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SOME NEW CHARACTERIZATION RESULTS ON EXPONENTIAL AND RELATED DISTRIBUTIONS

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ABSTRACT. It is well-known that most of the characterization results on exponential distribution are based on the solution of Cauchy functional equation and integrated Cauchy functional equation. Here, we consider the functional equation

$$F(x) = F(xy) + F(xQ(y)), \quad x, xQ(y) \in [0, \theta),\ y \in [0, 1],$$

where F and Q satisfy certain conditions, to give some new characterization results on exponential, power and Pareto distributions using the concepts of conditional random variables and order statistics.

1. Introduction

Because of importance of the exponential and geometric distributions, in many branches of statistics and applied probability, a large number of research articles appear in the literature characterizing these distributions based on different properties. The monographs of Galambos and Kotz (1978), Azarlov and Volodin (1986), and Rao and Shanbhag (1994) are devoted to characterizations of probability distributions, mainly on

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exponential and geometric distributions. Usually, the problem of characterizing a probability distribution function leads to solve a functional equation. It is mentioned in Rao and Shanbhag (1986) that most of the characteristic properties of the exponential and geometric distributions, based on conditional expectations and order statistics, can be obtained from "integrated Cauchy functional equation" under minimal assumptions. The monograph of Rao and Shanbhag (1994) provides a comprehensive study of the applications of the integrated Cauchy functional equation on characterizing exponential and geometric distributions based on different relations between ordered random variables. Asadi et al. (2001) applied integrated Cauchy functional equation to obtain several characterization results on exponential, power and Pareto distributions. Let X be a lifetime (non-negative) random variable with cumulative distribution function (cdf) F, and survival function $S = 1 - F$. The random variable X is said to have

- exponential distribution with mean λ if

$$S(x) = e^{-x/\lambda},\ x \geq 0, \quad \lambda > 0,$$

- power function distribution with parameter vector (α, θ) if

$$F(x) = \left(\frac{x}{\theta}\right)^{\alpha}, \quad 0 \leq x \leq \theta,\ \alpha > 0,\ \theta > 0,$$

- Pareto distribution with parameter vector (α, β) if

$$S(x) = \left(\frac{\beta}{x}\right)^{\alpha}, \quad x \geq \beta,\ \alpha > 0.$$

These distributions are of particular interest in statistical literature for their flexibility to model various data with different applications. Our purpose here is to give some characterization results on the above distributions. The results are applications of a functional equation which is recently solved by Aczel *et al.* (1999). Their result is stated in the following theorem.

Theorem 1.1. *Among the functions* $F : [0, \theta) \to \mathbf{R}_+ (= [0, \infty))$, $\theta \in (0, \infty]$, *and* $Q : [0, 1] \to \mathbf{R}_+$, *the functional equation*

$$F(x) = F(xy) + F(xQ(y)), \quad x, xQ(y) \in [0, \theta),\ y \in [0, 1], \tag{1.1}$$

has the trivial solutions

$$F = 0, \quad Q \ \ arbitrary\ (\leq 1\ if\ \theta < \infty), \tag{1.2}$$

and

$$(1.3)\qquad \begin{cases} F(x) = c > 0,\ x \in (0,\theta), & F(0) = 0, \\ Q(y) = 0,\ y \in (0,1], & Q(0) > 0 \ \textit{arbitrary} \\ & (\leq 1 \ \textit{if} \ \theta < \infty). \end{cases}$$

For all other solutions, there exist constants $k > 0$, $\alpha > 0$ such that

$$(1.4)\qquad F(x) = kx^{\alpha}, \qquad Q(y) = (1 - y^{\alpha})^{1/\alpha}.$$

If $\theta > 1$ and $F(1) = 1$, then $k = 1$.

Conversely, all pairs of functions of the form (1.2), (1.3) and (1.4) satisfy (1.1).

It is mentioned in Aczel et al. (1999) that the functions F and/or Q may map *into* the given ranges; and *onto* is not assumed. Neither is any regularity (monotonicity, continuity) supposed. Also, the same result follows when one assumes that Eq. (1.1) holds only for almost all pairs $(x, y) \in [0,\theta) \times [0,1]$ (with respect to planar Lebesgue measure).

The remainder of the paper is organized as follows: In Section 2, we obtain some characterization results on exponential, power and Pareto distributions based on functional equation in (1.1). Section 3 deals with some characterization results based on equality in distribution of some conditional random variables such as residual life random variable. In this section, characterizations based upon equality of expectation of some conditional random variables are also given. Section 4 is devoted to characterization results based on identity of distributions and equality of expectation of some functions of order statistics. The results of this section are extensions of the results obtained recently by Asadi (2006), and Tavangar and Asadi (2007).

2. **Characterizations based on relations satisfied by cumulative distribution function**

Assume that F is a cdf. In what follows, we will define the support of F (or a random variable having cdf F) as $(\alpha(F), \omega(F))$, where $\alpha(F) = \inf\{x : F(x) > 0\}$, and $\omega(F) = \sup\{x : F(x) < 1\}$. Note that, in general, the former may be $-\infty$, and the latter may be $+\infty$. If F satisfies (1.1), then, as a consequence of properties of a distribution function (such as the right continuity), two trivial solutions of equation (1.1) are excluded and we obtain the following result.

Theorem 2.1. *Let F be any cdf with support $[0, \theta)$, $\theta > 0$. Suppose that $Q : [0, \theta) \to \mathbf{R}_+$. The functional equation*

(2.1) $F(x) = F(xy) + F(xQ(y)), \quad x, xQ(y) \in [0, \theta),\ y \in [0, 1],$

holds if and only if F is a (rescaled) power function distribution with parameter vector (α, θ), for some constant $\alpha > 0$, and $Q(y) = (1 - y^\alpha)^{1/\alpha},\ 0 \leq y \leq 1$.

Remark 2.2. Equation (2.1) has a probabilistic interpretation as follows. Let X be a continuous non-negative random variable on $[0, \theta)$ with cdf F. If there is a function Q, which satisfies in the conditions of Theorem 2.1, such that

$$P[X > xy \mid X \leq x] = P[X \leq xQ(y) \mid X \leq x], \quad x, \in [0, \theta),\ y \in [0, 1],$$

then $Q(y) = (1 - y^\alpha)^{1/\alpha}$, and X has the power function distribution.

Most of the characterization results on exponential distribution are based on the Cauchy functional equation which is known in the statistical literature as the 'lack of memory property'. A non-negative random variable X is said to have the lack of memory property if, for all $x, y > 0$, its survival function S satisfies $S(x + y) = S(x)S(y)$. It is well-known that the only continuous distribution with this property is the exponential survival function. In the following theorem we give a characterization result on exponential distribution which is based upon the functional equation in Theorem 2.1.

Theorem 2.3. *Let F be any cdf with support $\mathbf{R}_+$, and $S = 1 - F$. Assume that $Q : \mathbf{R}_+ \to \mathbf{R}_+$. The functional equation*

(2.2) $\qquad S(x) = S(x + y) + S(x + Q(y)), \quad x, y \in [0, \infty),$

holds if and only if F is an exponential distribution with mean λ, for some $\lambda > 0$, and $Q(y) = -\lambda \log(1 - e^{-y/\lambda})$, $y > 0$.

Proof. The 'if' part of the theorem is straightforward and hence we prove the 'only if' part. To this end, we define the cdf G as $G(z) = S(-\log z),\ z \in [0, 1)$, where S is the survival function defined in the statement of the theorem. Let $u = e^{-x}$, $v = e^{-y}$, and $Q^*(v) = \exp\{-Q(-\log v)\}$. It is obvious that $Q^* : [0, 1] \to \mathbf{R}_+$. Now, Eq. (2.2) implies that $G(u) = G(uv) + G(uQ^*(v)),\ u, uQ^*(v) \in [0, 1),\ v \in [0, 1]$. That is, the pair of functions (G, Q^*) satisfies Eq. (2.1) with $\theta = 1$.

Therefore, using Theorem 2.1, we have $G(x) = x^{\alpha}$, $x \in [0,1)$, and $Q^*(y) = (1 - y^{\alpha})^{1/\alpha}$, $y \in [0,1]$, for some constant $\alpha > 0$. This means that F is an exponential cdf with mean $\lambda = 1/\alpha$, and Q is as stated in the theorem. Hence, the proof is complete. $\square$

Remark 2.4. The probabilistic interpretation of Eq. (2.2) is as follows. Let X be continuous non-negative random variable on $(0,\infty)$ with survival function S. If there exists a function Q which satisfies in conditions of Theorem 2.3, and

$$P[X \geq x + y \mid X \geq x] = P[X < x + Q(y) \mid X \geq x], \qquad x, y > 0,$$

then $Q(y) = -\lambda \log(1 - e^{-y/\lambda})$, for some $\lambda > 0$, and X has an exponential distribution with mean $1/\lambda$. It is also worth noting that $Q(y)$ here is equal to $Q(y) = \lambda \int_y^{\infty} r(x)dx$, where $r(x) = f(x)/F(x)$ is the reversed hazard rate, and f is the density of F.

The next theorem gives a characterization of the Pareto distribution which follows from Theorem 2.1. The proof, being the same as the proof of Theorem 2.3, is omitted.

Theorem 2.5. *Let F be any cdf with support $[\beta,\infty)$, and $S = 1 - F$. Suppose that $Q : [1,\infty) \to \mathbf{R}_+$. Then, the functional equation*

$$S(x) = S(xy) + S(xQ(y)), \quad x, xQ(y) \in [\beta,\infty),\ y \in [1,\infty),$$

holds if and only if F is a Pareto distribution with parameter vector (α,β), for some constant $\alpha > 0$, and $Q(y) = (1 - y^{-\alpha})^{-1/\alpha}$, $y \geq 1$.

3. Characterizations based on conditional random variables

Given any cdf F, the quantile (or generalized inverse) function $F^{\leftarrow}$ is defined by

$$F^{\leftarrow}(u) = \inf\{x : F(x) \geq u\}, \quad u \in (0,1).$$

It is known that $F^{\leftarrow}$, in general, does not preserve the relation " $<$ "; *i.e.*, it is not true that

$$x < y \quad \Leftrightarrow \quad F^{\leftarrow}(x) < F^{\leftarrow}(y).$$

For any cdf F, the following lemma, which can be proved easily, provides some results on $F^{\leftarrow}$, which we will use in the sequel.

Lemma 3.1. *For any cdf F,*
(i) $t < F^{\leftarrow}(u) \quad \Leftrightarrow \quad F(t) < u$,
(ii) $u \leq F(t) \quad \Leftrightarrow \quad F^{\leftarrow}(u) \leq t$.

Let X be an arbitrary random variable with cdf F. Then, $F(X)$ as well as $S(X) = 1 - F(X)$ have uniform $U(0,1)$ distributions. Hence, using the probability integral transform, we conclude that $F^{\leftarrow}[S(X)]$ is distributed as F, where $F^{\leftarrow}$ is the quantile function. That is $X \stackrel{d}{=} F^{\leftarrow}[S(X)]$, where $\stackrel{d}{=}$ stands for equality in distribution. A natural question that arises is whether there exists a strictly decreasing function Q, for which the relation $X \stackrel{d}{=} Q(X)$ holds. In this section, we obtain some solutions to this question for some special conditional random variables (such as the residual life random variable). First, we show that when the random variable of interest is the residual life random variable, the function Q is unique and the underlying distribution is exponential. This is given by the following theorem.

Theorem 3.2. *Let X be a non-negative random variable with the survival function S. Suppose that $Q : \mathbf{R}_+ \to \mathbf{R}_+$ is a strictly decreasing function. Let also $X_t = [X - t \mid X > t]$ be the residual life random variable. Then $X_t \stackrel{d}{=} Q(X_t)$, for almost all $t \in \mathbf{R}_+$ (with respect to Lebesgue measure) with $S(t) > 0$, if and only if S is the survival function of an exponential random variable with mean λ, for some constant $\lambda > 0$, and $Q(y) = -\lambda \log(1 - e^{-y/\lambda})$, $y > 0$.*

Proof. First, note that under the assumption $S(t) > 0$, the conditional random variable X_t is well-defined. Also, note that since every monotone function is measurable, $Q(X_t)$ is a random variable. We have,

$$\begin{aligned} P[Q(X_t) \geq x] &= 1 - P[X_t > Q^{-1}(x)] \\ &= 1 - \frac{P[X > t + Q^{-1}(x)]}{P[X > t]}. \end{aligned}$$

Let U be a random variable with uniform $U(0,1)$ distribution. From the probability integral transform, we have $X \stackrel{d}{=} F^{\leftarrow}(U)$. Now, it follows from Lemma 3.1 that

$$\begin{aligned} P[Q(X_t) \geq x] &= 1 - \frac{P[U > F(t + Q^{-1}(x))]}{P[U > F(t)]} \\ &= 1 - \frac{S(t + Q^{-1}(x))}{S(t)}, \end{aligned}$$

and

$$\begin{aligned} P[X_t \geq x] &= \frac{P[X \geq t + x]}{P[X > t]} \\ &= \frac{1}{S(t)}\{S(t + x) + F(t + x) - F((t + x)-)\}. \end{aligned}$$

We need to prove that

$$P[X_t \geq x] = \frac{S(t + x)}{S(t)}. \tag{3.1}$$

Let $D = \{x \in \mathbf{R}_+ \mid F$ has jump at $x\}$ denote the set of discontinuity points of F which is known to be countable. If D is an empty set, then the result is trivial. Hence, let $D = \{d_1, d_2, \ldots\}$, and define the set E_i's, $i = 1, 2, \ldots$, as $E_i = \{(t, x) \in \mathbf{R}_+ \times \mathbf{R}_+ \mid t + x = d_i\} = \{(d_i - x, x) \mid x \in [0, d_i]\}$. It is easy to observe that the E_i are measurable sets of planar Lebesgue measure zero which, in turn, implies that D is a set of planar Lebesgue measure zero. Therefore, Eq. (3.1), and consequently the following equation hold for almost all pairs $(t, x) \in \mathbf{R}_+ \times \mathbf{R}_+$ with respect to planar Lebesgue measure:

$$S(t) = S(t + x) + S(t + Q^{-1}(x)).$$

Now the result follows from Theorem 2.3.

Remark 3.3. It can be easily shown that Theorem 3.2 holds if we replace $X_t \stackrel{d}{=} Q(X_t)$ with $X_t \stackrel{d}{=} [Q(X - t) \mid X > t]$, where Q meets the requirements of the theorem.

A similar result characterizing a (rescaled) power function distribution is as follows.

Theorem 3.4. *Let X be a non-negative random variable with support $[0, \theta)$ having the cdf F. Assume that $Q : [0, 1] \to \mathbf{R}_+$ is a strictly decreasing function. Let also $X_{(t)} = [t^{-1}X \mid X \leq t]$. Then, $X_{(t)} \stackrel{d}{=} Q(X_{(t)})$, for almost all $t \in [0, \theta]$ (with respect to Lebesgue measure) with $F(t) > 0$, if* *and only if F is a (rescaled) power function distribution with parameter vector (α, θ), for some $\alpha > 0$, and $Q(y) = (1 - y^\alpha)^{1/\alpha}$, $0 \leq y \leq 1$.*

Proof. Along the lines of the proof of Theorem 3.2, we get

$$P[X_{(t)} \geq x] = 1 - \frac{F(tx)}{F(t)},$$

for almost all pairs $(t, x) \in [0, \theta] \times [0, 1]$ with respect to planar Lebesgue measure, and

$$P[Q(X_{(t)}) \geq x] = \frac{F(tQ^{-1}(x))}{F(t)},$$

for all $(t, x) \in [0, \theta] \times [0, 1]$. Now, the result follows from Theorem 2.1. $\square$

We can now state the next result characterizing the Pareto distribution. The proof is similar to the ones given for the above theorems and hence is omitted.

Theorem 3.5. *Let X be a non-negative random variable with support $[\beta, \infty)$, and denote by S its survival function. Assume that $Q : [0, 1] \to \boldsymbol{R}_+$ is a strictly decreasing function. Let also $X_{[t]} = [tX^{-1} \mid X > t]$. Then, $X_{[t]} \stackrel{d}{=} Q(X_{[t]})$, for almost all $t \in [\beta, \infty)$ (with respect to Lebesgue measure) with $S(t) > 0$, if and only if S is the survival function of a Pareto distribution with parameter vector (α, β), for some $\alpha > 0$, and $Q(y) = (1 - y^\alpha)^{1/\alpha}$, $0 \leq y \leq 1$.*

In the following theorem, we prove some results characterizing exponential, power and Pareto distributions based on some conditional expectations.

Theorem 3.6. *Let X be a non-negative random variable having a continuous cdf F, and survival function S.*

(i) Assume that the support of F is $(0, \theta)$, and $\lim_{t \to 0} F(t)/t^\alpha$ exists for constant $\alpha > 0$. Then,

$$E\{X \mid X \leq t\} = E\{(t^\alpha - X^\alpha)^{1/\alpha} \mid X \leq t\}, \; 0 \leq t \leq \theta, \tag{3.2}$$

if and only if F is a (rescaled) power function distribution with parameter vector (α, θ).

(ii) Assume that the support of F *is* $(0,\infty)$*, and* $\lim_{t\to\infty} e^{t/\lambda}S(t)$ *exists for constant* $\lambda > 0$*. Then,*

$$(3.3)\quad E\{X - t \mid X > t\} = E\{-\lambda \log(1 - e^{-(X-t)/\lambda}) \mid X > t\}, \quad t \geq 0,$$

if and only if F *is an exponential distribution with mean* λ*.*

(iii) Assume that the support of F *is* (β,∞)*, and* $\lim_{t\to\infty} t^{\alpha}S(t)$ *exists for constant* $\alpha > 0$*. Then,*

$$(3.4)\quad E\{X^{-1} \mid X > t\} = E\{(t^{-\alpha} - X^{-\alpha})^{1/\alpha} \mid X > t\},\ t \geq \beta,$$

if and only if F *is a Pareto distribution with parameter vector* (α, β)*.*

Proof. *(i)* The 'if' part is easy to verify. To prove the 'only if' part, note that Eq. (3.2) is equivalent to:

$$(3.5)\quad \int_0^t \{F(t) - F(x) - F((t^{\alpha} - x^{\alpha})^{1/\alpha})\}dx = 0,\ 0 \leq t \leq \theta.$$

Since F is continuous, given any $t \in [0,\theta]$, there exists a point $u_t \in (0,t)$ such that

$$(3.6)\qquad \frac{F(t)}{t^{\alpha}} = \frac{u_t^{\alpha}}{t^{\alpha}}\frac{F(u_t)}{u_t^{\alpha}} + \frac{t^{\alpha} - u_t^{\alpha}}{t^{\alpha}}\frac{F((t^{\alpha} - u_t^{\alpha})^{1/\alpha})}{t^{\alpha} - u_t^{\alpha}}.$$

Let μ_t^* be a probability measure which is concentrated on two points only such that it puts mass u_t^{α}/t^{α} at point $t - u_t$, and mass $1 - u_t^{\alpha}/t^{\alpha}$ at point $t - (t^{\alpha} - u_t^{\alpha})^{1/\alpha}$. Let $H(t) = F(t)/t^{\alpha},\ t \in [0,\theta]$. Then, Eq. (3.6) can be written as:

$$H(t) = \int_0^t H(t-u)\mu_t^*(du).$$

It follows from Theorem 1 of Fosam and Shanbhag (1997) that $H(t)$ is a positive constant independent of t. Hence, the proof is complete.

(ii) The proof of the 'if' part is straightforward and hence we prove the 'only if' part. To this end, note that Eq. (3.3) is equivalent to

$$\int_0^{\infty} \{S(t) - S(t+x) - S(t - \lambda \log(1 - e^{-x/\lambda}))\}dx = 0,\ t \geq 0,$$

which, after making appropriate transformations, can be written as:

$$\int_0^s \{S(-\log s) - S(-\log u) - S(-\log(s^{\alpha} - u^{\alpha})^{1/\alpha}\}\frac{1}{u}du = 0,\ 0 \leq s \leq 1,$$

with $\alpha = 1/\lambda$. Defining $G(s) = S(-\log s)$, $0 \le s \le 1$, we get an integral equation similar to Equation (3.5) with F replaced by G. The result then follows using the same arguments as the ones used to prove part *(i)* of the theorem.

(iii) The 'if' part is easy to verify and hence we prove the 'only if' part. One can show that Eq. (3.4) is equivalent to

$$\int_0^1 \{S(t) - S(t/x) - S([t(1-x^{\alpha})]^{-1/\alpha})\}dx = 0, \ t \ge \beta.$$

Let $G(z) = S(1/z)$, $0 < z < 1/\beta$. Upon making appropriate transformations, it is easily seen that Eq. (3.5) holds with F, and θ replaced by G, and $1/\beta$, respectively. This implies that G is the cdf of a (rescaled) power function distribution with parameter vector $(\alpha, 1/\beta)$. In view of the relation between S and G, the proof is then complete. □

4. **Characterizations based on order statistics**

Let $X_1, X_2, \ldots, X_n$ be independent random variables with a common cdf F. The order statistics relative to X_i are denoted by $X_{1:n}, X_{2:n}, \ldots,$ $X_{n:n}$. Order statistics have many applications in different branches of applied probability and statistics such as reliability, life-testing, goodness of fit tests, etc. (see, for example, David and Nagaraja, 2003). In the statistics literature, a large number of research work is devoted to characterizations of probability distributions based on order statistics. (Among others, we refer the reader to Rao and Shanbhag, 1994, Azlarov and Volodin, 1986, and Asadi, et al., 2001). In this section, we give some characterizations of the exponential, power, and Pareto distributions based on order statistics. The specialized versions of the results of this section have already appeared in Asadi (2006), and Tavangar and Asadi (2007). Before giving the main results of this section, we first prove the following lemma.

Lemma 4.1. *Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics from any cdf F. Then,*

(i) The survival function of $[X_{r:n} \mid X_{1:n} > t]$, $1 \le r \le n$, is given by

$$\bar{G}_{r,n}(x|t) = \sum_{i=0}^{r-1} \binom{n}{i} \{1 - \theta_t(x)\}^i \{\theta_t(x)\}^{n-i}, \ x > t,$$

where $\theta_t(x) = S(x)/S(t)$, and $S(x) = 1 - F(x)$, and *(ii) The cdf of* $[X_{r:n} \mid X_{n:n} \leq t]$ *is given by*

$$H_{r,n}(x|t) = \sum_{i=r}^{n} \binom{n}{i} \{\varphi_t(x)\}^i \{1 - \varphi_t(x)\}^{n-i}, \ x \leq t,$$

where $\varphi_t(x) = F(x)/F(t)$.

Proof. *(i)* The proof follows from the fact that $[X_{r:n} \mid X_{1:n} > t]$, $r = 1, 2, \ldots, n$, can be considered as the order statistics from conditional random variable $[X \mid X > t]$ with survival function $S(x)/S(t)$, $x > t$.

(ii) The proof follows by noting that $[X_{r:n} \mid X_{n:n} \leq t]$, $r = 1, 2, \ldots, n$, are the order statistics from conditional random variable $[X \mid X \leq t]$ with cdf $F(x)/F(t)$, $x \leq t$. □

Now, we can prove the following theorem.

Theorem 4.2. *Let* $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ *denote the order statistics from any cdf* F *with support* $\mathbf{R}_+$. *Let* $S = 1 - F$ *and assume that* $Q : \mathbf{R}_+ \to \mathbf{R}_+$ *is a strictly decreasing function. Then,*

$$[X_{r:n} - t \mid X_{1:n} > t] \stackrel{d}{=} [Q(X_{n-r+1:n} - t) \mid X_{1:n} > t], \tag{4.1}$$

for some $1 \leq r \leq n$, *and for almost all* $t \in \mathbf{R}_+$ *(with respect to Lebesgue measure) with* $S(t) > 0$, *if and only if* $Q(y) = -\lambda \log(1 - e^{-y/\lambda})$, $y > 0$, *and* F *is an exponential distribution with mean* λ, *for some* $\lambda > 0$.

Proof. First, we prove the 'only if' part of the theorem. Using Lemma 4.1, we get

$$\begin{aligned} P[X_{r:n} - t \geq x \mid X_{1:n} > t] &= \sum_{i=0}^{r-1} \binom{n}{i} \{1 - \theta_t(t+x)\}^i \{\theta_t(t+x)\}^{n-i} \\ &\quad + \{G_{r,n}(t+x|t) - G_{r,n}((t+x) - |t)\}, \end{aligned}$$

where $G_{r,n}(x|t)$ is the cdf of the conditional random variable $[X_{r:n} \mid X_{1:n} > t]$, and $\theta_t(t+x) = S(t+x)/S(t)$. In view of what we have observed in the proof of Theorem 3.2, we have,

$$\begin{aligned} P[X_{r:n} - t \geq x \mid X_{1:n} > t] &= \sum_{i=n-r+1}^{n} \binom{n}{i} \{\theta_t(t+x)\}^i \{1 - \theta_t(t+x)\}^{n-i} \\ &= \int_0^{\theta_t(t+x)} \frac{1}{B(r, n-r+1)} z^{n-r} (1-z)^{r-1} dz, \end{aligned}$$

for all $(t,x) \in \mathbf{R}_+ \times \mathbf{R}_+$, except on a set of planar Labesgue measure 0, where the last equality is from the relation between binomial sums and the incomplete beta function in which $B(.,.)$ denotes the complete beta function (see, for example, David and Nagaraja, 2003). On the other hand, we can use Lemma 4.1 again to obtain:

$$\begin{aligned}
&P[Q(X_{n-r+1:n} - t) \geq x \mid X_{1:n} > t] \\
&\quad = 1 - P[X_{n-r+1:n} > t + Q^{-1}(x) \mid X_{1:n} > t] \\
&\quad = 1 - \sum_{i=0}^{n-r} \binom{n}{i} \left\{1 - \theta_t(t + Q^{-1}(x))\right\}^i \left\{\theta_t(t + Q^{-1}(x))\right\}^{n-i} \\
&\quad = \sum_{i=n-r+1}^{n} \binom{n}{i} \left\{1 - \theta_t(t + Q^{-1}(x))\right\}^i \left\{\theta_t(t + Q^{-1}(x))\right\}^{n-i} \\
&\quad = \int_0^{1-\theta_t(t+Q^{-1}(x))} \frac{1}{B(r, n-r+1)} z^{n-r}(1-z)^{r-1} dz.
\end{aligned}$$

Now, from these results and Eq. (4.1), we obtain:

$$\theta_t(t+x) = 1 - \theta_t(t + Q^{-1}(x)),$$

for almost all $(t,x) \in \mathbf{R}_+ \times \mathbf{R}_+$, which leads to Eq. (2.2). Now, the result follows from Theorem 2.3. The 'if' part of the theorem is easy to verify and hence is omitted. Hence, the proof is complete. □

The following theorem proves a characterization of the power function distribution.

Theorem 4.3. *Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics from any cdf F with support $[0,\theta]$. Assume that $Q : [0,1] \to \mathbf{R}_+$ is a strictly decreasing function. Then,*

$$[X_{r:n} \mid X_{n:n} \leq t] \stackrel{d}{=} [t\, Q(t^{-1} X_{n-r+1:n}) \mid X_{n:n} \leq t],$$

for some $1 \leq r \leq n$, and for almost all $t \in [0,\theta]$ (with respect to Lebesgue measure) with $F(t) > 0$, if and only if $Q(y) = (1-y^\alpha)^{1/\alpha}$, $0 \leq y \leq 1$, and F is a (rescaled) power function distribution with parameter vector (α,θ), for some constant $\alpha > 0$.

Proof. To prove the 'only if' part of the theorem, note that one can apply Lemma 4.1 to obtain

$$\begin{aligned} P[X_{r:n} \geq x \mid X_{n:n} \leq t] &= 1 - \sum_{i=r}^{n} \binom{n}{i} \{\varphi_t(x)\}^i \{1 - \varphi_t(x)\}^{n-i} \\ &\quad + \{H_{r,n}(x|t) - H_{r,n}(x - |t)\}, \end{aligned}$$

where $\varphi_t(x) = F(x)/F(t)$, and $H_{r,n}(x|t)$ denotes the cdf of $[X_{r:n} \mid X_{n:n} \leq t]$. In view of the arguments already made, one can conclude that for almost all $(t, x) \in [0, \theta] \times [0, \theta]$ (with respect to planar Lebesgue measure),

$$\begin{aligned} P[X_{r:n} \geq x \mid X_{n:n} \leq t] &= 1 - \sum_{i=r}^{n} \binom{n}{i} \{\varphi_t(x)\}^i \{1 - \varphi_t(x)\}^{n-i} \\ &= \sum_{i=0}^{r-1} \binom{n}{i} \{\varphi_t(x)\}^i \{1 - \varphi_t(x)\}^{n-i}, \\ &= \sum_{i=n-r+1}^{n} \binom{n}{i} \{1 - \varphi_t(x)\}^i \{\varphi_t(x)\}^{n-i}, \\ &= \int_0^{1-\varphi_t(x)} \frac{1}{B(r, n-r+1)} z^{n-r} (1-z)^{r-1} dz. \end{aligned}$$

Also, we have,

$$\begin{aligned} &P\left[t\, Q(t^{-1} X_{n-r+1:n}) \geq x \mid X_{n:n} \leq t\right] \\ &\quad = \sum_{i=n-r+1}^{n} \binom{n}{i} \{\varphi_t(tQ^{-1}(x/t))\}^i \{1 - \varphi_t(tQ^{-1}(x/t))\}^{n-i} \\ &\quad = \int_0^{\varphi_t(tQ^{-1}(x/t))} \frac{1}{B(r, n-r+1)} z^{n-r} (1-z)^{r-1} dz. \end{aligned}$$

Now, the result follows from Theorem 2.1. The 'if' part of the theorem is trivial and hence its proof is omitted. The proof is now complete. □

Remark 4.4. The special case of Theorem 4.3, when $Q(y) = 1 - y$ and the underlying distribution is continuous, is investigated by Asadi (2006), and Tavangar and Asadi (2007).

Theorem 4.5. *Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the order statistics based on any cdf F with support $[\beta, \infty)$. Let also $S = 1 - F$. Assume that*

$Q : [0,1] \to \mathbf{R}_+$ is a strictly decreasing function. Then,

$$\left[\frac{t}{X_{r:n}} \;\middle|\; X_{1:n} > t\right] \overset{d}{=} \left[Q\left(\frac{t}{X_{n-r+1:n}}\right) \;\middle|\; X_{1:n} > t\right],$$

for almost all $t \in [\beta, \infty)$, (with respect to Lebesgue measure) with $S(t) > 0$, if and only if $Q(y) = (1 - y^{\alpha})^{1/\alpha}$, $0 \le y \le 1$, and F is a Pareto distribution with parameter vector (α, β), for some constant $\alpha > 0$.

Proof. Using Lemma 4.1, we can write

$$\begin{aligned} P\left[\frac{t}{X_{r:n}} > x \;\middle|\; X_{1:n} > t\right] &= 1 - \sum_{i=0}^{r-1} \binom{n}{i} \{\theta_t(t/x)\}^{n-i}\{1 - \theta_t(t/x)\}^{i} \\ &\quad - \{G_{r,n}(t/x|t) - G_{r,n}((t/x) - |t)\}, \\ &= 1 - \sum_{i=n-r+1}^{n} \binom{n}{i} \{\theta_t(t/x)\}^{i}\{1 - \theta_t(t/x)\}^{n-i} \\ &= 1 - \int_0^{\theta_t(t/x)} \frac{1}{B(r, n-r+1)} z^{n-r}(1-z)^{r-1} dz. \end{aligned}$$

and

$$\begin{aligned} &P\left[Q\left(\frac{t}{X_{n-r+1:n}}\right) > x \;\middle|\; X_{1:n} > t\right] \\ &\quad = \sum_{i=0}^{n-r} \binom{n}{i} \{\theta_t(t/Q^{-1}(x))\}^{n-i}\{1 - \theta_t(t/Q^{-1}(x))\}^{i} \\ &\quad = 1 - \int_0^{1-\theta_t(t/Q^{-1}(x))} \frac{1}{B(r, n-r+1)} z^{n-r}(1-z)^{r-1} dz. \end{aligned}$$

The result then follows from Theorem 2.5 . (we omit the details). $\square$

The following theorem gives a characterization of the rescaled beta distribution. The proof is similar to the proof of Theorem 4.3 and hence is omitted.

Theorem 4.6. *Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the order statistics from any cdf F with support $[0, \theta)$. Let also $S = 1 - F$. Assume that $Q : [0,1] \to \boldsymbol{R}_+$ is a strictly decreasing function. Then,*

$$[X_{r:n} \mid X_{1:n} > t] \overset{d}{=} [1 - (1-t)\, Q((1-t)^{-1}(1 - X_{n-r+1:n})) \mid X_{1:n} > t],$$

for some $1 \le r \le n$, *and for almost all* $t \in [0,\theta]$ *(with respect to Lebesgue measure) with* $S(t) > 0$, *if and only if* $Q(y) = (1-y^{\beta})^{1/\beta}$, $0 \le y \le 1$, *and* F *is a rescaled beta distribution of the form* $F(x) = 1-(1-x/\theta)^{\beta}$, $0 \le x < \theta$, *for some constant* $\beta > 0$.

By imposing some restrictions on the underlying cdf, one can obtain the following result which is stronger than that given in this section. The proof is omitted, since it follows easily from proofs of Theorems 4.2, 4.3, and 4.5, and the argument made for the proof of Theorem 3.6.

Theorem 4.7. *Let* $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ *denote the order statistics based on a continuous cdf* F *and survival function* S.

(i) Assume that the support of F *is* $(0,\infty)$, *and* $\lim e^{t/\lambda}S(t)$ *exists as* $t \to \infty$, *for some constant* $\lambda > 0$. *Then,*

$$E\{X_{r:n} - t \mid X_{1:n} > t\} = E\{-\lambda \log(1 - e^{-(X_{n-r+1:n}-t)}) \mid X_{1:n} > t\},\ t > 0,$$

for some $1 \le r \le n$, *if and only if* F *is an exponential distribution with mean* λ.

(ii) Assume that the support of F *is* $(0,\theta)$, *and* $\lim F(t)/t^{\alpha}$ *exists as* $t \to 0$, *for some constant* $\alpha > 0$. *Then,*

$$E\{X_{r:n} \mid X_{n:n} \le t\} = E\{(t^{\alpha} - X^{\alpha}_{n-r+1:n})^{1/\alpha} \mid X_{n:n} \le t\},\ 0 \le t \le \theta,$$

for some $1 \le r \le n$, *if and only if* F *is a power function distribution with parameter vector* (α,θ).

(iii) Assume that the support of F *is* (β,∞), *and* $\lim t^{\alpha}S(t)$ *exists as* $t \to \infty$, *for some* $\alpha > 0$. *Then,*

$$E\{X^{-1}_{r:n} \mid X_{1:n} > t\} = E\{(t^{-\alpha} - X^{-\alpha}_{n-r+1:n})^{1/\alpha} \mid X_{1:n} > t\},\ t \ge \beta,$$

for some $1 \le r \le n$, *if and only if* F *is a Pareto distribution with parameter vector* (α,β).

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