln \kappa_1} e^{-\kappa_2 t} V_{\rho(t_0)}(\tilde{x}(t_0)) \\
&= \kappa_1^{N_0} e^{-\kappa_2 t + \frac{t}{\tau_a}\ln \kappa_1} V_{\rho(t_0)}(\tilde{x}(t_0))
\end{aligned}
$$

which is equivalent to

$$
V_{\rho(t)}(\tilde{x}(t)) < \kappa_1^{N_0} e^{-\left(\kappa_2 - \frac{1}{\tau_a}\ln \kappa_1\right)t} V_{\rho(t_0)}(\tilde{x}(t_0)) \tag{17}
$$

Therefore, according to Eq. (10), inequality (17) implies that $V_{\rho(t)}(\tilde{x}(t))$ converges exponentially to zero as t tends to infinity, which in turns means that $\tilde{x}(t)$ converges exponentially to zero as t tends to infinity.

Remark 2. Equation (14) is always constructed as the positive definite character of Lyapunov function $V_k(\tilde{x}(t))$, and similar technique is also used in Ref. [41].

Theorem 1 provides an LMI condition to robust sliding-mode switched observer (6) under Assumptions 1 and 2. Since the algebraic equation (4) can be solved by the optimization problem (5) with LMI constraint, we can directly derive the following Corollary 1 from Eqs. (5) and (9) as follows.

Corollary 1. *Under Assumptions 1 and 2, observer (6) with sliding-mode control law (7) can asymptotically estimate the continuous state vector x provided that for given scalars $\mu_2 > \mu_1 > 0$, there exist symmetric positive definite matrices P_k, S and matrices L_k, G_k such that the following optimization problem:*

$$
\begin{cases}
\min \delta \\
P_k > I \\
P_k > \mu_1 S \\
P_k < \mu_2 S \\
P_k A_k + \Gamma_k C_k + (P_k A_k + \Gamma_k C_k)^{\mathrm{T}} < 0 \\
\begin{bmatrix} \delta I & D_k^{\mathrm{T}} P_k - G_k C_k \\ (D_k^{\mathrm{T}} P_k - G_k C_k)^{\mathrm{T}} & \delta I \end{bmatrix} > 0
\end{cases}
$$

has feasible solution, then $Q_k = -[(A_k - L_k C_k)^{\mathrm{T}} P_k + P_k (A_k - L_k C_k)]$ and the observer gain matrices $L_k = -P_k^{-1}\Gamma_k$, where τ_a is sufficiently large such that Eq. (10) satisfies.

Remark 3. Some works of the existing literature have considered the problem of the state estimation for switched systems without unknown inputs [5,16–18,21–23]. Bejarano and Pisano [2] and Zhang et al. [16] adopted the CLF approach. This paper incorporates the ADT switching and MLF approach to investigate the state estimation issue for a class of switched linear systems with unknown inputs. So, it is different from Refs. [2,5,16–18,21–23] on both system structure and techniques since the MLF approach is relatively less conservative than the CLF one, and the switched system considered in this paper is with unknown inputs.

3.2 Reduced-Order Observer Design.

In Sec. 3.1, a full-order sliding-mode switched observer is proposed to estimate the continuous state of uncertain linear switched system. In this section, in order to reconstruct the continuous state of switched system, the underlining idea is to develop a reduced-order switched observer which can directly eliminate the influences of the unknown inputs. We first give the orthogonal procedure.

There exist invertible matrices $\mathrm{N}_k \in R^{p\times p}$ such that $C_k = \mathrm{N}_k \widehat{C}_k$, where $\widehat{C}_k \in R^{p\times n}$, $\widehat{C}_k \widehat{C}_k^{\mathrm{T}} = I_p$. We extend matrices $\widehat{C}_k$ such that it becomes an orthogonal matrices $\mathrm{T}_k = \begin{bmatrix} \widehat{C}_k^{\mathrm{T}} & \mathrm{M}_k^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, where $\mathrm{M}_k \in R^{(n-p)\times n}$. If we take the state transformation $\bar{x}(t) = \mathrm{T}_k x(t)$, the system (1) is, $\forall t \in [t_i, t_{i+1})$, $\rho(t) = \rho(t_i) = k \in \Lambda$ and $\rho(t_{i-1}) = k' \in \Lambda$, then equivalent to

$$
\begin{cases}
\dot{\bar{x}}(t) = \bar{A}_k \bar{x}(t) + \bar{B}_k u(t) + \bar{D}_k \eta(t) \\
\bar{x}(t_i^+) = \mathrm{T}_k \mathrm{T}_{k'}^{-1} \bar{x}(t_i^-) \\
y(t) = \bar{C}_k \bar{x}(t)
\end{cases} \tag{18}
$$

where $\bar{A}_k = \mathrm{T}_k A_k \mathrm{T}_k^{\mathrm{T}}$, $\bar{B}_k = \mathrm{T}_k B_k$, $\bar{D}_k = \mathrm{T}_k D_k$ and $\bar{C}_k = C_k \mathrm{T}_k^{\mathrm{T}} = \mathrm{N}_k \begin{bmatrix} I_p & 0 \end{bmatrix}$.

The state $\bar{x}(t)$ will jump at the switching instants $t = t_i$ as the time-varying state transformation $\bar{x}(t) = \mathrm{T}_k x(t)$, so the state features can be expressed, for the transformed system (18) and the switching time t_i, as follows:

$$
\bar{x}(t_i^+) = \mathrm{T}_k x(t_i^+) = \mathrm{T}_k x(t_i^-) = \mathrm{T}_k \mathrm{T}_{k'}^{-1} \bar{x}(t_i^-)
$$

which is the second equation of Eq. (18).

Lemma 2. *If there exist symmetric positive definite matrices $P_k, Q_k \in R^{n\times n}$, matrices $L_k \in R^{n\times p}$ and $G_k \in R^{q\times p}$ such that Eq. (4) holds, then we can find $\bar{P}_k = \mathrm{T}_k P_k \mathrm{T}_k^{\mathrm{T}}$, $\bar{Q}_k = \mathrm{T}_k Q_k \mathrm{T}_k^{\mathrm{T}}$, $\bar{L}_k = \mathrm{T}_k L_k$ and $\bar{G}_k = G_k$, such that*

$$
\begin{cases}
(\bar{A}_k - \bar{L}_k \bar{C}_k)^{\mathrm{T}} \bar{P}_k + \bar{P}_k (\bar{A}_k - \bar{L}_k \bar{C}_k) = -\bar{Q}_k \\
\bar{D}_k^T \bar{P}_k = \bar{G}_k \bar{C}_k
\end{cases} \tag{19}
$$

hold, where $k \in \Lambda = \{1, 2, \ldots, N\}$.

For $\forall \rho(t) = k \in \Lambda$, we decompose matrices $\bar{A}_k$, $\bar{B}_k$, $\bar{D}_k$, $\bar{P}_k$, and $\bar{Q}_k$ into block matrices as follows:

$$
\bar{A}_k = \begin{bmatrix} \bar{A}_{k,1} & \bar{A}_{k,2} \\ \bar{A}_{k,3} & \bar{A}_{k,4} \end{bmatrix}, \quad \bar{B}_k = \begin{bmatrix} \bar{B}_{k,1} \\ \bar{B}_{k,2} \end{bmatrix}, \quad \bar{D}_k = \begin{bmatrix} \bar{D}_{k,1} \\ \bar{D}_{k,2} \end{bmatrix},
$$

$$
\bar{P}_k = \begin{bmatrix} \bar{P}_{k,1} & \bar{P}_{k,2} \\ \bar{P}_{k,2}^T & \bar{P}_{k,3} \end{bmatrix}, \quad \bar{Q}_k = \begin{bmatrix} \bar{Q}_{k,1} & \bar{Q}_{k,2} \\ \bar{Q}_{k,2}^T & \bar{Q}_{k,3} \end{bmatrix}
$$

where $\bar{A}_{k,1}$, $\bar{P}_{k,1}$, $\bar{Q}_{k,1} \in R^{p\times p}$, $\bar{B}_{k,1} \in R^{p\times m}$, and $\bar{D}_{k,1} \in R^{p\times q}$. As the special structure of $\bar{C}_k$, we find that the block in the intersection of the second row and the second column in the first equation of Eq. (19) is

$$
\bar{A}_{k,2}^{\mathrm{T}} \bar{P}_{k,2} + \bar{A}_{k,4}^{\mathrm{T}} \bar{P}_{k,3} + \bar{P}_{k,2}^{\mathrm{T}} \bar{A}_{k,2} + \bar{P}_{k,3} \bar{A}_{k,4} = -\bar{Q}_{k,3}
$$

Denote $\bar{\mathrm{K}}_k = -\bar{P}_{k,3}^{-1} \bar{P}_{k,2}^{\mathrm{T}}$, the above equation implies that

$$
\left(\bar{A}_{k,4} - \bar{\mathrm{K}}_k \bar{A}_{k,2}\right)^{\mathrm{T}} \bar{P}_{k,3} + \bar{P}_{k,3}\left(\bar{A}_{k,4} - \bar{\mathrm{K}}_k \bar{A}_{k,2}\right) = -\bar{Q}_{k,3} \tag{20}
$$

By the second equation of Eq. (19), there is

$$
\begin{bmatrix} \bar{D}_{k,1} \\ \bar{D}_{k,2} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \bar{P}_{k,1} & \bar{P}_{k,2} \\ \bar{P}_{k,2}^T & \bar{P}_{k,3} \end{bmatrix} = \bar{G}_k \mathrm{N}_{\rho(t)} \begin{bmatrix} I_p & 0 \end{bmatrix}
$$

which means that the following equation:

$$
\begin{bmatrix} \bar{P}_{k,2}^T & \bar{P}_{k,3} \end{bmatrix} \bar{D}_k = 0 \tag{21}
$$

holds. By taking a state transformation of

$$\chi(t)=\begin{bmatrix}\chi_1(t)\\ \chi_2(t)\end{bmatrix}=\Xi_k\begin{bmatrix}\bar{x}_1(t)\\ \bar{x}_2(t)\end{bmatrix} \tag{22}$$

where $\chi_1(t)\in R^p$, $\chi_2(t)\in R^{n-p}$, and $\Xi_k=\begin{bmatrix}I_p & 0\\ -\bar{\mathrm{K}}_k & I_{n-p}\end{bmatrix}$, we can obtain, for $\rho(t_i)=k\in\Lambda$ and $\rho(t_{i-1})=k'\in\Lambda$, that

$$\chi(t_i^+)=\Xi_k\bar{x}(t_i^+)=\Xi_k\bar{x}(t_i^-)=\Xi_k\Xi_{k'}^{-1}\chi(t_i^-) \tag{23}$$

So, a system decomposition for switched system (18), $\forall t\in[t_i,t_{i+1})$, $\rho(t)=\rho(t_i)=k\in\Lambda$ and $\rho(t_{i-1})=k'\in\Lambda$, can be derived from Eqs. (21) to (23) that

$$\begin{cases}\dot{\chi}_1(t)=(\bar{A}_{k,1}+\bar{A}_{k,2}\bar{\mathrm{K}}_k)\chi_1(t)+\bar{A}_{k,2}\chi_2(t)+\bar{B}_{k,1}u(t)+\bar{D}_{k,1}\eta(t)\\ \dot{\chi}_2(t)=(\bar{A}_{k,4}-\bar{\mathrm{K}}_k\bar{A}_{k,2})\chi_2(t)+\begin{bmatrix}-\bar{\mathrm{K}}_k & I_{n-p}\end{bmatrix}\bar{B}_k u(t)+\left(\bar{A}_{k,3}+\bar{A}_{k,4}\bar{\mathrm{K}}_k-\bar{\mathrm{K}}_k(\bar{A}_{k,1}+\bar{A}_{k,2}\bar{\mathrm{K}}_k)\right)\chi_1(t)\\ \chi_2(t_i^+)=(-\bar{\mathrm{K}}_k+\bar{\mathrm{K}}_{k'})\chi_1(t_i^-)+\chi_2(t_i^-)\\ y(t)=\mathrm{N}_k\chi_1(t)\end{cases} \tag{24}$$

We need to reconstruct the states of the switched system (24) such that the states of the original switched system (1) can be obtained by the state transformation $\bar{x}=\mathrm{T}_k x$. Thus, a reduced-order observer is considered to reach the above purpose as follows:

$$\begin{cases}\dot{\hat{\chi}}_2(t)=(\bar{A}_{k,4}-\bar{\mathrm{K}}_k\bar{A}_{k,2})\hat{\chi}_2(t)+\begin{bmatrix}-\bar{\mathrm{K}}_k & I_{n-p}\end{bmatrix}\bar{B}_k u(t)+\left(\bar{A}_{k,3}+\bar{A}_{k,4}\bar{\mathrm{K}}_k-\bar{\mathrm{K}}_k(\bar{A}_{k,1}+\bar{A}_{k,2}\bar{\mathrm{K}}_k)\right)\mathrm{N}_k^{-1}y(t)\\ \hat{\chi}_2(t_i^+)=(-\bar{\mathrm{K}}_k+\bar{\mathrm{K}}_{k'})\mathrm{N}_{k'}^{-1}y(t_i^-)+\hat{\chi}_2(t_i^-)\\ \hat{\bar{x}}(t)=\begin{bmatrix}\mathrm{N}_k^{-1}y(t)\\ \hat{\chi}_2(t)+\bar{\mathrm{K}}_k\mathrm{N}_k^{-1}y(t)\end{bmatrix}\end{cases} \tag{25}$$

THEOREM 2. *Consider system (1) fulfilling Assumption 1, the observer system (25) is a reduced-order observer of switched system (24) provided that for given scalars $\mu_2>\mu_1>0$, there exists a symmetric positive definite matrix $S\in R^{n\times n}$ such that Eq. (9) holds, and τ_a' is sufficiently large according to*

$$\tau_a'>\frac{\ln(\kappa_1)}{\kappa_3} \tag{26}$$

where $\kappa_3=\inf_{\ell\in\Lambda}\left(\lambda_{\min}\left(\bar{Q}_{\ell,3}\right)/\lambda_{\max}\left(\bar{P}_{\ell,3}\right)\right)$.

Proof. If the observer error is set as $\tilde{\chi}_2(t)=\chi_2(t)-\hat{\chi}_2(t)$, we can then get the observer error dynamic system between Eqs. (24) and (25) as follows:

$$\dot{\tilde{\chi}}_2(t)=(\bar{A}_{k,4}-\bar{\mathrm{K}}_k\bar{A}_{k,2})\tilde{\chi}_2(t) \tag{27}$$

Consider the Lyapunov function candidate

$$V_k'=\tilde{\chi}_2^{\mathrm{T}}(t)\bar{P}_{k,3}\tilde{\chi}_2(t) \tag{28}$$

its derivative along the error dynamic system (27), based on Eq. (20), is

$$\begin{aligned}\dot{V}_k'&=\tilde{\chi}_2^{\mathrm{T}}(t)\left[(\bar{A}_{k,4}-\bar{\mathrm{K}}_k\bar{A}_{k,2})^{\mathrm{T}}\bar{P}_{k,3}+\bar{P}_{k,3}(\bar{A}_{k,4}-\bar{\mathrm{K}}_k\bar{A}_{k,2})\right]\tilde{\chi}_2(t)\\ &=-\tilde{\chi}_2^{\mathrm{T}}(t)\bar{Q}_{k,3}\tilde{\chi}_2(t)\le-\lambda_{\min}\left(\bar{Q}_{k,3}\right)\tilde{\chi}_2^{\mathrm{T}}(t)\tilde{\chi}_2(t)\\ &\le-\frac{\lambda_{\min}\left(\bar{Q}_{k,3}\right)}{\lambda_{\max}\left(\bar{P}_{k,3}\right)}V_k'(\tilde{\chi}_2(t))\le-\inf_{\ell\in\Lambda}\left(\frac{\lambda_{\min}\left(\bar{Q}_{\ell,3}\right)}{\lambda_{\max}\left(\bar{P}_{\ell,3}\right)}\right)V_k'(\tilde{\chi}_2(t))\\ &=-\kappa_3V_k'(\tilde{\chi}_2(t))\end{aligned}$$

So, for any $t\in[t_i,t_{i+1})$ and $\rho(t_i)=k$, we can obtain

$$V'_{\rho(t_i)}(\tilde{\chi}_2(t))\le e^{-\kappa_3(t-t_i)}V'_{\rho(t_i)}(\tilde{\chi}_2(t_i)) \tag{29}$$

We can derive from Eq. (9) and $\bar{P}_k=\mathrm{T}_kP_k\mathrm{T}_k^{\mathrm{T}}$ that

$$\mu_1\mathrm{Z}<\bar{P}_k<\mu_2\mathrm{Z} \tag{30}$$

where $\mathrm{Z}=\mathrm{T}_kS\mathrm{T}_k^{\mathrm{T}}$. If we decompose matrix Z into the following block matrix $\mathrm{Z}=\begin{bmatrix}\mathrm{Z}_1 & \mathrm{Z}_2\\ \mathrm{Z}_2^T & \mathrm{Z}_3\end{bmatrix}$, where $\mathrm{Z}_1\in R^{p\times p}$ and $\mathrm{Z}_3\in R^{(n-p)\times(n-p)}$, then Eq. (30) shows that:

$$\mu_1\mathrm{Z}_3<\bar{P}_{k,3}<\mu_2\mathrm{Z}_3 \tag{31}$$

holds. So, we can further derive from Eqs. (29) and (31) that $\forall(k,k')\in\Lambda\times\Lambda, k\ne k'$

$$\begin{aligned}V'_{\rho(t_i)}(\tilde{\chi}_2(t))&\le e^{-\kappa_3(t-t_i)}\frac{V'_{\rho(t_i)}(\tilde{\chi}_2(t_i))}{V'_{\rho(t_i^-)}(\tilde{\chi}_2(t_i^-))}V'_{\rho(t_i)}(\tilde{\chi}_2(t_i^-))\\ &=e^{-\kappa_3(t-t_i)}\frac{\tilde{\chi}_2^{\mathrm{T}}(t_i)\bar{P}_{k,3}\tilde{\chi}_2(t_i)}{\tilde{\chi}_2^{\mathrm{T}}(t_i^-)\bar{P}_{k',3}\tilde{\chi}_2(t_i^-)}V'_{\rho(t_i^-)}(\tilde{\chi}_2(t_i^-))\\ &<e^{-\kappa_3(t-t_i)}\frac{\mu_2\tilde{\chi}_2^{\mathrm{T}}(t_i)\mathrm{Z}_3\tilde{\chi}_2(t_i)}{\mu_1\tilde{\chi}_2^{\mathrm{T}}(t_i^-)\mathrm{Z}_3\tilde{\chi}_2(t_i^-)}V'_{\rho(t_i^-)}(\tilde{\chi}_2(t_i^-))\\ &=\kappa_1e^{-\kappa_3(t-t_i)}V'_{\rho(t_i^-)}(\tilde{\chi}_2(t_i^-))\end{aligned}$$

which means that

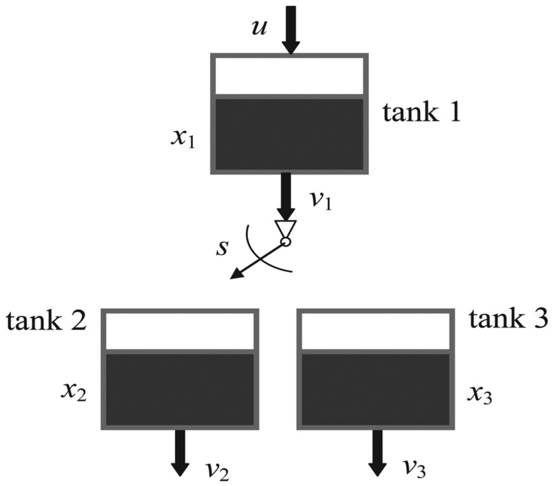

Fig. 1 Three-tank system

$$V'_{\rho(t_i)}(\tilde{\chi}_2(t_i)) < \kappa_1 V'_{\rho(t_i^-)}(\tilde{\chi}_2(t_i^-)) = \kappa_1 \tilde{\chi}_2^{\mathrm{T}}(t_i^-)\bar{P}_{\rho(t_i^-),3}\tilde{\chi}_2(t_i^-) \tag{32}$$

Similar to the proof of Theorem 1, by iterating Eqs. (29) and (32) from $i=0$ to $i=N_{\rho(t)}(t,0)$, we can obtain that

$$V'_{\rho(t)}(\tilde{\chi}_2(t)) < \kappa_3^{N_0} e^{-\left(\kappa_3 - \frac{1}{\tau'_a}\ln\kappa_1\right)t} V'_{\rho(t_0)}(\tilde{\chi}_2(t_0))$$

which means that $V'_{\rho(t)}(\tilde{\chi}_2(t))$ converges exponentially to zero as t tends to infinity. Thus, the estimation error $\tilde{\chi}_2(t)$ in Eq. (27) converges exponentially to zero as t tends to infinity.

After the estimation of $\chi_2(t)$ is derived from reduced-order switched observer (25), if Eqs. (9) and (26) hold, it is easy to compute the estimation of $\bar{x}$ by the state transformation (22), and it is

$$\hat{\bar{x}}(t) = \Xi_k^{-1}\hat{\chi}(t) = \begin{bmatrix} I_p & 0 \\ -\bar{\mathrm{K}}_k & I_{n-p} \end{bmatrix}^{-1}\hat{\chi}(t) = \begin{bmatrix} I_p & 0 \\ \bar{\mathrm{K}}_k & I_{n-p} \end{bmatrix}\begin{bmatrix} \hat{\chi}_1(t) \\ \hat{\chi}_2(t) \end{bmatrix}$$
$$= \begin{bmatrix} \mathrm{N}_k^{-1}y(t) \\ \hat{\chi}_2(t) + \bar{\mathrm{K}}_k\mathrm{N}_k^{-1}y(t) \end{bmatrix}$$

Based on the transformation $\bar{x}(t) = \mathrm{T}_k x(t)$, the state estimation of original switched system (1) can be obtained as follows:

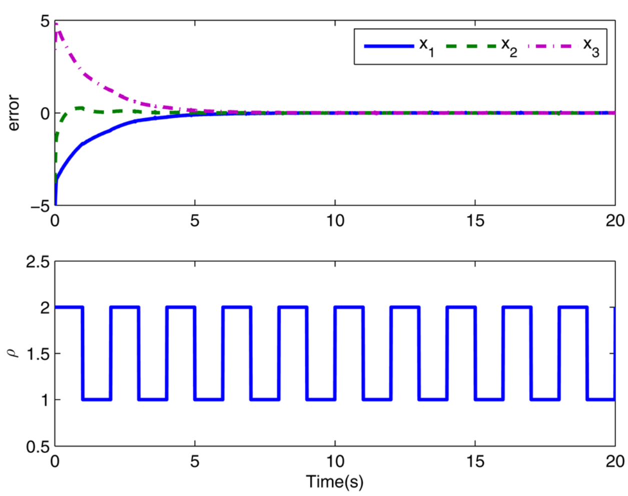

Fig. 2 State estimation error curves and switching signal $\rho(t)$ with τ_a=1

$$\hat{x}(t) = \mathrm{T}_k^{-1}\hat{\bar{x}}(t) = \mathrm{T}_k^{-1}\begin{bmatrix} \mathrm{N}_k^{-1}y(t) \\ \hat{\chi}_2(t) + \bar{\mathrm{K}}_k\mathrm{N}_k^{-1}y(t) \end{bmatrix} \tag{33}$$

It should be pointed out that the design procedure of reduced-order observer shows that the conditions under which a full-order observer exists also guarantee the existence of a reduced-order observer. The design method of the reduced-order observer that is dependent on Assumption 1 and matrix inequality (9) is then presented.

4 Simulation

4.1 System Description. In this section, we present an example to highlight the performances of the proposed robust sliding-mode observer and reduced-order observer developed in this paper. We consider the simplified three-tank system which is used to control the liquid level in the production process of chemical industry and is depicted in Fig. 1.

In Fig. 1, the injection rate of tank 1 is u, the liquid level heights of the three tanks are x_1, x_2 and x_3, respectively. v_1, v_2, and v_3 stand for the effluent rates of the tanks. The relationship between liquid inflow or outflow rate and liquid level height is linear. When the selector s switches to the tank 2, the variation of liquid level heights can be described as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -v_1 & 0 & 0 \\ v_1 & -v_2 & 0 \\ 0 & 0 & -v_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\eta$$

and

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -v_1 & 0 & 0 \\ 0 & -v_2 & 0 \\ v_1 & 0 & -v_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\eta$$

for the case that the selector s switches to the tank 3, where $\eta = 3.6\cos(4.8t)$ is the unknown inputs, which stands for the model uncertainty. If the values of effluent rates are set as $v_1 = 2.85$, $v_2 = 3.6$, and $v_3 = 1.36$, respectively, then the three-tank system is governed by the following state equations with the form of system (1):

$$A_1 = \begin{bmatrix} -2.85 & 0 & 0 \\ 2.85 & -3.6 & 0 \\ 0 & 0 & -1.36 \end{bmatrix}, A_2 = \begin{bmatrix} -2.85 & 0 & 0 \\ 0 & -3.6 & 0 \\ 2.85 & 0 & -1.36 \end{bmatrix},$$
$$B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad D_1 = D_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The output matrices are assumed as $C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and the known input u is set as $u = 12\sin(10.5t)$.

4.2 Full-Order Observer. For given parameters $\mu_1 = 1$ and $\mu_2 = 1.8$, by solving the optimization problem in Corollary 1, we can obtain that

Table 1 Different parameters and figures

ℓ_1	ℓ_2	κ_1	κ_2	$(\ln\kappa_1)/\kappa_2$	τ_a	Figures
1	1.8	1.8	0.8511	0.6906	1	2
1	1.6	1.6	0.4163	1.1289	1.5	3
1	1.2	1.2	0.3960	0.4604	0.5	4

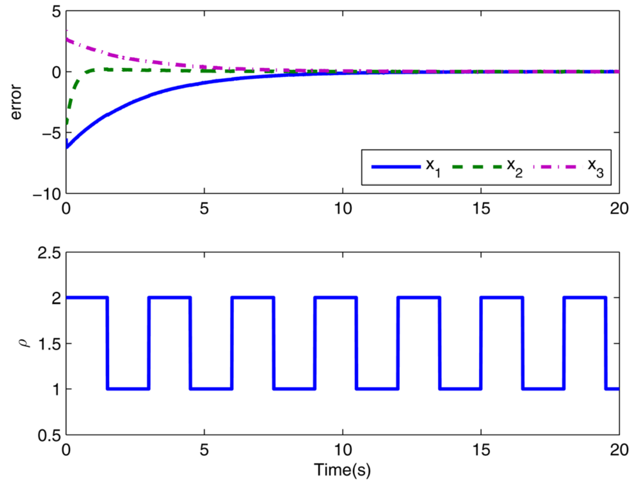

Fig. 3 State estimation error curves and switching signal $\rho(t)$ with τ_a=1.5

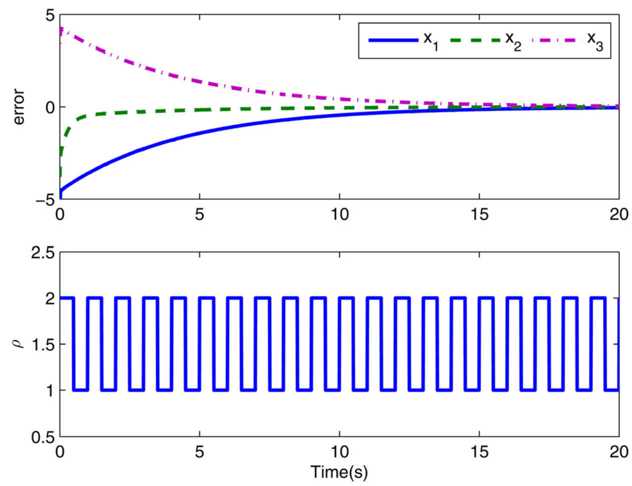

Fig. 4 State estimation error curves and switching signal $\rho(t)$ with τ_a=0.5

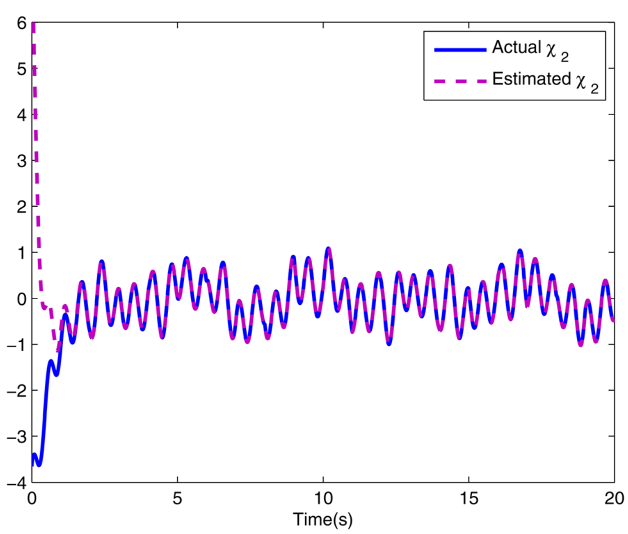

Fig. 5 Estimation for state χ_2

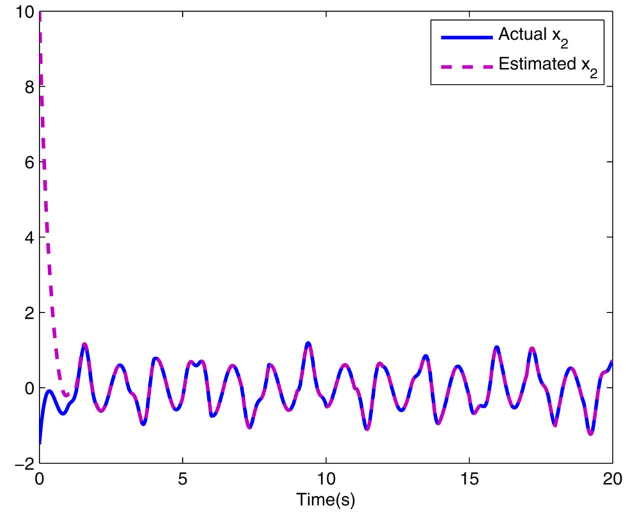

Fig. 6 Estimation for state x_2

$$P_1 = \begin{bmatrix} 6.8014 & -1.5314 & 0.2301 \\ -1.5304 & 2.693 & -1.1615 \\ 0.2301 & -1.1615 & 5.2325 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 6.3301 & -1.7014 & 1.0151 \\ -1.7014 & 2.7495 & -1.048 \\ 1.0151 & -1.048 & 4.2832 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 6.578 & 0.0561 & -0.0217 \\ 0.0561 & 19.3893 & 0.0241 \\ -0.0217 & 0.0241 & 6.5987 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 6.4374 & 0.1649 & 0 \\ 0.1649 & 19.7961 & 0 \\ 0 & 0 & 6.5994 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} -1.689 & 0.173 \\ 6.0083 & 2.1331 \\ 0.9971 & -0.2635 \end{bmatrix}, G_2 = \begin{bmatrix} -1.4405 & 1.38 \\ 2.8197 & 2.6416 \\ 1.9575 & -0.2703 \end{bmatrix}$$

and the gain matrices $L_1 = \begin{bmatrix} 5.5001 \\ 4.3011 \end{bmatrix}^{\mathrm{T}}$ and $L_2 = \begin{bmatrix} 5.6438 \\ 4.2503 \end{bmatrix}^{\mathrm{T}}$. The parameters $\kappa_1 = 1.8$ and $\kappa_2 = 0.8511$ can be easily obtained, so the value of $(\ln\kappa_1)/\kappa_2$ is 0.6906. Let us generate a possible ADT witching sequences with $\tau_a = 1$, as shown in the bottom of Fig. 2. In the simulation, the initial state values and initial estimation values are set as $x(0) = \begin{bmatrix} 2.5 & -1.5 & 1.8 \end{bmatrix}^{\mathrm{T}}$ and $\hat{x}(0) = \begin{bmatrix} 8 & 2.3 & -1.6 \end{bmatrix}^{\mathrm{T}}$, respectively. Then, applying the observers in Eq. (6) with the sliding-mode law (7), the error curves of state response between the estimated systems and the observers are depicted in the top of Fig. 2, from which we can see that the estimation performance is perfect under the designed ADT switching signal.

Remark 4. In order to further investigate the parameter influence on the performance of state estimation, many different parameters are tested, and the values of them and the serial number of simulation figures are shown in Table 1. We can derive from Table 1 that the ADT τ_a can be allowed to choose the different values, as the different parameter ℓ_2 is given. In the case of different parameters, the curves of state estimation error and switching signal are plotted in Figs. 2, 3, and 4, respectively. From Figs. 2 to 4, we can see that different parameters can affect not only the choice of ADT τ_a but also the convergence speed of the state estimation error dynamics. How to optimally choose parameters such that the designed observer is with preferable performance will be our next consideration.

4.3 Reduced-Order Observer. By applying Smith orthogonal procedure to C_1 and C_2, we have

$$\mathrm{T}_1 = \mathrm{T}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathrm{N}_1 = \mathrm{N}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which make the transformed system (18) be satisfied. We can further obtain the matrices $\bar{A}_{k,1}$, $\bar{A}_{k,2}$, $\bar{A}_{k,3}$, $\bar{A}_{k,4}$, $\bar{B}_k$, $\bar{Q}_{k,3}$ for $k=1$, 2, and matrices

$$\bar{P}_{1,2} = \bar{P}_{2,2} = \begin{bmatrix} -1.2272 \\ -1.2293 \end{bmatrix}, \bar{P}_{1,3} = \bar{P}_{2,3} = [2.4569]$$

which lead to the fact that $\bar{\mathrm{K}}_1 = \bar{\mathrm{K}}_2 = \begin{bmatrix} 0.4996 & 0.5004 \end{bmatrix}$. It is easily to compute the parameter κ_3 and its result is $\kappa_3 = \inf_{\ell \in \Lambda} \left(\lambda_{\min}\left(\bar{Q}_{\ell,3}\right) / \lambda_{\max}\left(\bar{P}_{\ell,3}\right)\right) = 7.2$, so $\ln(\kappa_1)/\kappa_3 = 0.0816$ which guarantees the ADT switching sequences with $\tau_a' = 1 > 0.0816$. The dimension of the reduced-order switched observer (25) is 1 since $n - p = 1$. If the initial value of $\hat{\chi}_2$ is set as $\hat{\chi}_2(0) = 8$, the estimation of state $\chi_2(t)$ can be derived from the reduced-order switched observer (25), and the estimation curves are given in Fig. 5. After this, the state estimation of original switched system (1) can be obtained by Eq. (33), and Fig. 6 presents the simulation result for $x_2(t)$. From Figs. 5 and 6, we know that the effectiveness of state trajectory tracking is satisfactory.

5 Conclusions

The issues of state estimation are discussed for switched linear system with unknown inputs in this paper. Under the assumption that the switching signals are synchronous between the observers and estimated system, a full-order switched observer with sliding-mode control laws is developed by MLF approach, and the stability condition of the observer error dynamics involves an ADT constraint on the switching sequence. With the help of state transformation, the reduced-order switched system which is without unknown inputs is constructed and a reduced-order switched observer is proposed such that the purpose of reconstructing the states of the original switched system is realized. The sufficient conditions of the existence of both the full-order observer and reduced-order one are obtained by solving the optimization problem with the LMI constraint. It is also pointed out that the conditions under which a full-order switched observer exists also guarantee the existence of a reduced-order switched observer.

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References

[1] Bako, L., 2011, "Identification of Switched Linear Systems Via Sparse Optimization," Automatica, **47**(4), pp. 668–677.

[2] Bejarano, F. J., and Pisano, A., 2011, "Switched Observers for Switched Linear Systems With Unknown Inputs," IEEE Trans. Autom. Control, **56**(3), pp. 681–686.

[3] Koenig, D., Marx, B., and Jacquet, D., 2008, "Unknown Input Observers for Switched Nonlinear Discrete Time Descriptor Systems," IEEE Trans. Autom. Control, **53**(1), pp. 373–379.

[4] Ríos, H., Davila, J., and Fridman, L., 2012, "High-Order Sliding Mode Observers for Nonlinear Autonomous Switched Systems With Unknown Inputs," J. Franklin Inst., **349**(10), pp. 2975–3002.

[5] Hernandez, S. M., and García, R. A., 2014, "An Observer for Switched Lipschitz Continuous Systems," Int. J. Control, **87**(1), pp. 207–222.

[6] Lin, H., and Antsaklis, P. J., 2009, "Stability and Stabilizability of Switched Linear Systems: A Survey of Recent Results," IEEE Trans. Autom. Control, **54**(2), pp. 308–322.

[7] Wang, X., and Shao, C., 2013, "Finite-Time Input-to-State Stability and Optimization of Switched Nonlinear Systems," ASME J. Dyn. Syst. Meas. Control, **135**(4), p. 041018.

[8] Jasim, N. F., 2013, "Observability and Distinguishability Properties and Stability of Arbitrarily Fast Switched Systems," ASME J. Dyn. Syst. Meas. Control, **135**(5), p. 051011.

[9] Sun, Z., 2006, "Stabilization and Optimization of Switched Linear Systems," Automatica, **42**(5), pp. 783–788.

[10] Zhao, X., Liu, X., Yin, S., and Li, H., 2014, "Improved Results on Stability of Continuous-Time Switched Positive Linear Systems," Automatica, **50**(2), pp. 614–621.

[11] Zhao, X., Li, H., and Niu, B., 2015, "Switching Stabilization for a Class of Slowly Switched Systems," IEEE Trans. Autom. Control, **60**(1), pp. 221–226.

[12] Deaecto, G. S., Daafouz, J., and Geromel, J. C., 2012, "H_2 and H_∞ Performance Optimization of Singularly Perturbed Switched Systems," SIAM J. Control Optim., **50**(3), pp. 1597–1615.

[13] Chatterjee, D., and Liberzon, D., 2011, "Stabilizing Randomly Switched Systems," SIAM J. Control Optim., **49**(5), pp. 2008–2031.

[14] Zhou, L., Ho, D. W. C., and Zhai, G., 2013, "Stability Analysis of Switched Linear Singular Systems," Automatica, **49**(5), pp. 1481–1487.

[15] Chen, Y., Fei, S., and Zhang, K., 2014, "Stabilisation for Switched Linear Systems With Time-Varying Delay and Input Saturation," Int. J. Syst. Sci., **45**(3), pp. 532–546.

[16] Zhang, J., Han, Z., Zhu, F., and Zhao, Z., 2014, "Absolute Exponential Stability and Stabilization of Switched Nonlinear Systems," Syst. Control Lett., **66**, pp. 51–57.

[17] Shen, J., and Hu, J., 2012, "Stability of Discrete-Time Switched Homogeneous Systems on Cones and Conewise Homogeneous Inclusions," SIAM J. Control Optim., **50**(4), pp. 2216–2253.

[18] Aleksandrov, A. Y., Kosov, A. A., and Platonov, A. V., 2012, "On the Asymptotic Stability of Switched Homogeneous Systems," Syst. Control Lett., **61**(1), pp. 127–133.

[19] Xiang, W., and Xiao, J., 2014, "Stabilization of Switched Continuous-Time Systems With all Modes Unstable Via Dwell Time Switching," Automatica, **50**(3), pp. 940–945.

[20] Li, Z., Gao, H., Agarwal, R., and Kaynak, O., 2013, "H_∞ Control of Switched Delayed Systems With Average Dwell Time," Int. J. Control, **86**(12), pp. 2146–2158.

[21] Duan, C., and Wu, F., 2014, "Analysis and Control of Switched Linear Systems Via Dwell-Time Min-Switching," Syst. Control Lett., **70**, pp. 8–16.

[22] Yu, X., Wu, C., Liu, F., and Wu, L., 2013, "Sliding Mode Control of Discrete-Time Switched Systems With Time-Delay," J. Franklin Inst., **350**(1), pp. 19–33.

[23] Sun, Z., and Ge, S. S., 2005, "Analysis and Synthesis of Switched Linear Control Systems," Automatica, **41**(2), pp. 181–195.

[24] Zhao, X., Zheng, X., Niu, B., and Liu, L., 2015, "Adaptive Tracking Control for a Class of Uncertain Switched Nonlinear Systems," Automatica, **41**, pp. 185–191.

[25] Xiang, W., and Xiao, J., 2012, "Stability Analysis and Control Synthesis of Switched Impulsive Systems," Int. J. Robust Nonlinear Control, **22**(13), pp. 1440–1459.

[26] Davoodi, M. R., Golabi, A., Talebi, H. A., and Momeni, H. R., 2012, "Simultaneous Fault Detection and Control Design for Switched Linear Systems: A Linear Matrix Inequality Approach," ASME J. Dyn. Syst. Meas. Control, **134**(6), p. 061010.

[27] Dong, X., and Zhao, J., 2013, "Robust Output Tracking Control for a Class of Uncertain Switched Nonlinear Systems," ASME J. Dyn. Syst. Meas. Control, **135**(3), p. 031001.

[28] Alessandri, A., and Coletta, P., 2001, "Switching Observers for Continuous-Time and Discrete-Time Linear Systems," American Control Conference, Arlington, TX, pp. 2516–2521.

[29] Pettersson, S., 2006, "Designing Switched Observers for Switched Systems Using Multiple Lyapunov Functions and Dwell-Time Switching," 2nd IFAC Conference on Analysis and Design of Hybrid Systems, Alghero, Italy, pp. 18–23.

[30] Tian, Y., Floquet, T., Belkoura, L., and Perruquetti, W., 2011, "Algebraic Switching Time Identification for a Class of Linear Hybrid Systems," Nonlinear Anal. Hybrid Syst., **5**(2), pp. 233–241.

[31] Bejarano, F. J., Pisano, A., and Usai, E., 2011, "Finite-Time Converging Jump Observer for Switched Linear Systems With Unknown Inputs," Nonlinear Anal. Hybrid Syst., **5**(2), pp. 174–188.

[32] Xiang, M., and Xiang, Z., 2013, "Observer Design of Switched Positive Systems With Time-Varying Delays," Circuits Syst. Signal Process., **32**(5), pp. 2171–2184.

[33] Zhao, X., Liu, H., Zhang, J., and Li, H., 2015, "Multiple-Mode Observer Design for a Class of Switched Linear Systems," IEEE Trans. Autom. Sci. Eng., **12**(1), pp. 272–280.

[34] Tanwani, A., Shim, H., and Liberzon, D., 2013, "Observability for Switched Linear Systems: Characterization and Observer Design," IEEE Trans. Autom. Control, **58**(4), pp. 891–904.

[35] Zhang, L., Zhuang, S., and Shi, P., 2015, "Non-Weighted Quasi-Time-Dependent H_∞ Filtering for Switched Linear Systems With Persistent Dwell-Time," Automatica, **54**, pp. 201–209.
[36] Germani, A., Manes, C., and Palumbo, P., 2009, "State Estimation of Stochastic Systems With Switching Measurements: A Polynomial Approach," Int. J. Robust Nonlinear Control, **19**(14), pp. 1632–1655.
[37] Yang, J., Chen, Y., and Wang, X., 2015, "Active Mode Identification and Continuous State Estimation for Switched Linear Systems With Unknown Inputs and Slow Switching Signal," Circuits Syst. Signal Process., **34**(7), pp. 2193–2211.
[38] Barbot, J. P., Saadaoui, H., Djemai, M., and Manamanni, N., 2007, "Nonlinear Observer for Autonomous Switching Systems With Jumps," Nonlinear Anal. Hybrid Syst., **1**(4), pp. 537–547.
[39] Yang, J., Chen, Y., Zhu, F., Yu, K., and Bu, X., 2015, "Synchronous Switching Observer for Nonlinear Switched Systems With Minimum Dwell Time Constraint," J. Franklin Inst., **352**(11), pp. 4665–4681.
[40] Corless, M., and Tu, J., 1998, "State and Input Estimation for a Class of Uncertain Systems," Automatica, **34**(6), pp. 757–764.
[41] Xiang, W., and Xiao, J., 2013, "Finite-Time Stability and Stabilisation for Switched Linear Systems," Int. J. Syst. Sci., **44**(2), pp. 384–400.