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Research Article

Flexural Free Vibrations of Multistep Nonuniform Beams

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This paper presents an exact approach to investigate the flexural free vibrations of multistep nonuniform beams. Firstly, one-step beam with moment of inertia and mass per unit length varying as $I(x) = \alpha_1(1 + \beta x)^{r+4}$ and $m(x) = \alpha_2(1 + \beta x)^r$ was studied. By using appropriate transformations, the differential equation for flexural free vibration of one-step beam with variable cross section is reduced to a four-order differential equation with constant coefficients. According to different types of roots for the characteristic equation of four-order differential equation with constant coefficients, two kinds of modal shape functions are obtained, and the general solutions for flexural free vibration of one-step beam with variable cross section are presented. An exact approach to solve the natural frequencies and modal shapes of multistep beam with variable cross section is presented by using transfer matrix method, the exact general solutions of one-step beam, and iterative method. Numerical examples reveal that the calculated frequencies and modal shapes are in good agreement with the finite element method (FEM), which demonstrates the solutions of present method are exact ones.

1. Introduction

Beams with nonuniform cross section are widely used in various engineering fields, such as bridges, tall buildings, and helicopter rotor blades. A large number of studies can be found in literature about the free vibrations of nonuniform beams. Vibration problems of beams with nonuniform cross section are often described by partial differential equations and in most cases it is extremely difficult to find their closed form solutions. Consequently, a wide range of approximate and numerical solutions such as Rayleigh-Ritz, Galerkin, finite difference, finite element, and spectral finite element methods have been used to obtain the natural vibration characteristics of variable-section beams [1–4].

Huang and Li investigated the free vibration of axially functionally graded beams with variable flexural rigidity and mass density [5]. A novel and simple approach was presented to solve modal shapes and corresponding frequencies through transforming traditional fourth-order governing differential equation into Fredholm integral equation. Koplow et al. proposed an analytical solution for the dynamic response of Euler-Bernoulli beams with step changes in cross section, which was verified by experimental tests and receptance coupling methods [6]. Firouz-Abadi et al. studied the transverse free vibrations of a class of variable-cross section beams using Wentzel-Kramers-Brillouin (WKB) approximation [7]. The governing equation of motion for the Euler-Bernoulli beam including axial force distribution was utilized to obtain a singular differential equation in terms of the natural frequency of vibration and a WKB expansion series was applied to find the solution. Inaudi and Matusevich investigated longitudinal vibration problems of variable-cross section rods using an improved power series method [8]. This method introduced domain partition implementation in matrix formulation, as an alternative to other power series techniques in vibration analysis. Therefore, the method solved linear differential equations efficiently up to a desired degree of accuracy and remedies two limitations of the conventional power series method. The Adomian decomposition method (ADM) is employed to investigate the free vibrations of tapered Euler-Bernoulli beams with a continuously exponential variation of width and a constant thickness along the length under various boundary conditions [9]. Duan and Wang demonstrated the free vibration of beams with multiple step changes using the modified discrete singular convolution (DSC) [10]. The jump conditions at the steps were used to overcome the difficulty in using ordinary DSC for dealing with ill-posed problems. A transfer matrix method and the Frobenius method were adopted by J. W. Lee and J. Y. Lee to solve the free vibration characteristics of a tapered Bernoulli-Euler beam and obtain the power series solution for bending vibrations [11].

Nevertheless, besides all advantages of such numerical methods, exact solutions can provide adequate insight into the physics of the problems and convenience for parametric studies. The other advantage of exact solutions is their significance in the field of inverse problems. An exact solution can be more useful than numerical solutions to design the characteristics and damage identification of a structure.

Wang derived the closed form solutions for free vibration of a flexural bar with variably distributed stiffness but uniform mass [12]. Using a systematic approach, Abrate obtained a closed form solution of longitudinal vibration for rods whose cross section varies as $A(x) = A_0(1 + a[x/L])^2$ [13]. Kumar and Sujith found the exact solutions for longitudinal vibration of nonuniform rods whose cross section varies as $A = (a + bx)^n$ and $A = A_0 \sin^2(a + bx)$ [14]. Li presented an exact approach for free longitudinal vibrations of one-step nonuniform rod with classical and nonclassical boundary conditions [15]. The approach assumed that the distribution of mass is arbitrary, and distribution of longitudinal stiffness is expressed as a functional relation with mass distribution and vice versa. Li et al. obtained exact solutions of flexural vibration for beam-like structures whose moment of inertia and mass per unit length vary as $EI(x) = a(1+\beta x)^{n+4}$, $m(x) = \alpha(1 + \beta x)^n$ and $EI(x) = a \cdot \exp(-bx)$, $m(x) = \alpha \cdot \exp(-bx)$, respectively [16, 17].

The components whose moment of inertia $I(x)$ and mass per unit length $m(x)$ satisfy $I(x) = \alpha_1(1 + \beta x)^{r+4}$ and $m(x) = \alpha_2(1 + \beta x)^r$ are widely used in civil engineering, such as the bridge with cross section height varying as $h(x) = h_0(1+\beta x)^2$. This kind of beam-like structure is usually solved by dividing it into several segments whose $I(x)$ and $m(x)$ satisfy different distributions. The paper derived the exact general solution of one-step nonuniform beam firstly and then obtained the exact solution of multistep nonuniform beam by combining the transfer matrix method and exact solutions of one-step beam. The exact solutions of present method not only can provide convenience for parametric studies, but also are very useful for inverse problems such as damage identification.

2. Modal Shape Function of One-Step Beam

2.1. Beams with Uniform Cross Section. The governing differential equation for undamped free flexural vibration of beam with uniform cross section can be written as [18]

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + m\frac{\partial^2 y(x,t)}{\partial t^2} = 0, \quad (1)$$

where E is Young's modulus, I is moment of inertia, m is mass per unit length, and $y(x,t)$ is transverse displacement at position x and time t.

Assuming the beam performs a harmonic free vibration at equilibrium position, that is,

$$y(x,t) = \phi^u(x)e^{j\omega t}, \quad (2)$$

where $\phi^u(x)$ is mode shape function of uniform beam, ω is corresponding natural frequency.

Inserting (2) into (1) obtains

$$EI\frac{d^4\phi^u(x)}{dx^4} - \omega^2 m\phi^u(x) = 0. \quad (3)$$

Equation (3) leads to the modal shape function of beam with uniform cross section:

$$\phi^u(x) = \sum_{i=1}^{4} B_i S_i^u(x), \quad (4)$$

where $S_1^u(x) = \sinh kx$, $S_2^u(x) = \cosh kx$, $S_3^u(x) = \sin kx$, $S_4^u(x) = \cos kx$, are B_i $(i = 1, 2, 3, 4)$ are integration constants, $k^4 = \omega^2 m/EI$.

2.2. Beams with Variable Cross Section. The governing differential equation for undamped free flexural vibration of beam with variable cross section can be written as [19]

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right] + m(x)\frac{\partial^2 y(x,t)}{\partial t^2} = 0, \quad (5)$$

where E is Young's modulus, $I(x)$ is bending moment of inertia at position x, $m(x)$ is mass per unit length at position x, and $y(x,t)$ is transverse displacement at position x and time t.

Assuming the beam performs a harmonic free vibration at equilibrium position, that is,

$$y(x,t) = \phi(x)e^{j\omega t}, \quad (6)$$

where $\phi(x)$ is mode shape function of beam, ω is natural frequency.

Substituting (6) into (5) arrives at

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\phi(x)}{dx^2}\right] - \omega^2 m(x)\phi(x) = 0. \quad (7)$$

The moment of inertia $I(x)$ and mass per unit length $m(x)$ of beam are assumed to vary as

$$I(x) = \alpha_1(1+\beta x)^{r+4},$$
$$m(x) = \alpha_2(1+\beta x)^r, \quad (8)$$
$$(\alpha_1, \alpha_2, \beta \neq 0),$$

where α_1, α_2, and β are arbitrary constants but not equal to 0 and r is a positive integer.

Substituting (8) into (7) yields

$$0 = E\alpha_1 (1+\beta x)^4 \frac{d\phi^4(x)}{dx^4} + 2E\alpha_1\beta(r+4)(1+\beta x)^3 \frac{d\phi^3(x)}{dx^3} + E\alpha_1\beta^2(r+4)(r+3)(1+\beta x)^2 \frac{d\phi^2(x)}{dx^2} - \omega^2\alpha_2\phi(x). \quad (9)$$

Let

$$z = \ln(1+\beta x), \quad D = \frac{d}{dz}. \quad (10)$$

It can be derived that

$$\frac{d\phi^k(x)}{dx^k}(1+\beta x)^k = \beta^k D(D-1)(D-2)\cdots(D-k+1)\phi(z), \quad (k=1,2,\ldots,n). \quad (11)$$

Introducing (11) into (9), one arrives at

$$0 = E\alpha_1\beta^4 D(D-1)(D-2)(D-3)\phi(z) + 2E\alpha_1\beta^4(r+4)D(D-1)(D-2)\phi(z) + E\alpha_1\beta^4(r+4)(r+3)D(D-1)\phi(z) - \omega^2\alpha_2\phi(z). \quad (12)$$

Equation (12) can be simplified as

$$0 = \Big[D^4 + (2(r+4)-6)D^3 + ((r+4)(r+3)-6(r+4)+11)D^2 + (4(r+4)-(r+4)(r+3)-6)D - \frac{\omega^2\alpha_2}{E\alpha_1\beta^4}\Big]\cdot\phi(z). \quad (13)$$

Let

$$a_1 = 2(r+4)-6, \quad a_2 = (r+4)(r+3)-6(r+4)+11, \quad a_3 = 4(r+4)-(r+4)(r+3)-6, \quad a_4 = -\frac{\omega^2\alpha_2}{E\alpha_1\beta^4}. \quad (14)$$

Equation (13) can be written as

$$\left(D^4 + a_1D^3 + a_2D^2 + a_3D + a_4\right)\phi(z) = 0. \quad (15)$$

The characteristic equation of (15) is

$$d^4 + a_1d^3 + a_2d^2 + a_3d + a_4 = 0. \quad (16)$$

The four roots of (16) are

$$d_{1,2} = \frac{g - a_1/2 \pm \sqrt{(a_1/2-g)^2 - 4(t/2-f)}}{2}, \quad d_{3,4} = \frac{-g - a_1/2 \pm \sqrt{(a_1/2+g)^2 - 4(t/2+f)}}{2}, \quad (17)$$

where

$$t = 3\sqrt{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + 3\sqrt{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \frac{a_2}{3}, \quad t \text{ is a real root},$$

$$p = a_1a_3 - \frac{1}{3}a_2^2 - 4a_4, \quad q = \frac{1}{3}a_1a_2a_3 - \frac{2}{27}a_2^3 - a_1^2a_4 + \frac{8}{3}a_2a_4 - a_3^2, \quad f = \left|\left(\frac{t^2}{4} - a_4\right)^{1/2}\right|, \quad g = \frac{(a_1t/4 - a_3/2)}{f}. \quad (18)$$

The modal shape function of one-step beam with variable cross section can be written as

$$\phi(x) = c_1e^{d_1z} + c_2e^{d_2z} + c_3e^{d_3z} + c_4e^{d_4z}, \quad (19)$$

where c_1, c_2, c_3, and c_4 are four undetermined coefficients.

Since $a_4 < 0$, then $f > t/2$, and $t/2 - f < 0$, it can be found that d_1 and d_2 are two real roots in the first equation of (17). From the second equation in (17), it can be found that when $(a_1/2+g)^2 > 4(t/2+f)$, d_3 and d_4 are also two real roots and when $(a_1/2+g)^2 < 4(t/2+f)$, d_3 and d_4 are two conjugate complex roots.

Modal shapes of one-step beam are determined by the roots as follows:

(1) When d_1, d_2, d_3, and d_4 are all real roots, substituting $z = \ln(1+\beta x)$ into (19), one obtains

$$\phi(x) = B_1(1+\beta x)^{d_1} + B_2(1+\beta x)^{d_2} + B_3(1+\beta x)^{d_3} + B_4(1+\beta x)^{d_4}. \quad (20)$$

(2) When d_1 and d_2 are real roots, while d_3 and d_4 are two conjugate complex roots, let

$$s_1 = \frac{-g - a_1/2}{2}, \quad s_2 = \left|\frac{\sqrt{4(t/2+f) - (a_1/2+g)^2}}{2}\right|; \quad (21)$$

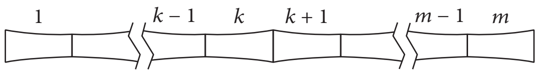

Figure 1: Multistep beam.

the roots d_3 and d_4 can be expressed as

$$\begin{aligned} d_3 &= s_1 + s_2 j, \\ d_4 &= s_1 - s_2 j, \end{aligned} \tag{22}$$

where $j = \sqrt{-1}$.

Substituting $z = \ln(1+\beta x)$ and (22) into (19) yields

$$\begin{aligned} \phi(x) = {} & B_1 (1+\beta x)^{d_1} + B_2 (1+\beta x)^{d_2} \\ & + B_3 (1+\beta x)^{s_1} \cos s_2 \ln(1+\beta x) \\ & + B_4 (1+\beta x)^{s_1} \sin s_2 \ln(1+\beta x). \end{aligned} \tag{23}$$

In (20) and (23), B_i $(i = 1,2,3,4)$ have the same meanings as those in (4) and they are integration constants. The solutions of (20) and (23) are the exact general solutions only at the condition of $\beta \neq 0$. This is because z will be equal to zero in (10) if for $\beta = 0$. And the derivation of constant term for the second equation in (10) has no mathematical meaning. Meanwhile, from the numerical point of view, small value of β $(\beta \to 0)$ is adopted and the uniform beam is approached; then, it could be solved by (20) or (23). However, compared to solutions of uniform beam by (4), the solutions of approximate uniform beam are not exact general solutions but numerical solutions.

Equations (20) and (23) can be expressed with a uniform style as

$$\phi(x) = \sum_{i=1}^{4} B_i S_i(x); \tag{24}$$

when the modal shape function is (20), $S_i(x) = (1+\beta x)^{d_i}$ for $i = 1,2,3,4$, and when it is (23), $S_i(x) = (1+\beta x)^{d_i}$ for $i = 1,2$, and $S_3(x) = (1+\beta x)^{s_1} \cos s_2 \ln(1+\beta x)$, $S_4(x) = (1+\beta x)^{s_1} \sin s_2 \ln(1+\beta x)$.

3. Transfer Relationship for Undetermined Coefficients of Multistep Beam Modal Shapes

As shown in Figure 1, the multistep beam is divided into m segments. For an arbitrary beam segment k $(k = 1,2,\ldots,m)$ with length l_k, a local Cartesian coordinated system is established with the origin locating at the left end of segment, $I_k(x)$ is bending moment of inertia for the kth segment, and $\phi_k(x)$ is mode shape function for the kth segment, where x is defined as belonging to the kth segment in the local Cartesian coordinated system.

Taking the kth segment, for example, the continuity of deformations, equilibrium of moments, and shear forces are satisfied at the right end of the kth segment and left end of the $(k+1)$th segment; that is,

$$\begin{aligned} & \phi_k(x)\big|_{x=l_k} = \phi_{k+1}(x)\big|_{x=0}, \\ & \phi_k'(x)\big|_{x=l_k} = \phi_{k+1}'(x)\big|_{x=0}, \\ & E\left[I_k(x)\cdot\phi_k''(x)\right]\big|_{x=l_k} = E\left[I_{k+1}(x)\cdot\phi_{k+1}''(x)\right]\big|_{x=0}, \\ & \left.\frac{d\left[EI_k(x)\cdot\phi_k''(x)\right]}{dx}\right|_{x=l_k} \\ & \quad = \left.\frac{d\left[EI_{k+1}(x)\cdot\phi_{k+1}''(x)\right]}{dx}\right|_{x=0}, \end{aligned} \tag{25}$$

where prime denotes the derivative with respect to local coordinate x.

Substituting modal shape functions of the kth and $(k+1)$th segments ((4) or (24)) into (25) yields

$$\begin{aligned} & \sum_{i=1}^{4} B_i^k S_i^k(l_k) = \sum_{i=1}^{4} B_i^{k+1} S_i^{k+1}(0), \\ & \sum_{i=1}^{4} B_i^k S_i'^k(l_k) = \sum_{i=1}^{4} B_i^{k+1} S_i'^{k+1}(0), \\ & I_k(l_k) \sum_{i=1}^{4} B_i^k S_i''^k(l_k) = I_{k+1}(0) \sum_{i=1}^{4} B_i^{k+1} S_i''^{k+1}(0), \\ & I_k'(l_k) \sum_{i=1}^{4} B_i^k S_i''^k(l_k) + I_k(l_k) \sum_{i=1}^{4} B_i^k S_i'''^k(l_k) \\ & \quad = I_{k+1}'(0) \sum_{i=1}^{4} B_i^{k+1} S_i''^{k+1}(0) \\ & \qquad + I_{k+1}(0) \sum_{i=1}^{4} B_i^{k+1} S_i'''^{k+1}(0), \end{aligned} \tag{26}$$

where B_i^k denotes the ith undetermined coefficient of modal shape function of the kth segment and S_i^k is the ith term in modal shape function of the kth segment.

Equation (26) can be rewritten into a matrix form:

$$\mathbf{H}_k \mathbf{U}_k = \mathbf{H}_{k+1} \mathbf{U}_{k+1}, \tag{27}$$

where

$$\begin{aligned} \mathbf{U}_k &= \left\{B_1^k \;\; B_2^k \;\; B_3^k \;\; B_4^k\right\}^T, \\ \mathbf{U}_{k+1} &= \left\{B_1^{k+1} \;\; B_2^{k+1} \;\; B_3^{k+1} \;\; B_4^{k+1}\right\}^T, \\ \mathbf{H}_k &= \left[\mathbf{H}_{k1} \;\; \mathbf{H}_{k2} \;\; \mathbf{H}_{k3} \;\; \mathbf{H}_{k4}\right], \\ \mathbf{H}_{k+1} &= \left[\mathbf{H}_{(k+1)1} \;\; \mathbf{H}_{(k+1)2} \;\; \mathbf{H}_{(k+1)3} \;\; \mathbf{H}_{(k+1)4}\right], \end{aligned} \tag{28}$$

in which

$$\mathbf{H}_{ki} = \begin{bmatrix} S_i^k(l_k) \\ S_i'^k(l_k) \\ I_k(l_k) S_i''^k(l_k) \\ I_k'(l_k) S_i''^k(l_k) + I_k(l_k) S_i'''^k(l_k) \end{bmatrix},$$
$$\mathbf{H}_{(k+1)i} = \begin{bmatrix} S_i^{k+1}(0) \\ S_i'^{k+1}(0) \\ I_{k+1}(0) S_i''^{k+1}(0) \\ I_{k+1}'(0) S_i''^{k+1}(0) + I_{k+1}(0) S_i'''^{k+1}(0) \end{bmatrix}. \quad (29)$$

Let

$$\mathbf{T}_k = \mathbf{H}_{k+1}^{-1}\mathbf{H}_k; \quad (30)$$

then,

$$\mathbf{U}_{k+1} = \mathbf{H}_{k+1}^{-1}\mathbf{H}_k\mathbf{U}_k = \mathbf{T}_k\mathbf{U}_k. \quad (31)$$

Equation (31) represents the transfer relationship of four undetermined coefficients for modal shapes of the kth and $(k+1)$th segments. $\mathbf{T}_k$ is the transfer matrix. In accordance with (31), transfer relationship between the first segment and the last segment can be expressed as

$$\mathbf{U}_m = \mathbf{T}_{m-1}\mathbf{T}_{m-2}\cdots\mathbf{T}_2\mathbf{T}_1\mathbf{U}_1 = \mathbf{T}\mathbf{U}_1, \quad (32)$$

where $\mathbf{T} = \mathbf{T}_{m-1}\mathbf{T}_{m-2}\cdots\mathbf{T}_2\mathbf{T}_1$.

4. Solutions of Natural Frequencies and Modal Shapes

Based on the modal shape function of one-step beam ((4) or (24)) and transfer relationships for undetermined coefficients of modal shapes for each segment ((32)), natural frequencies and modal shapes of multistep beam can be determined by boundary conditions of the beam. The cantilever and simply supported boundaries are considered in this paper, respectively.

For cantilever multistep beam, the boundary conditions are

$$\begin{gathered} \phi_1(0) = 0, \\ \phi_1'(x)\big|_{x=0} = 0, \\ EI_m(x)\cdot\phi_m''(x)\big|_{x=l_m} = 0, \\ E\frac{d\left[I_m(x)\cdot\phi_m''(x)\right]}{dx}\bigg|_{x=l_m} = 0. \end{gathered} \quad (33)$$

Inserting the modal shape functions into (33) and utilizing transfer relationship in (32), one arrives at

$$\begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_m\mathbf{T} \end{bmatrix}\mathbf{U}_1 = 0, \quad (34)$$

where

$$\mathbf{C}_1 = \begin{bmatrix} S_1^1(0) & S_2^1(0) & S_3^1(0) & S_4^1(0) \\ S_1'^1(0) & S_2'^1(0) & S_3'^1(0) & S_4'^1(0) \end{bmatrix},$$
$$\mathbf{C}_m = \begin{bmatrix} S_1''^m(l_m) & S_2''^m(l_m) & S_3''^m(l_m) & S_4''^m(l_m) \\ S_1'''^m(l_m) & S_2'''^m(l_m) & S_3'''^m(l_m) & S_4'''^m(l_m) \end{bmatrix}. \quad (35)$$

For simply supported multistep beam, the boundary conditions are

$$\begin{gathered} \phi_1(0) = 0, \\ EI_1(x)\cdot\phi_1''(x)\big|_{x=0} = 0, \\ \phi_m(l_m) = 0, \\ EI_m(x)\cdot\phi_m''(x)\big|_{x=l_m} = 0. \end{gathered} \quad (36)$$

Similar to (34), one obtains

$$\begin{bmatrix} \mathbf{C}_1' \\ \mathbf{C}_m'\mathbf{T} \end{bmatrix}\mathbf{U}_1 = 0, \quad (37)$$

where

$$\mathbf{C}_1' = \begin{bmatrix} S_1^1(0) & S_2^1(0) & S_3^1(0) & S_4^1(0) \\ S_1''^1(0) & S_2''^1(0) & S_3''^1(0) & S_4''^1(0) \end{bmatrix},$$
$$\mathbf{C}_m' = \begin{bmatrix} S_1^m(l_m) & S_2^m(l_m) & S_3^m(l_m) & S_4^m(l_m) \\ S_1''^m(l_m) & S_2''^m(l_m) & S_3''^m(l_m) & S_4''^m(l_m) \end{bmatrix}. \quad (38)$$

Equations (34) and (37) can be written into a uniform style

$$\mathbf{C}\mathbf{U}_1 = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} \begin{Bmatrix} B_1^1 \\ B_2^1 \\ B_3^1 \\ B_4^1 \end{Bmatrix} = \mathbf{0}, \quad (39)$$

where the elements c_{ij} $(i, j = 1, 2, 3, 4)$ in $\mathbf{C}$ are functions of natural frequencies ω for multistep beam.

Existence of nontrivial solutions of (39) leads to the following frequency equation

$$|\mathbf{C}| = 0. \quad (40)$$

If the frequency region $[\omega_{ai}, \omega_{bi}]$ for the ith natural frequency ω_i of multistep beam and the expressions of modal shape functions for each segment ((20) or (23)) can be determined, then the expressions of elements in matrix $\mathbf{C}$ can be uniquely determined. The ith natural frequency ω_i of multistep beam can be calculated by (40) directly, and undetermined coefficients in modal shape function of the first segment can be solved by (39); then, the undetermined coefficients for other segments can be obtained by using transfer relationship in (32).

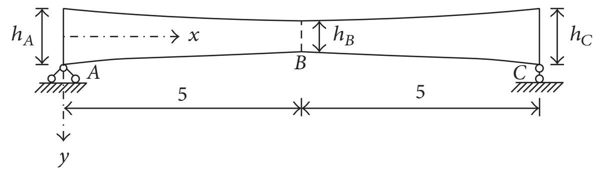

Figure 2: Two-step beam.

If the frequency region or expressions of modal shape function for each segment ((20) or (23)) cannot be determined, the natural frequencies and modal shapes can be solved by the half-interval method [20]. The detailed process for determining natural frequencies is as follows. Firstly, an initial value of the ith natural frequency ω_{ai} is assumed, and the expressions for modal shape function of each segment corresponding to ω_{ai} can be determined ((20) or (23)); then, the coefficient matrix $\mathbf{C}$ corresponding to ω_{ai} is obtained and the determinant of matrix $\mathbf{C}$ is calculated (denoted as $D_{ai} = |\mathbf{C}(\omega_{ai})|$). Then, a new value $\omega_{bi} = \omega_{ai} + \Delta\omega$ with $\Delta\omega$ (e.g., $\Delta\omega = 0.5$) representing the increment of ω is assumed, and the same calculations are repeated to determine the new determinant corresponding to ω_{bi} (denoted as $D_{bi} = |\mathbf{C}(\omega_{bi})|$). If D_{ai} and D_{bi} have opposite signs, there is at least one natural frequency in the interval $[\omega_{ai}, \omega_{bi}]$; otherwise, let $\omega_{ai} = \omega_{bi}$ and continue the above procedures until D_{ai} and D_{bi} have opposite signs. For the next step, let $\omega_{ci} = (\omega_{ai} + \omega_{bi})/2$; if D_{ai} and D_{ci} have the same sign, let $\omega_{ai} = \omega_{ci}$; otherwise, let $\omega_{bi} = \omega_{ci}$; a new interval $[\omega_{ai}, \omega_{bi}]$ is obtained. The same calculations are repeated to determine the new interval $[\omega_{ai}, \omega_{bi}]$ until the rank of matrix $\mathbf{C}$ is equal to three. The accurate values of ω are obtained, respectively, using the half-interval method.

5. Numerical Examples

To verify the correctness of present analytical method, several numerical examples are studied in this paper. The materials used in these examples are all the same; that is, Yong's modulus is 3.25×10^{10} Pa and density is $2500\,\text{kg/m}^3$. Poisson's ratio is $\nu = 0.3$ in a two-dimensional FEM (2D-FEM) analysis.

5.1. Reliability of the Proposed Method

5.1.1. Validation by Finite Element Method. In order to illustrate the proposed method, a two-step beam with 10 m span, as shown in Figure 2, is used as numerical model and solved by one-dimensional FEM (1D-FEM), 2D-FEM, and the present analytical method, respectively.

Cross section of this beam is rectangular with constant width 0.3 m. The heights of sections at A, B, and C are expressed by h_A, h_B, and h_C, respectively. $h_A = h_C$, $h_B = 0.7h_A$. And the height of each step varies nonlinearly as

A to B:

$$h_{AB}(x) = h_A\left(1 + \beta_{AB}x\right)^2, \quad (0 \le x \le 5),$$
$$\beta_{AB} = \frac{\sqrt{h_B/h_A} - 1}{5} = \frac{\sqrt{7/10} - 1}{5}; \tag{41}$$

B to C:

$$h_{BC}(\xi) = h_B\left(1 + \beta_{BC}\xi\right)^2, \quad (\xi = x - 5,\ 5 \le x \le 10),$$
$$\beta_{BC} = \frac{\sqrt{h_C/h_B} - 1}{5} = \frac{\sqrt{10/7} - 1}{5}. \tag{42}$$

For 1D-FEM. The finite element method with two-nodes beam element. Each node of the beam element has two degree-of-freedoms, that is, transverse displacement and rotation angle. The cubic Hermite polynomial is adopted as shape function for beam element. Therefore, the shape functions N_i $(i = 1, 2, 3, 4)$ can be expressed as

$$N_1 = \frac{\left(l^3 - 3lx^2 + 2x^3\right)}{l^3},$$
$$N_2 = \frac{\left(l^2x - 2lx^2 + x^3\right)}{l^2},$$
$$N_3 = \frac{\left(3lx^2 - 2x^3\right)}{l^3},$$
$$N_4 = \frac{\left(x^3 - lx^2\right)}{l^2},$$
$$0 \le x \le l. \tag{43}$$

In (43), l is the length of beam element and $l = 0.5$ m.

Let $\mathbf{N} = \{N_1\ N_2\ N_3\ N_4\}$; stiffness matrix and mass matrix of the beam element can be evaluated as

$$\mathbf{K}^e = \int_0^l \mathbf{B}^T EI(x)\,\mathbf{B}\,dx,$$
$$\mathbf{M}^e = \int_0^l m(x)\,\mathbf{N}^T\mathbf{N}\,dx, \tag{44}$$

where $\mathbf{B} = d^2\mathbf{N}/dx^2$ is the strain matrix of beam element. Using above stiffness matrix and mass matrix, finite element analysis is carried on by the codes written in MATLAB.

For 2D-FEM. A finite element analysis is carried out using ANSYS software. A 2D-FEM model of two-step beam is built with 8-node PLANE183 elements as shown in Figure 3.

Figure 3: A two-dimensional ANSYS model for 2D-FEM analysis ($L/h_A = 10$).

The ratios of span and height at A (L/h_A) vary from 10 to 100. h_A, h_B, and h_C are variables with L/h_A, and β_{AB} and β_{BC} are constants. Relative errors of the first two-order natural frequencies obtained by 1D-FEM and 2D-FEM with respect to present analytical method are shown in Figure 4.

As can be seen from Figure 4, the relative errors between 1D-FEM and present method remain at a very small lever all the time no matter how L/h_A changes, and the relative

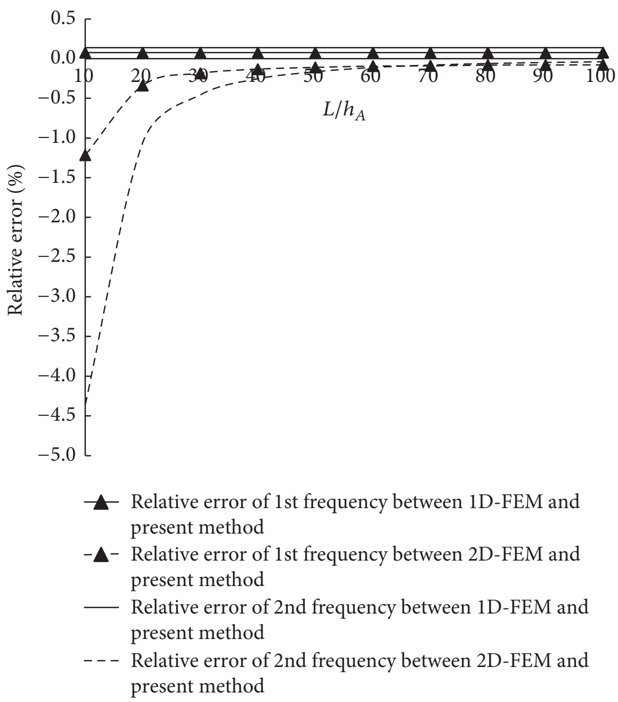

Figure 4: Relative error of 1D-FEM and 2D-FEM.

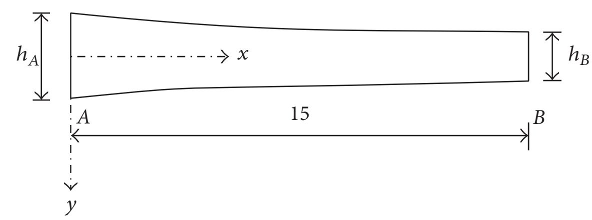

Figure 5: One-step beam with small value of β.

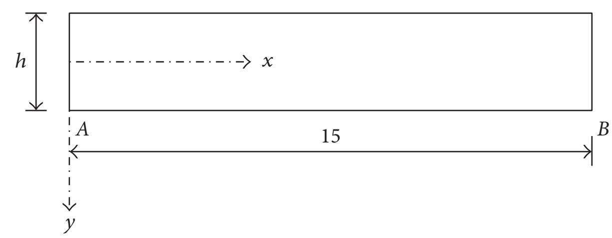

Figure 6: One-step uniform beam.

errors between 2D-FEM and the present method decrease rapidly with L/h_A increasing. This is because 1D-FEM and the present method are employed without considering the effect of shear deformation and the stiffness and mass matrixes are formed by integration method for the variable moment of inertia and mass in 1D-FEM modeling. However, the shear deformation is inevitable for a two-dimensional FEM model. When L/h_A reaches a certain lever, the error between 2D-FEM and the present method becomes smaller. For example, when $L/h_A = 60$, the frequencies obtained by 2D-FEM and the present method are very close and relative error of the first two order frequencies is about −0.1%. With regard to beam-like structural free vibration, it can be treated as plane stress problem, so a 2D-FEM analysis is more accurate than a 1D-FEM analysis in theory. For easy of calculation, the beam-like structure can be treated as Euler-Bernoulli beam when L/h_A reaches a certain lever and the effect of shear deformation could be ignored. And the object of this paper is not considering the effect of shear deformation, so the solutions of the proposed method are exact ones based on Euler-Bernoulli beam theory. This leads to different relative errors of 1D-FEM and 2D-FEM shown in Figure 4. It also reveals that the present method possesses favorable accuracy for the larger ratio of span and sectional height.

5.1.2. Numerical Simulation for Small Value of β. From the numerical point of view, small value of β ($\beta \rightarrow 0$) is adopted, and the uniform beam is approached. In order to verify the explanations, a one-step beam with small value of β (shown in Figure 5) and a one-step uniform beam (shown in Figure 6) are used for numerical simulation by the present method ((20) and (23) for nonuniform beams, (4) for uniform beams).

One-Step Beam with Small Value of β. The span is 15 m. Cross section of this beam is rectangular with constant width 0.3 m.

The heights of sections at A and B are $h_A = 0.25$ m and $h_B = 0.249$ m, respectively, and the height varies nonlinearly from A to B as

$$h(x) = h_A(1+\beta x)^2, \quad (0 \le x \le 15),$$
$$\beta = \frac{\sqrt{h_B/h_A} - 1}{15} = \frac{\sqrt{249/250} - 1}{15}, \quad (\beta \longrightarrow 0). \tag{45}$$

Therefore,

$$I(x) = \frac{bh^3(x)}{12} = \frac{bh_A^3(1+\beta x)^6}{12},$$
$$m(x) = \rho b h(x) = \rho b h_A(1+\beta x)^2, \tag{46}$$
$$(0 \le x \le 15).$$

One-Step Uniform Beam. The beam is also 15 m span with constant width 0.3 m, and the height is 0.249 m.

The one-step beams with different boundary conditions have been analyzed.

Case 1. A cantilever beam model.

Case 2. A simply supported beam model.

The first three natural frequencies of one-step beam with small value of β ($\beta \rightarrow 0$) calculated by (20) or (23) are compared with those of one-step uniform beam calculated by (4); the results are listed in Tables 1 and 2.

As can be seen from the results, the relative errors between nonuniform beam with small value of β and uniform

Table 1: Calculation results of natural frequencies for Case 1.

Beam type	One-step beam with small value of β	One-step uniform beam with height 0.249 m	Relative error (%)
1st (rad/s)	4.0680	4.0499	−0.4451
2nd (rad/s)	25.4507	25.3805	−0.2755
3rd (rad/s)	71.2273	71.0663	−0.2261

Table 2: Calculation results of natural frequencies for Case 2.

Beam type	One-step beam with small value of β	One-step uniform beam with height 0.249 m	Relative error (%)
1st (rad/s)	11.3912	11.3684	−0.2002
2nd (rad/s)	45.5647	45.4734	−0.2003
3rd (rad/s)	102.5205	102.3152	−0.2003

beam are very small. It reveals that the uniform beam is approached when β is small enough ($\beta \to 0$). However, for small value of β representing a uniform beam, the solutions are not exact general solutions but numerical solutions.

5.2. One-Step Beam. A one-step beam with 15 m span, as shown in Figure 5, is used for numerical simulation. Cross section of this beam is rectangular with constant width 0.3 m. The heights of sections at A and B are $h_A = 0.25$ m and $h_B = 0.175$ m, respectively, and the sectional height varies nonlinearly from A to B as

$$h(x) = h_A(1+\beta x)^2, \quad (0 \le x \le 15),$$
$$\beta = \frac{\sqrt{h_B/h_A}-1}{15} = \frac{\sqrt{7/10}-1}{15}. \tag{47}$$

Therefore,

$$I(x) = \frac{bh^3(x)}{12} = \frac{bh_A^3(1+\beta x)^6}{12},$$
$$m(x) = \rho bh(x) = \rho bh_A(1+\beta x)^2, \quad (0 \le x \le 15). \tag{48}$$

The one-step beam with different boundary conditions has been analyzed.

Case 3. A cantilever beam model.

Case 4. A simply supported beam model.

The first three natural frequencies of one-step beam with different boundary conditions calculated by the proposed method and FEM are listed in Tables 3 and 4; modal shapes are shown in Figures 7 and 8.

As can be seen from the results calculated by the proposed method and FEM, relative errors for the first three natural frequencies of the one-step beam with different boundary conditions are very small. It reveals that the solutions for one-step beam with variable cross section by the proposed method are exact ones. The reasons lie in that only the differential equations of motion for one-step beam with variable cross section are used to make the solutions, and no other assumptions are introduced.

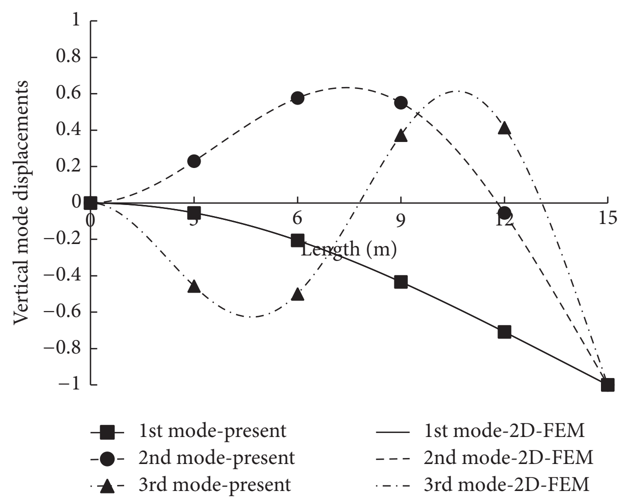

Figure 7: Modal shapes of Case 3.

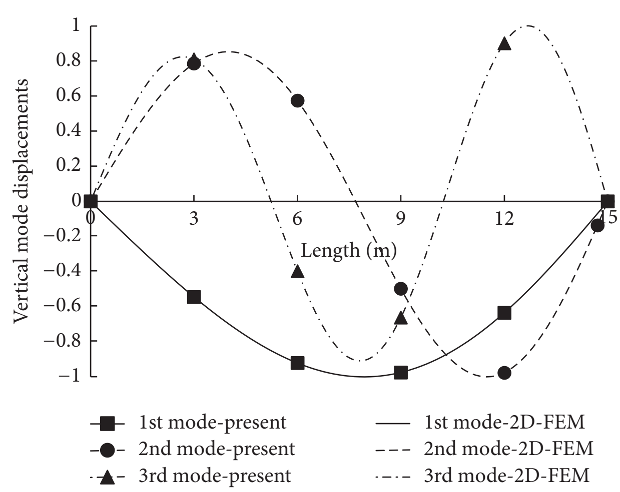

Figure 8: Modal shapes of Case 4.

5.3. Multistep Beam. To verify the correctness of present method for free vibration analysis of multistep beams, a two-step simply supported beam with variable cross section (as shown in Figure 9) is used for numerical simulation firstly. The cross section of this beam is rectangular. The segments

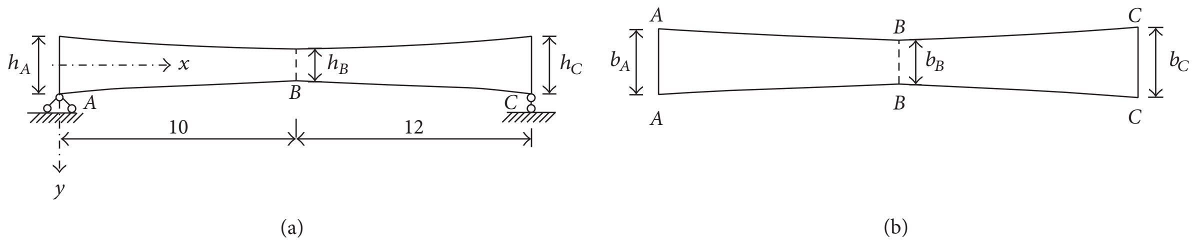

FIGURE 9: Two-step simply supported beam: (a) elevation graph; (b) plane graph.

TABLE 3: Calculation results of natural frequencies for Case 3.

Methods	Present (rad/s)	1D-FEM (rad/s)	2D-FEM (rad/s)	Relative error (%)	
				1D-FEM	2D-FEM
1st	4.2234	4.2229	4.2232	−0.0120	−0.0047
2nd	22.8615	22.8680	22.8406	0.0283	−0.0914
3rd	61.3271	61.3522	61.1800	0.0410	−0.2399

TABLE 4: Calculation results of natural frequencies for Case 4.

Methods	Present (rad/s)	1D-FEM (rad/s)	2D-FEM (rad/s)	Relative error (%)	
				1D-FEM	2D-FEM
1st	9.5349	9.5398	9.5316	0.0511	−0.0350
2nd	38.3529	38.3716	38.3029	0.0487	−0.1303
3rd	86.2536	86.2964	85.9980	0.0495	−0.2964

AB and BC have variable heights and widths. In the two-dimensional FEM analysis, a fine mesh of elements is used and the average thickness of element is adopted for element thickness.

For segment AB, the heights of sections A and B are h_A = 0.36 m and h_B = 0.24 m, respectively; the widths of sections A and B are b_A = 0.3 m and b_B = 0.2449 m, respectively. The height and width from A to B vary as

$$h_{AB}(x) = h_A(1+\beta_{AB}x)^2, \quad b_{AB}(x) = b_A(1+\beta_{AB}x), \quad (0 \le x \le 10), \quad \beta_{AB} = \frac{\sqrt{h_B/h_A}-1}{10} = \frac{\sqrt{2/3}-1}{10}. \tag{49}$$

Therefore,

$$I_{AB}(x) = \frac{b_{AB}(x)\cdot h_{AB}^3(x)}{12} = \frac{b_A h_A^3(1+\beta_{AB}x)^7}{12}, \quad m_{AB}(x) = \rho b_{AB}(x)\cdot h_{AB}(x) = \rho b_A h_A (1+\beta_{AB}x)^3, \quad (0 \le x \le 10). \tag{50}$$

For segment BC, the heights of sections B and C are h_B = 0.24 m and h_C = 0.38 m, respectively; the widths of sections B and C are b_B = 0.2449 m and b_C = 0.3082 m, respectively. The height and width from B to C vary as

$$h_{BC}(\xi) = h_B(1+\beta_{BC}\xi)^2, \quad b_{BC}(\xi) = b_B(1+\beta_{BC}\xi), \quad (\xi = x-10,\ 10 \le x \le 22), \quad \beta_{BC} = \frac{\sqrt{h_C/h_B}-1}{12} = \frac{\sqrt{19/12}-1}{12}. \tag{51}$$

Therefore,

$$I_{BC}(\xi) = \frac{b_{BC}(\xi)\cdot h_{BC}^3(\xi)}{12} = \frac{b_B h_B^3(1+\beta_{BC}\xi)^7}{12}, \quad m_{BC}(\xi) = \rho b_{BC}(\xi)\cdot h_{BC}(\xi) = \rho b_B h_B(1+\beta_{BC}\xi)^3, \quad (0 \le \xi = x-10 \le 12). \tag{52}$$

The first three natural frequencies of two-step simply supported beam are calculated by the proposed method and FEM, which are listed in Table 5. The modal shapes are shown in Figure 10.

Secondly, a three-step simply supported beam with variable cross section, as shown in Figure 11, is used for numerical simulation. Cross section of this beam is rectangular with constant width 0.3 m. The segments A-B and C-D are variable sections, and B-C is uniform cross section.

Table 5: Calculation results of natural frequencies of two-step simply supported beam.

Methods	Present (rad/s)	1D-FEM (rad/s)	2D-FEM (rad/s)	Relative error (%)	
				1D-FEM	2D-FEM
1st	5.7666	5.7712	5.7570	0.0805	−0.1656
2nd	25.1551	25.1761	25.1157	0.0838	−0.1566
3rd	57.1504	57.2114	56.9501	0.1068	−0.3490

Table 6: Calculation results of natural frequencies of three-step simply supported beam.

Methods	Present (rad/s)	1D-FEM (rad/s)	2D-FEM (rad/s)	Relative error (%)	
				1D-FEM	2D-FEM
1st	5.2110	5.2132	5.2089	0.0414	−0.0402
2nd	21.6022	21.5989	21.5683	−0.0155	−0.1571
3rd	50.2561	50.3032	50.1738	0.0938	−0.1638

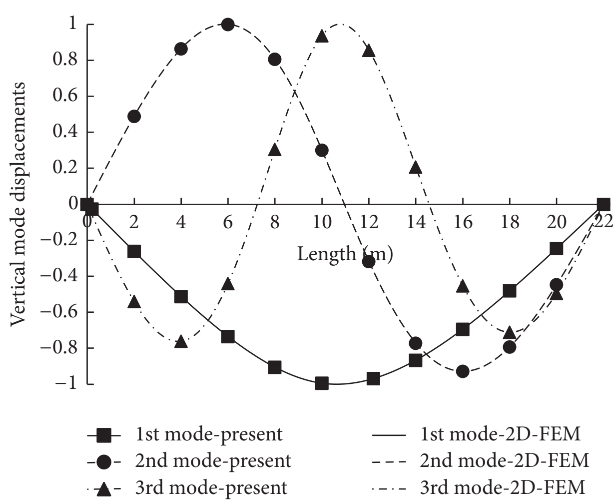

Figure 10: Modal shapes of two-step simply supported beam.

For segment AB, the heights of sections A and B are $h_A = 0.3$ m and $h_B = 0.2$ m, respectively. The height of section from A to B varies as

$$h_{AB}(x) = h_A(1+\beta_{AB}x)^2, \quad (0 \le x \le 5),$$
$$\beta_{AB} = \frac{\sqrt{h_B/h_A}-1}{5} = \frac{\sqrt{2/3}-1}{5}. \tag{53}$$

Therefore,

$$I_{AB}(x) = \frac{bh_{AB}^3(x)}{12} = \frac{bh_A^3(1+\beta_{AB}x)^6}{12},$$
$$m_{AB}(x) = \rho b h_{AB}(x) = \rho b h_A(1+\beta_{AB}x)^2, \tag{54}$$
$$(0 \le x \le 5).$$

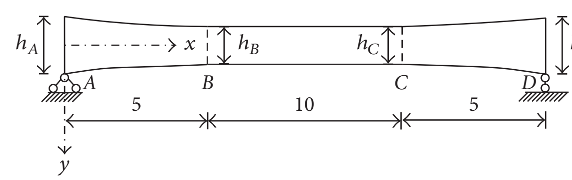

Figure 11: Three-step simply supported beam.

For segment CD, the heights of sections C and D are $h_C = 0.2$ m and $h_D = 0.3$ m. The height of section from C to D varies as

$$h_{CD}(\xi) = h_C(1+\beta_{CD}\xi)^2, \quad (0 \le \xi = x - 15 \le 5),$$
$$\beta_{CD} = \frac{\sqrt{h_D/h_C}-1}{5} = \frac{\sqrt{3/2}-1}{5}. \tag{55}$$

Therefore,

$$I_{CD}(\xi) = \frac{bh_{CD}^3(\xi)}{12} = \frac{bh_C^3(1+\beta_{CD}\xi)^6}{12},$$
$$m_{CD}(\xi) = \rho b h_{CD}(\xi) = \rho b h_C(1+\beta_{CD}\xi)^2, \tag{56}$$
$$(0 \le \xi = x - 15 \le 5).$$

The first three natural frequencies of three-step simply supported beam calculated by the proposed method and FEM are listed in Table 6, and the modal shapes are shown in Figure 12.

From Tables 5 and 6 and Figures 10 and 12, for two-step and three-step simply supported beams, it can be found that the first three natural frequencies and modal shapes calculated by the proposed method are in good agreement with those calculated by FEM. It reveals that the solutions for multistep beam with variable cross section by the proposed method are exact ones. This is because only transfer matrix method and exact solutions of one-step beam with variable cross section are used to solve the equations of vibration for multistep beam with variable cross section, and no other assumptions are introduced.

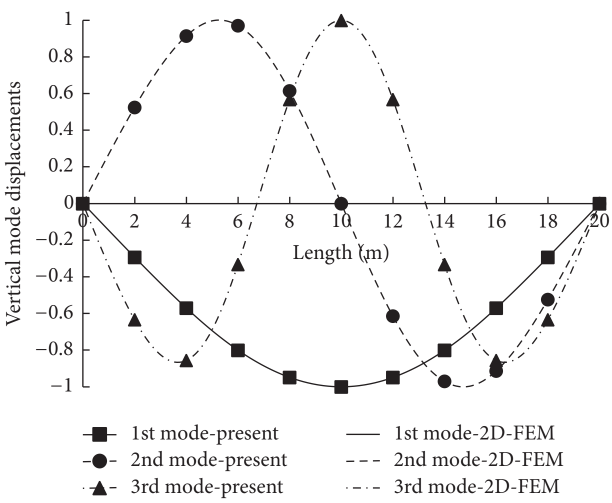

Figure 12: Modal shapes of three-step simply supported beam.

6. Conclusions

In this paper, an exact approach to investigate the flexural free vibrations of multistep nonuniform beams is presented. The differential equation for flexural free vibration of one-step beam with $I(x) = \alpha_1(1+\beta x)^{r+4}$ and $m(x) = \alpha_2(1+\beta x)^r$ is reduced to a four-order differential equation with constant coefficients by using appropriate transformations. And the general solutions of one-step beam are obtained, whose modal shape functions are found to have two kinds of expressions. Then, combining transfer method, iterative method, and the general solutions of one-step beam, the solving method for natural frequencies and modal shapes of multistep beam is formulated. Numerical simulations on multistep nonuniform beam are used to verify the feasibility. The following conclusions can be obtained:

(1) The comparison of relative errors among the present analytical method and finite element methods reveals that the solutions of proposed method are exact ones based on Euler-Bernoulli beam theory. The present method is suitable for the larger ratio of span and sectional height and possesses favorable accuracy.

(2) The relative errors between nonuniform beam with small value of β and uniform beam are very small. It reveals that for small value of β representing a uniform beam, the solutions are not exact general solutions but numerical solutions.

(3) Numerical studies of one-step cantilever beam and simply supported beam indicate that natural frequencies and modal shapes calculated by the proposed method are very close to the FEM results, which demonstrates the solutions of one-step beam are exact ones. And numerical examples of two-step and three-step simply supported beams show that the calculated frequencies and modal shapes are also in good agreements with FEM results, which also demonstrate the solutions of presented method for multistep beam with variable cross section are exact ones. The reasons lie in that only transfer matrix method and general solution of one-step beam are used to obtain the solution equation of natural vibration characteristics, and no other assumptions are introduced.

The exact solutions of proposed method can provide adequate insight into the physics of problems and convenience for parametric studies. Furthermore, they are very useful for inverse problems such as structural damage identification. Therefore, it is always desirable to obtain exact solutions for such problems. However, when the ratio of span and sectional height is small, relative errors between FEM and present method may be rather larger. The deficiency of this paper is not considering shear deformation in vibration analysis of nonuniform beams, leading to larger error for the small ratio of span and sectional height. This also has become the topic which the author further deliberated from now on.

Competing Interests

The authors declare that they have no competing interests regarding the publication of this paper.

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