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Generalized Nonpolynomial Spline Method by Fractional Order

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Abstract

In this paper, consisted in finding the nonpolynomial spline function and generalized it by fractional order. Other propose of this function was interpolating spline fractional derivatives, with their effective applications to numerically solving fractional boundary value problems. We also discussed the rate of convergence of the method with fractional order. The error bounds and convergence of our difference schemes for nonpolynomial spline with fractional order are theoretically established, this is done by computer program with the aid of the Maptlab 13 for all the above prescribed methods, and the numerical results for boundary value problems are also presented.

Keywords: *Caputo Derivative, Non-polynomial spline, Convergence analysis.*

1 Introduction

Recently, differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering [2, 10, 11]. Most fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques must be used. The variational iteration method [12, 14] and the fractional polynomial method [3, 4] are relatively new approaches to provide an analytical approximation to linear and nonlinear problems, and they are particularly valuable as tools for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization.

As is well known, the difficulty of solving fractional differential equations is essentially because fractional calculus is non-local operators. This non-local property means that the next state a system not only depends on its current state but also on its historical states starting from the initial time. This property is closer to reality and is the main reason why fractional calculus has become more and more useful and popular. In other words, this non-local property is good for modeling reality, but a challenge for numerical computations. During the recent years, much effort has been devoted to the numerical investigations of fractional calculus via fractional spline function of a polynomial form (see [3, 5, 6, 7, 8]).

In this paper, we consider a new fractional spline of non-polynomial form to solve the generalized Bagley-Torvik equation of the form [15, 16, 17, 18]:

$$(D^{2\alpha} + \eta D^{\alpha} + \mu)\, y(x) = f(x),\ \alpha = 1.5,\ \ x \in [a, b] \tag{1}$$

Subject to boundary conditions

$$y(a) = y(b) = 0, \tag{2}$$

where η, μ are all real constants and m=1 or 2. The function f(x) is continuous on the interval [a, b] and the operator D^{α} represents the Caputo fractional derivative. When$\alpha = 1$, then equation (1) is reduced to the classical second order boundary value problem.

Such problems arise in the theory of the fractional calculus and a number of other scientific applications. In general, it is difficult to obtain the analytical solution of Equations (1)-(2) for arbitrary choices of η, μ and f(x). We usually resort to a numerical method for obtaining an approximate solution of the problem equations (1) and (2).

In this paper, we have derived a new fractional spline scheme using a non-polynomial spline for the solution of Equations (1)-(2). Finally, some numerical evidence is included to show the practical applicability and superiority of our methods.

2 Preliminaries

In this section, we recall some basic facts in fractional calculus. There are many ways to define fractional integral and fractional derivative. In this paper we will use Riemann- Liouville fractional integral and Caputo fractional derivative.

Let α be a positive real and *f*(*x*) be a function defined on the right side of *a*, then

Definition 1: *[2, 13] The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by*

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi) d\xi, \qquad n-1 < \alpha < n \in \mathbb{N},$$

where Γ is the gamma function.

Definition 2*[2, 15] The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} \frac{d^n}{d\xi^n} f(\xi) d\xi, \qquad n-1 < \alpha < n \in \mathbb{N}.$$

Definition 3: *[1] The Caputo fractional derivative of order $\alpha > 0$ is defined by*

$$_C D_a^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} \frac{d^n}{d\xi^n} f(\xi) d\xi, \qquad n-1 < \alpha < n \in \mathbb{N}.$$

Definition 4: *[1, 2, 15] The Grünwald definition for fractional derivative is:*

$$^G D_a^\alpha f(x) = lim_{n\to\infty} \sum_k^n g_{\alpha,k} y(x-kh), \tag{3}$$

where the Grünwald weights are: $g_{\alpha,k} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}$. (4)

3 Consistency Relations

In this section, we obtain an approximate solution of the fractional differential equation (1)-(2) using non-polynomial fractional spline functions. For this purpose, we introduce a finite set of grid points x_i by dividing the interval $[a, b]$ into n- equal parts.

$$x_i = a + ih, x_0 = a, x_n = b, h = \frac{b-a}{n},\ i = 0(1)n. \tag{5}$$

Let $y(x)$ be the exact solution of the equation (1) and S_i be an approximation to $y_i = y(x_i)$ obtained by the segment $P_i(x)$ passing through the points (x_i, S_i) and

(x_{i+1}, S_{i+1}) then in each subinterval the fractional spline segment $P_i(x)$ has the form:

$$P_i(x) = a_i + b_i(x - x_i)^\alpha + c_i\, sin_\alpha\, k(x - x_i)^\alpha + d_i\, cos_\alpha\, k(x - x_i)^{3/2}, i = 0(1)n. \quad (6)$$

where a_i, b_i, c_i and d_i are constants and k is the frequency of the trigonometric functions which will be used to raise the accuracy of the method. For convenience consider the following relations:

$$P_i(x_i) = y_i,\ P_i(x_{i+1}) = y_{i+1}, D^{2\alpha}P_i(x_i) = M_i,\ D^{2\alpha}P_i(x_{i+1}) = M_{i+1},\ i = 0(1)n. \quad (7)$$

Via a straightforward calculation we obtain the values of a_i, b_i, c_i and d_i as follows:

$$a_i = y_i + \frac{M_i}{k^2}, \quad (8)$$

$$b_i = \frac{y_{i+1} - y_i}{h^\alpha} + \frac{M_{i+1} - M_i}{\theta k}, \quad (9)$$

$$c_i = \frac{M_i \cos_\alpha \theta - M_{i+1}}{k^2 \sin_\alpha \theta}, \quad (10)$$

$$d_i = -\frac{M_i}{k^2}, \quad (11)$$

where $\theta = kh^\alpha$ and for $i = 0(1)n - 1$.

Using the continuity conditions $D^{2\alpha}P_{i-1}(x_i) = D^{2\alpha}P_i(x_i)$ we have the following consistency relations:

$$\frac{1}{h^{2\alpha}}(y_{i+1} - 2y_i + y_{i-1}) = \lambda M_{i+1} + 2\beta M_i + \lambda M_{i-1},\ i = 2(1)n - 1, \quad (12)$$

where

$$\lambda = \frac{1}{\theta^2}(\Gamma(\alpha + 1)\theta\, csc_\alpha\, \theta - 1) and \beta = \frac{1}{\theta^2}(1 - \Gamma(\alpha + 1)\theta\, cot_\alpha\, \theta),$$

where $\theta = kh^\alpha$ and

$$M_i = f_i - \mu S_i - \eta D^\alpha S(x)|_{x=x_i},\ i = 0(1)n, \quad (13)$$

with $f_i = f(x_i)$. Now to determine $D^\alpha S(x)|_{x=x_i},\ i = 0(1)n$, we use the fact that

$$(w + 1)^r = \sum_{k=0}^{\infty} \binom{r}{k} w^k, \qquad \text{for each} \quad |w| \le 1, p > 0,$$

Where $\binom{r}{k} = \frac{(-1)^k \Gamma(k-r)}{\Gamma(-r)\Gamma(k+1)}$.

If we set $w = -1$ then the above summation will be vanished. From which we may approximate of the fractional term$D^{\frac{3}{2}}S(x)\Big|_{x=x_i}$, $i = 0(1)n$, as follows:

$$D^{\alpha}S(x)|_{x=x_i} \approx h^{-\alpha} \sum_k^i g_{\alpha,k} S(x_i - kh), i = 0(1)n, \quad (14)$$

where the Grünwald weights $g_{\alpha,k}$ are given in equation (4).

4 Convergence Analysis

Here we investigate the error analysis of the spline method described in section 3. Let $Y = (y_i)$, $S = (s_i), T = (t_i)$ and $E = (e_i) = Y - S$ be $n - 1$ dimensional column vectors. Then, we can write the system given by (13) as follows:

$$PS = h^{2\alpha} BM, \quad (15)$$

where the matrices P and B are given below

$$P_{i,j} = \begin{cases} -2, & for i = j = 1(1)n - 1; \\ 1, & for \ |i - j| = 1; \\ 0, & otherwise. \end{cases}$$

The tridiagonal matrix B is given by

$$B = \begin{pmatrix} 2\beta & \lambda & & & & \\ \lambda & 2\beta & \lambda & & & \\ & \ddots & \ddots & \ddots & & \\ & & & \lambda & 2\beta & \lambda \\ & & & & \lambda & 2\beta \end{pmatrix}.$$

The vector M can be written as:

$$M = F - \mu S - \eta h^{-\alpha} G \quad (16)$$

Where the vectors F and the matrix G are given below respectively:

$$F = (f_1, f_2, \dots, f_{n-2}, f_{n-1})^t \quad (17)$$

and

$$G=\begin{pmatrix} g_{\alpha,0} & & & & & \\ g_{\alpha,1} & g_{\alpha,1} & & & & \\ \vdots & \vdots & \ddots & & & \\ g_{\alpha,n-3} & g_{\alpha,n-4} & \cdots & g_{\alpha,1} & g_{\alpha,0} & \\ g_{\alpha,n-3} & g_{\alpha,n-4} & \cdots & g_{\alpha,2} & g_{\alpha,1} & g_{\alpha,0} \end{pmatrix}.$$

where $g_{\alpha,k}$ are the Grünwald weights and given in equation (4).

Substituting from equation (17) into equation (15) we get:

$(P+\mu h^{2\alpha}B+\eta h^{\alpha}BG)S=h^{2\alpha}BF$, and

$$(P+\mu h^{2\alpha}B+\eta h^{\alpha}BG)Y=h^{2\alpha}BF+T.$$

Hence

$$T=(P+\mu h^{2\alpha}B+\eta h^{\alpha}BG)E. \tag{18}$$

Our main purpose now is to derive a bound on $\|E\|$. From the equation (18) we can write the error term as

$$E=(I+\mu h^{2\alpha}P^{-1}B+\eta h^{\alpha}P^{-1}BG)^{-1}P^{-1}T,$$

which implies that

$$\|E\|=\|(I+\mu h^{2\alpha}P^{-1}B+\eta h^{\alpha}P^{-1}BG)^{-1}\|\cdot\|P^{-1}\|\cdot\|T\|. \tag{19}$$

In order to derive the bound on $\|E\|$, the following two lemmas are needed.

Lemma 1: *[9] If N is a square matrix of order n and* $\| N \| < 1$ *, then* $(I+N)^{-1}$ *exists and*

$$\|(I+N)^{-1}\|<\frac{1}{1-\|N\|}.$$

Lemma 2: *The matrix* $(P+\mu h^{2\alpha}B+\eta h^{\alpha}BG)$ *given in Eq. (18) is nonsingular if*

$$(\mu+2\eta m h^{-\alpha})w<1, \qquad where\ w=\frac{1}{8}((b-a)^2+h^2).$$

Proof: Let

$$N=\mu h^{2\alpha}P^{-1}B+\eta h^{\alpha}P^{-1}BG. \tag{20}$$

It was shown, in [4], that

$$\|P^{-1}\| \leq \frac{h^{-2}}{8}((b-a)^2+h^2) = wh^{-2}, \tag{21}$$

and from the system B we have for $\lambda+\beta = \frac{1}{\Gamma(2\alpha+1)}$ and $\lambda \neq \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}$ that

$$\|B\| = 1, \tag{22}$$

and from the system G, we have

$$\|G\| = \sum_{i=0}^{n-2} |g_{\alpha,k}|,$$

which, together with the fact that $g_{\alpha,0} = 1$ and $g_{\alpha,1} = -\alpha$, leads to

$$\|G\| \leq 2m,\ \forall\ (m-1) < \alpha < m. \tag{23}$$

Substituting equations (21)–(23) into equation (20) and then using our assumption we obtain

$$\| N \| < 1. \tag{24}$$

Then by lemma 1, $(I+\mu h^{2\alpha}P^{-1}B+\eta h^{\alpha}P^{-1}BG)^{-1}$ exists and

$$(I+\mu h^{2\alpha}P^{-1}B+\eta h^{\alpha}P^{-1}BG)^{-1} \leq \frac{1}{1-\mu h^{2\alpha}\|P^{-1}\|\|B\|-\eta h^{\alpha}\|P^{-1}\|\|B\|\|G\|}. \tag{25}$$

This completes proof of the lemma.■

As a result of the above lemma, the discrete boundary value problem (15) has a unique solution if $(\mu+2\eta m h^{-\alpha})w < 1$. Expanding (12) in fractional Taylor's series about x_i we obtain

$$\|T\| = \xi_1 h^{4\alpha} M_4, \tag{26}$$

where

$$M_4 = \max_{a \leq x \leq b} |D^{4\alpha} y(x)|.$$

Hence using equation (19) we have

$$\|E\| = \frac{\|P^{-1}\|\|T\|}{1-\mu h^{2\alpha}\|P^{-1}\|\|B\|-\eta h^{\alpha}\|P^{-1}\|\|B\|\|G\|} \cong O(h^{4\alpha-1}). \tag{27}$$

In view of lemma 2, we can conclude the following theorem:

Theorem 1: *Let y(x) be the exact solution of the continuous boundary value problem (1)-(2) and let* $y(x_i)$, $i = 1(1)n-1$, *satisfy the discrete boundary value* *problem (15). Moreover, if we set* $e_i = y_i - s_i$*, then* $\|E\| \cong O(h^{4\alpha-1})$ *as given by equation (27), neglecting all errors due to round off.*

5 Numerical Illustrations and Discussion

To illustrate our method and to demonstrate its convergence and applicability of our presented methods computationally, we have solved two fractional boundary value problems for different values of $\alpha,\ \lambda$ and all calculations are implemented with MATLAB 12.

Example 1: *Consider the boundary value problem*

$(D^{2\alpha} + 0.5D^{\alpha} + 1)y(x) = f(x)$, where

$$f(x) = x^{4-2\alpha}\left(\frac{120x}{\Gamma(6-\alpha)} - \frac{24}{\Gamma(5-\alpha)}\right) + x^4(x-1) + 0.5x^{4-2\alpha}\left(\frac{120x}{\Gamma(6-\alpha)} - \frac{24}{\Gamma(5-\alpha)}\right),$$

Subject to $y(0) = y(1) = 0$. The exact solution of this problem is $y(x) = x^4(x-1)$.

The numerical solution for $\eta = 0.5, \mu = 1, n = 8$ and $\alpha = 1.5$ is represented in Table 1, and for $\eta = 0.5, \mu = 1, n = 8, \lambda = \beta = 0.25$ and $\alpha = 1$ the the approximate values are given in Table 2. Also, absolute errors for each case are demonstrated and the exact and numerical solutions are demonstrated in Figure 1. for $h = 0.25$ and $\alpha = 1$.

Table 1: Exact, approximate and absolute error

$\alpha = 1.5$, and $\lambda + \beta = \frac{1}{12}$			
x	**Exact Solution**	**Approximation Solution**	**Error**
0	0	0	0
0.125	-0.0002140	0.00009092	1.22702513E-04
0.250	-0.0029297	0.00005791	2.87176769E-03
0.375	-0.0123596	-0.0002619	1.20976282E-02
0.500	-0.0312500	-0.0009829	3.02670898E-02
0.625	-0.0572200	-0.0020054	5.52150044E-02
0.750	-0.0791000	-0.0028265	7.62749965E-02
0.875	-0.0732730	-0.0023161	7.09565864E-02
0	0	0	0

Table 1: Exact, approximate and absolute error

$\alpha = 1$, and $\lambda = \beta = 0.25$			
x	**Exact Solution**	**Approximation Solution**	**Error**
0	0	0	0
0.125	-0.000213623046875	0.000669849021779	8.834720686540414E-04
0.250	-0.002929687500000	-0.001578011755589	1.351675744410663E-03
0.375	-0.012359619140625	-0.010343254460196	2.016364680429414E-03
0.500	-0.031250000000000	-0.027737358431649	3.512641568350627E-03
0.625	-0.057220458984375	-0.050748995989373	6.471462995002224E-03
0.750	-0.079101562500000	-0.067549138705254	1.155242379474605E-02
0.875	-0.073272705078125	-0.053705159914945	1.956754516318006E-02
0	0	0	0

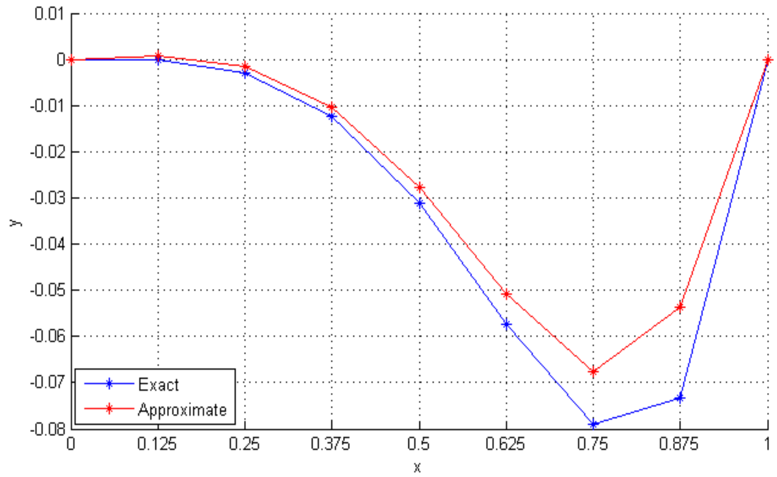

Figure 1: Exact and approximate solutions of Example 1 with h=0.25

Example 2: *Consider the fractional differential equation*

$$D^{2\alpha}y(x) + \eta D^{\alpha}y(x) + \mu y(x) = f(x), \tag{29}$$

Where

$$f(x) = \mu x^3(x-1) + 120x^{5-\alpha}\left(\frac{\eta}{\Gamma(6-\alpha)} - \frac{x^{-\alpha}}{\Gamma(6-\alpha)}\right) + 5040x^{7-\alpha}\left(\frac{\eta}{\Gamma(8-\alpha)} - \frac{x^{-\alpha}}{\Gamma(8-\alpha)}\right),$$

subject to $y(0) = y(1) = 0$. The exact solution of this problem is

$$y(x) = x^7 - x^5.$$

The numerical results obtained, for different values of $\alpha, \beta, \mu, \lambda$ and for $0 \leq x \leq 1$, are shown in Table 3 and 4, together with absolute errors, to illustrate the accuracy of the proposed method. Also, absolute errors for each case are demonstrated and the exact and numerical solutions are given in Figure 2 for $\alpha = 1$.

Table 3: Exact, approximate and absolute error

$\mu = 1, \eta = 0.5, \alpha = 1.5$, and $\lambda = \beta = \frac{1}{12}$			
x	**Exact Solution**	**Approximation Solution**	**Error**
0	0	0	0
0.125	-0.00003004	-0.00956430	9.534259850E-03
0.250	-0.00091552	-0.01995814	1.904261930E-02
0.375	-0.00637292	-0.03206469	2.569176706E-02
0.500	-0.02343750	-0.04607803	2.264053716E-02
0.625	-0.05811452	-0.05944937	1.334851175E-03
0.750	-0.10382080	-0.06402991	3.979088570E-02
0.875	-0.12021303	-0.04230987	7.790315638E-02
0	0	0	0

Table 4: Exact, approximate and absolute error

$\mu = 1, \eta = 0.5, \alpha = 1, \lambda = \frac{1}{14}$, and $\beta = \frac{3}{7}$			
x	**Exact Solution**	**Approximation Solution**	**Absolute Error**
0	0	0	0
0.125	-0.0000300407409667	0.005406581671185	5.436622412151732E-03
-0.250	-0.000915527343750	0.009593364167635	1.050889151138525E-02
0.375	-0.006372928619385	0.008700135962566	1.507306458195120E-02
0.500	-0.023437500000000	-0.004576612592262	1.886088740773831E-02
0.625	-0.058114528656006	-0.036598772088711	2.151575656729532E-02
0.750	-0.103820800781250	-0.080967938337654	2.285286244359583E-02
0.875	-0.120213031768799	-0.096837697958757	2.337533381004159E-02
0	0	0	0

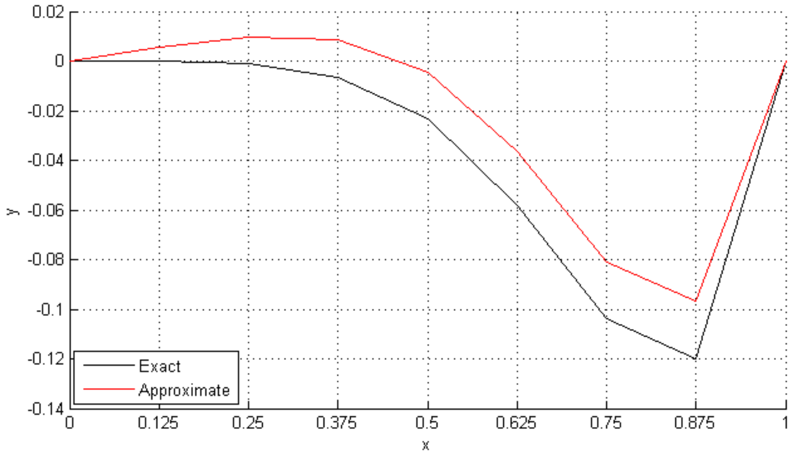

Figure 2: Exact and approximate solutions of Example 2 with h=0.125

6 Conclusion

A non-polynomial fractional spline method has been considered for the numerical solution of fractional boundary value problem form (1)-(2). In general, the method used in paper has been proved effectiveness in solving fractional differential equations numerically and also solved on two problems for different value of $\eta, \mu, \alpha, \lambda$ and β; further more, the results obtained are very encouraging. The method is simple and easy to apply.

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