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Design Under Uncertainty: Balancing Expected Performance and Risk

The problem of quantifying uncertainty in the design process is approached indirectly. Nonquantifiable variability resulting from lack of knowledge is treated as epistemic uncertainty and quantifiable variability caused by random influences is treated as aleatory uncertainty. The emphasis in this approach is on the effects of epistemic uncertainty, left unquantified, on design performance. Performance is treated as a random function of the epistemic uncertainties that are considered as independent variables, and a design decision is based on the mean and variance of design performance. Since the mean and variance are functions of the uncertainties, multicriteria decision methods are employed to determine the preferred design. The methodology is illustrated on a three-spring model with stochastic forcing and two uncertain damping coefficients. Based on the example, the concept of balancing expected performance and risk is explored in an engineering context. Risk is quantified using aleatory uncertainty for fixed values of epistemic uncertainty. The study shows the unique features of this approach in which risk-based design decisions are made under both aleatory and epistemic uncertainties without assuming a distribution for epistemic uncertainty. [DOI: 10.1115/1.4002836]

1 Introduction

While uncertainty is understood as the inability to determine the true state of affairs of a system, there are several theories from different fields, which are in support of or in conflict with each other, on how to define and model it. The engineering community distinguishes between aleatory and epistemic uncertainty [1]. Aleatory uncertainty results from inherent variations associated with the system or its environment. Epistemic uncertainty results from lack of knowledge about the system or the environment due to scarce data, limited understanding, or fault events. The former is commonly modeled with probability theory while for the latter, the use of probability theory is limited and mathematical theories such as Bayesian estimation, evidence theory, possibility theory, and interval analysis are applied.

To account for uncertainty, the conventional (deterministic) optimization-based framework for engineering design has been extended. Type I and type II robust design approaches have been proposed to model variations in design evaluation due to variations in random parameters and variables [2,3]. Variations in design feasibility have been captured by reliability-based design in which design variables are random and the design feasibility is modeled by the probability of constraint satisfaction [4–9]. In Ref. [10], the reliability-based approach has been extended with interval variables that cannot be represented by any distribution. The authors also convincingly motivate the use of interval variables.

In engineering optimization, many authors use also fuzzy sets and possibility distributions to handle uncertainty [11]. A combined probability/possibility approach using random and fuzzy variables to account for both aleatory and epistemic uncertainties is proposed in Ref. [12]. Effects of epistemic and aleatory uncertainty are quantified in Ref. [13] using concepts of the evidence theory.

Another approach is based on Monte Carlo simulations, which generate random sample behaviors that imitate uncertainties inherent in any process [14–16]. For multicriteria design problems, the multiattribute utility theory has been used in the presence of uncertainty and risk [3,17–19]. Multidisciplinary optimization approaches have been extended to account for uncertainty but with limited modeling capabilities for interactions between disciplines or components [20–22].

In the operations research community, uncertainty has been modeled with stochastic programming [23,24], which uses probabilistic information about the problem data, and robust optimization [25,26], which integrates goal programming formulations with a scenario-based description of the data. A recent detailed study of different perspectives of uncertainty and risk is presented in Ref. [27] while a review of approaches to risk in optimization under uncertainty is given in Ref. [28].

Knight [29], a pioneer in studies on uncertainty and risk in economics, clearly distinguished between the epistemic and aleatory uncertainties long before they were called so. According to Knight, *uncertainty* is system nonquantifiable randomness arising when a system cannot be completely described because of a lack of understanding or limitation of knowledge. On the other hand, *risk* is known as system quantifiable randomness caused by stochastic variability resulting from inherent fluctuations that the system experiences with respect to time, space, or individual characteristics. We are guided by Knight and declare a variable epistemic only if not enough statistical data and engineering judgments are available about it. While other authors (e.g., Du [13]) try to quantify epistemic uncertainty, we define epistemic variables as such that cannot be quantified due to lack of information of any kind.

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We propose using random functions to model decision problems associated with design under uncertainty. In stochastic analysis, a random function is an indexed collection of vector random variables $Y(b)$, $b \in \mathbb{B}$ defined on the probability space and taking values in R^m, where $\mathbb{B} \subseteq R^n$ is an indexed parameter set. When $n=1$ and $m=1$, the concept of random function reduces to the *random process* that, in engineering, is often referred to as the times series. When $n=2$ and $m=1$, the concept of random function reduces to the *random field* (referred to as $Y(s,r)$ later) that has been used to model static images, random topographies (landscapes), composition variations of an inhomogeneous material (concrete strength, soil permeability), etc. If $m \geq 2$, we talk about *vector fields*.

In our approach, nonquantifiable variability resulting from lack of knowledge is treated as epistemic uncertainty and quantifiable variability caused by random influences is treated as aleatory uncertainty. Our approach is new in the sense that what is unknown (i.e., epistemic uncertainty) is left necessarily unmodeled but is related to the observed behavior of the system. The domain $\mathbb{B}$ represents the space of epistemic uncertainties determined by exogenous unknown parameters that do not follow any stochastic principles. The exogenous parameters might be completion time, availability of resources, physical parameters (temperature, humidity), interest rates, degree of hazard, demands/requirements, manufacturing environment, operating conditions, etc. On the other hand, the random field $Y(b)$ models the criterion evaluating system performance that is variable due to inherent fluctuations that follow stochastic principles. In the case presented in Sec. 4, a random field $Y(b_1,b_2)$ is introduced to model a criterion for designing a three-spring system with friction coefficients b_1 and b_2 being the epistemic uncertainties.

A benefit of our modeling epistemic uncertainties as independent variables is that system responses become random functions of these epistemic uncertainties. That is, the system response $Y(b)$ for fixed values of discretized epistemic uncertainties (b_1,b_2) becomes an aleatory uncertainty. However, it is challenging to estimate the joint probability density function of the overall system response. Generally, the system response $Y(b)$ is not normally distributed. In Sec. 4.3, we present an approach to converting a non-normal random field into a normal random field. This approach enables our methodology to quantify aleatory uncertainties using just the mean and the standard deviation of the field independently of the type of its original distribution.

This methodology has already been applied to design problems in automotive vehicle design: a preliminary version [30,31] and an advanced version [32]. In the current paper, we present a simple design problem under uncertainty and risk with a finite number of feasible designs. The example, despite being simple, illustrates most of the features of our methodology and points to theoretical and methodological gaps to be filled in. Under different modeling assumptions but using the same design criterion, we develop several decision models of this design problem. In each case, the design criterion is converted to a decision criterion that guides the choice of a preferred design. In particular, in a stochastic case, risk is quantified using aleatory uncertainty for fixed values of epistemic uncertainty and the concept of balancing expected performance and risk is used as a decision criterion. Methods of multicriteria decision-making (MCDM) are used to identify a design that is preferred in the entire range of uncertainties. Conceptually, our epistemic uncertainties correspond to the interval variables in Ref. [10] but our modeling and methodological approaches are different. Our model makes use of random fields rather than the reliability-based optimization and our methodology identifies a design that performs best over the entire space of uncertainties rather than only for the worst case combination.

In Sec. 2, we present a physics-based model of the design problem and introduce a design criterion. Design in a deterministic case under uncertainty is discussed in Sec. 3 while Sec. 4 presents a stochastic case. This paper is concluded in Sec. 5.

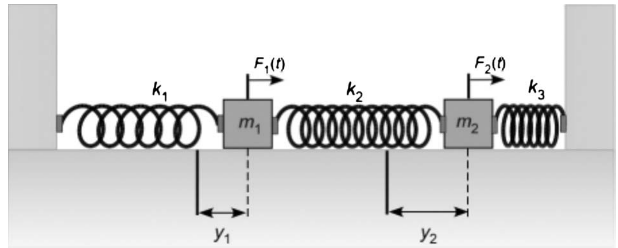

Fig. 1 Three-spring system

2 Design Problem

We believe that design problems always arise in a context dictated by a higher-level decision maker (DM) who passes the design context down to the designer. The design context is given by objectives and constraints that often come from previous design decisions. Even within a tight design context, the designer may be faced with external disturbances and uncertainties that are unknown to the higher-level DM. Allowing for both uncertainty and quantifiable randomness, and also for communication across decision levels implied by "engineering context" requires a new approach.

We start with a physics-based model of a three-spring system to quantify its stochastic performance under uncertainty and use this model solely for illustration to mathematically quantify this performance. We then develop decision models of the three-spring system, which is at the heart of our methodology and is used in the subsequent decision phase to identify a preferred design alternative.

Note that in actual applications in industry, physics-based models may not be available or may be available only for the simplest components in which case the performance to build a decision model could be obtained from archived data, simulations of virtual systems, or from informed guesses for preliminary designs.

2.1 A Physics-Based Deterministic Model. Consider the process level model of a three-spring system as our "real" system, i.e., the source of observations used to estimate second order statistics of the system performance. As depicted in Fig. 1, masses m_1 and m_2 with friction coefficients b_1 and b_2, respectively, are connected with three springs with coefficients k_1,k_2,k_3. When external time-dependent forces $F_1(t)$ and $F_2(t)$ are applied to the masses, the latter move and their displacements, $y_1(t)$ and $y_2(t)$, are observed.

The objective of the higher-level DM is to achieve a "balanced" system, which is not precisely defined by this DM. This general design criterion along with the external forces, viewed as important at the higher level, is passed down to the designer. The designer must find masses m_1 and m_2, so that a feasibility condition specified by an earlier design decision holds, for example $m_1+m_2=3$, and the system is balanced when the forces are applied.

The designer, working at the level of process design, makes use of the following mathematical model:

$$m_1 y_1'' = -k_1 y_1 - b_1 y_1' + k_2(y_2 - y_1) + F_1$$

$$m_2 y_2'' = -k_3 y_2 - b_2 y_2' - k_2(y_2 - y_1) + F_2$$

The first and second derivatives of the displacements are denoted by y_i' and y_i'', respectively, for $i=1,2$. Renaming the variables as $x_1=y_1$, $x_2=y_1'$, $x_3=y_2$, $x_4=y_2'$, and assuming for simplicity that $F_2=0$, the model can be written as

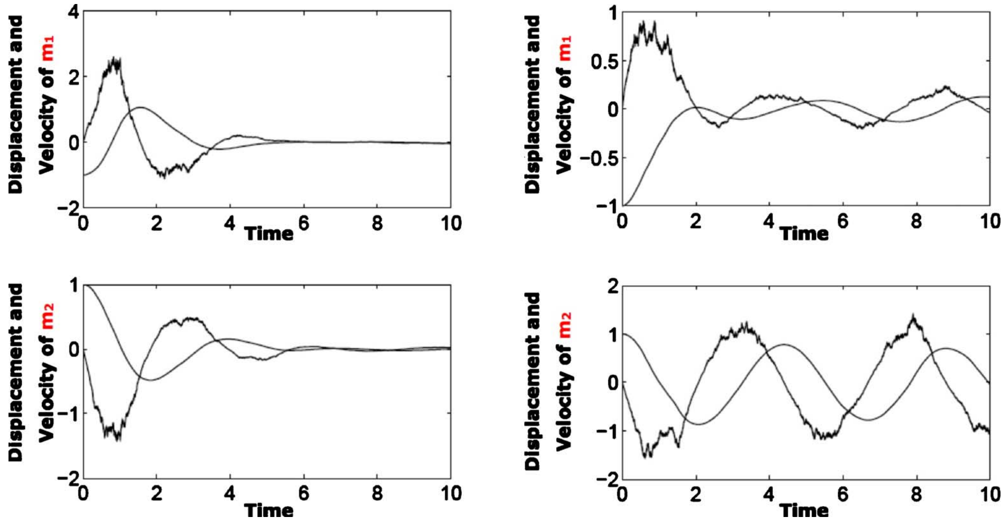

Fig. 2 Simulation for the same designs ($m_1 = m_2 = 1.5$) with two different uncertainty pairs: left (b_1, b_2) = (1.5, 1.5); right (b_1, b_2) = (4.0, 0.25). Initial conditions for displacement and velocity are $y_1(0) = -1$, $y_2(0) = 1$, and $y_1'(0) = y_2'(0) = 0$, respectively.

$$dX = (AX + \hat{F}_1)dt \tag{1}$$

where $X^T = [x_1, x_2, x_3, x_4]$, $\hat{F}_1^T = [0, (1/m_1)F_1, 0, 0]$, and A is the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(k_1 + k_2)/m_1 & -b_1/m_1 & k_2/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & 0 & -(k_2 + k_3)/m_2 & -b_2/m_2 \end{bmatrix}$$

This deterministic model is now extended to account for uncertainty and randomness.

2.2 A Physics-Based Stochastic Model With Uncertainty. We recognize that the deterministic model is not realistic because the performance of the three-spring system depends on the values of the friction coefficients, the external forces applied to the masses, and other environmental factors. At this modeling stage, uncertainty and quantifiable randomness are introduced to the model. Uncertainty enters the model through the friction coefficients b_1 and b_2 that are assumed to be uncorrelated. It is assumed that these coefficients are unknown and have no known probability distribution and can be modeled only with intervals of uncertainty. Quantifiable randomness is modeled with Wiener processes of time.

The standard Wiener process $W(s)$ (also known as the Brownian motion process) has five basic properties: (i) $W(0) = 0$; (ii) sample paths are almost surely continuous; (iii) for $0 \le s < r$, the increments $W(r) - W(s)$ are random variables and are normally distributed with the zero mean and variance $r - s$; (iv) disjoint increments are independent; and (v) the process is normal.

The external force $F_1(t)$ is random and modeled with a Wiener process $W_1(t)$, i.e., $dF_1 = -F_1 dt + w dW_1$ where $w > 0$ is a scaling factor and dW_1 is the differential of W_1. Additionally, the system experiences external disturbances exerted on the masses m_1 and m_2, which are also modeled with Wiener processes, $W_2(t)$ and $W_4(t)$. The ordinary differential equation (A) becomes the stochastic differential equation

$$dX = (AX + \hat{F}_1)dt + B(X)dW \tag{2}$$

where $B(X)dW = [0, uX_2 dW_2, 0, vX_4 dW_4]^T$ with $u, v > 0$ being scaling factors, and dW_2 and dW_4 being the differentials of W_2 and W_4, respectively. For illustrative purposes assume that $k_1 = 3$, $k_2 = 1$, $k_3 = 2$, $X(0) = [-1, 0, 1, 1]^T$, and $(w, u, v) = (1/10, 1/2, 1/3)$.

2.3 A Design Criterion. The final task in the modeling process is to define a criterion that will effectively measure the design performance and according to which feasible designs could be compared.

To obtain a better understanding of how to measure the system performance, we simulate the external disturbances and record the displacement and velocity of the masses. We use the Milstein algorithm [33] to simulate the stochastic differential Eq. (2). Using this algorithm, the convergence is known to be second order for the computed statistics of the system output.

For each choice of sample disturbances, the velocity of each mass is a random process and we cannot predict the velocity at a later time given the velocities up to and including the velocity at time t_1. Based on the simulations, we only know the distribution of displacements and velocities at a later time. Figure 2 depicts the displacement and velocity of both masses ($m_1 = m_2 = 1.5$) for two different pairs of friction coefficients, $(b_1, b_2) = (1.5, 1.5)$ and $(b_1, b_2) = (4.0, 0.25)$. This example illustrates different performances of the same design at two different uncertainty pairs and points to the difficulty of comparing designs that exhibit stochastic behavior.

Since the general objective is to achieve a balanced system, one may think of the energy dissipated by the masses m_1 and m_2. Figure 3 depicts the energy dissipated by the masses whose ve-

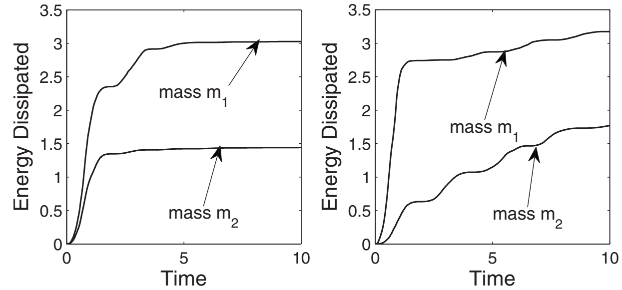

Fig. 3 Simulation for the same designs ($m_1 = m_2 = 1.5$) with two different uncertainty pairs: left: (b_1, b_2) = (1.5, 1.5); right: (b_1, b_2) = (4.0, 0.25)

locity and displacement are shown in Fig. 2. To calculate the energy, we rewrite spring equations (A) that balance forces on the two masses

$$m_1 y_1'' = -(k_1 + k_2) y_1 - b_1 y_1' + k_2 y_2 + F_1(t)$$

$$m_2 y_2'' = k_2 y_1 - (k_2 + k_3) y_2 - b_2 y_2' + F_2(t)$$

The two forces acting in the direction opposite the motion, dissipating the system's initial energy, are $b_1 y_1'$ and $b_2 y_2'$. The increments of energy lost, equivalent to the work done by these two forces, are $b_1 y_1' dy_1$ and $b_2 y_2' dy_2$. Calculate the energy dissipated by the mass m_1 and m_2, respectively, as follows:

$$H_1(b_1,b_2) = \int_0^\infty b_1 y_1' dy_1(t) = \int_0^\infty b_1 x_2^2 dt$$

and

$$H_2(b_1,b_2) = \int_0^\infty b_2 y_2' dy_2(t) = \int_0^\infty b_2 x_4^2 dt$$

The initial potential energy of the system is the total work done on the masses m_1 and m_2 by the system, namely, $H_1(b_1,b_2) + H_2(b_1,b_2)$. Eventually, the system comes to rest with zero potential energy. The initial energy is dissipated by the friction of the masses. A reasonable notion of balance is that $H_1(b_1,b_2) = H_2(b_1,b_2)$, i.e., the masses dissipate equal amounts of the initial potential energy. Assume that this criterion is a reasonable interpretation of the upper level objective of the system being balanced. Introduction of this design criterion is a creative act on the part of the designer.

The *design criterion* is a number $Y(b_1,b_2)$ calculated from the following expression:

$$Y(b_1,b_2) = |H_1(b_1,b_2) - H_2(b_1,b_2)| \tag{3}$$

where the integrals above are calculated on the finite interval [0,10]. The objective of the design problem is to find the best performing design among finitely many designs allowing different combinations of the masses m_1 and m_2. For demonstration, assume five feasible designs satisfying the constraint $m_1 + m_2 = 3$ and given by the pairs (m_1, m_2) as follows:

$$d_1 = (0.5, 2.5), \quad d_2 = (1.0, 2.0), \quad d_3 = (1.5, 1.5)$$

$$d_4 = (2.0, 1.0), \quad d_5 = (2.5, 0.5)$$

In effect, criterion Y is a function of (m_1, m_2, b_1, b_2). Referencing one of the five designs, particular choices of m_1 and m_2, we use Y_i, $i = 1, 2, \ldots, 5$. For particular values of the uncertainties (b_1, b_2), we use $Y(b_1, b_2)$ or for a particular design $Y_i(b_1, b_2)$.

The design problem can now be formulated as the following multiobjective optimization problem (MOP):

$$\min\{Y(m_1,m_2,b_1,b_2) | (b_1,b_2) \in \mathbb{B}\}$$

$$s.t. (m_1,m_2) \in \{\text{set of feasible designs}\} \tag{4}$$

where $\mathbb{B}$ denotes the uncertainty space. The multiple objectives of this MOP are related to the design criterion calculated at multiple points $b = (b_1, b_2)$ of the uncertainty space.

In the following sections, we solve the three-spring design problem by solving this MOP under different assumptions. We examine a deterministic case with epistemic uncertainty and two stochastic cases, one without and the other with epistemic uncertainty. In each case, we use either the proposed design criterion or a decision criterion being a deterministic counterpart of the design criterion. We solve the MOP following the standard methodology of first finding the feasible designs that are efficient in the Pareto sense, and then identifying a preferred design from among the efficient ones by employing MCDM methods. Note that we do not solve an optimization problem but rather a selection problem to identify efficient designs. In effect, our approach does not depend upon the solvability of any optimization problem, which is a typical limitation of many decision methods.

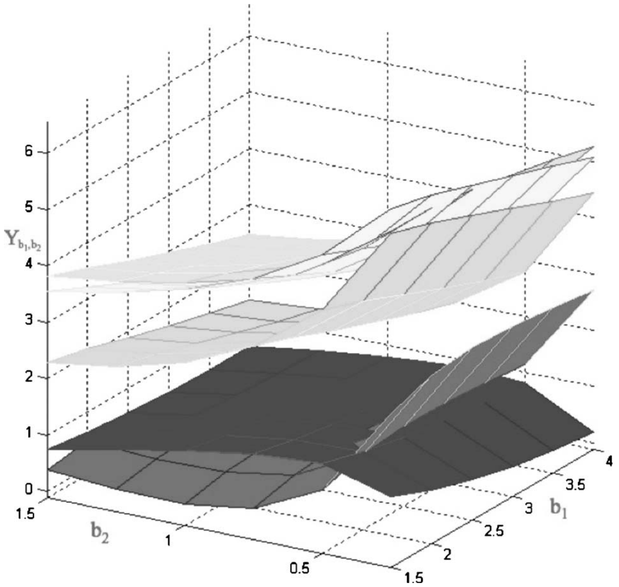

Fig. 4 Performance surfaces for the feasible designs. The two lowest surfaces represent the performances of design 4 (very dark gray) and design 5 (black).

3 Design in a Deterministic Case Under Epistemic Uncertainty

The three-spring system is assumed to be deterministic, i.e., the scaling factors $w = u = v = 0$. Let the intervals $[b_1^l, b_1^u]$ and $[b_2^l, b_2^u]$, where $b_1^l < b_1^u$ model the intervals of uncertainty for the friction coefficients b_1 and b_2, respectively. Since these coefficients are assumed to be uncertain, it is absolutely impossible to state that one value is more likely than another or that they are equally likely. Thus, each combination of values attained by b_1 and b_2 are considered independent. Naturally, it is impossible to compute the design criterion for all these combinations since an interval of the real line contains uncountably many numbers. The design criterion, however, is a continuous function of b_1 and b_2, so each interval of uncertainty can be discretized into sufficiently small intervals and the design criterion can be computed at the grid points of the discretization. Henceforth, each interval of uncertainty is represented by its grid points.

Assume the uncertain friction coefficients b_1 and b_2 take values in the intervals $[b_1^l, b_1^u] = [1.5, 4.0]$ and $[b_2^l, b_2^u] = [0.25, 1.5]$, respectively. The MOP given by Eq. (4) becomes

$$\min\{Y(m_1,m_2,b_1,b_2) | b_1 \in [1.5,4.0], b_2 \in [0.25,1.5]\}$$

$$s.t. (m_1,m_2) \in \{\text{set of feasible designs}\} \tag{5}$$

Due to the discretization, the number of grid points in each interval of uncertainty is finite and so is the number of objectives in this MOP. For each feasible design, we calculate design criterion (3) at all grid points using deterministic Eq. (1) and construct a performance surface made up of the computed values of the design criterion. Figure 4 depicts the performance surfaces for all five feasible designs.

Solving the MOP involves identifying the feasible designs that are efficient (in the Pareto sense).

DEFINITION 3.1. *Design iis said to dominate design j provided for all $b \in \mathbb{B}: Y_i(b) \le Y_j(b)$ with at least one strict inequality. Design i is said to be efficient if it is not dominated by any other*

Table 1 Distances and volumes between the nondominated surface of designs 4 and 5 and the ideal surface

	d_4	d_5
Max-ordering	2.9514	1.4812
ℓ_1	14.5550	10.1747
ℓ_2	5.5897	2.3435
ℓ_∞	2.5385	0.9796
Volume	0.5618	0.9376

feasible design. The performance surface associated with efficient design is called nondominated.

Figure 4 shows that designs 4 and 5 dominate all other feasible designs and are efficient while their surfaces are nondominated.

3.1.1 Preference Rule. If there exists a single efficient design, then this design is the *preferred design*. Otherwise, in order to find a preferred design, we apply two MCDM methods. First, we apply the max-ordering approach [34] and the efficient design that solves the min-max problem

$$\min_{i=4,5} \max_{b\in\mathbb{B}} Y_i(b) \tag{6}$$

becomes preferred.

Second, we apply the minimum-norm approach [34]. We determine the lower envelope of all nondominated surfaces, which we refer to as the ideal surface $Y^{\text{ideal}}(b)$:

$$Y^{\text{ideal}}(b) = \min_{i=1,\ldots,5} Y_i(b)$$

Note that the ideal surface may not be associated with any individual efficient design but some or even all designs may contribute to it. We then find the preferred design whose nondominated surface is the closest to the ideal surface by solving

$$\min_{i=4,5}\{\|Y_i(b) - Y^{\text{ideal}}(b)\|, b \in \mathbb{B}\} \tag{7}$$

where $\|\cdot\|$ denotes the norm. Additionally, as another preference rule, we calculate the volume between each nondominated surface and the ideal surface. Table 1 reports the optimal objective values (distances) for the designs 4 and 5 for problem (6), problem (7) with ℓ_1, ℓ_2 and ℓ_∞-norm, and also the volume. Based on the first four measures, design 5 is preferred while design 4 is preferred with respect to the volume measure.

In this paper, the efficient set is presented solely for demonstration purposes because the MCDM methods for finding a preferred design could be applied directly to the feasible rather than efficient set of designs.

4 Design in a Stochastic Case

Designs based on stochastic Eq. (2) recognize quantifiable variability affecting the three-spring system coming from the stochastic external force $F_1(t)$ and the external disturbances exerted on the masses m_1 and m_2. The system performance is stochastic and is examined without and with epistemic uncertainty originating from the design parameters b_1 and b_2. The methods of MCDM used in the deterministic case above are extended to the corresponding classes of decision problems. Notice that there is overlap between the approaches to the three classes of problems.

4.1 Design With No Epistemic Uncertainty. Unlike the deterministic case, there is no epistemic uncertainty but quantifiable random variability. For fixed values of the friction coefficients $b=(b_1,b_2)$ and a design (m_1,m_2), performance $Y(b)=|H_1(b)-H_2(b)|$ is a random variable. Simulating the stochastic differential Eq. (2) 20,000 times, we calculate the sample mean μ_k of the performance $Y_k(b), k=1,\ldots,5$ and estimate the probability $P(Y_j \geq \mu_k)$ for $j=1,\ldots,5$.

DEFINITION 4.1. *Risk associated with design j relative to design k is the probability that the performance of design j exceeds the expected performance of design k,* $\text{risk}_{kj}=P(Y_j\geq\mu_k)$.

Also, introduce a decision surrogate

$$f_{kj} = \text{risk}_{kj}\mu_k$$

which is a measure of how badly design j performs relative to design k or the "exposure" of design j relative to design k. Taking the decision problem as a multicriteria decision problem, each f_{kj}, $k=1,2,\ldots,n$, can be thought of as the performance of design j with respect to the kth criterion.

DEFINITION 4.2. *Design j is said to dominate design i if* $f_{kj}\leq f_{ki}$ *for all k, and* $f_{kj}<f_{ki}$ *for at least one k,* $k=1,2,3,4,5$. *Design j is said to be efficient if there does not exist another design that dominates design j.*

If there exists a single efficient design, then this design is the *preferred design*.

Given the five feasible designs and fixing $b_1=1.5$ and $b_2=1.5$, we have a 5×5 matrix of f_{kj} values:

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} \end{bmatrix} = \begin{bmatrix} 0.4549 & 0.4087 & 0.2660 & 0.1231 & 0.0883 \\ 0.4605 & 0.4113 & 0.2678 & 0.1240 & 0.0890 \\ 0.7275 & 0.5925 & 0.3835 & 0.1848 & 0.1276 \\ 0.9372 & 0.8288 & 0.5897 & 0.3311 & 0.2475 \\ 0.9535 & 0.8631 & 0.6351 & 0.3714 & 0.2980 \end{bmatrix}$$

Column j represents the decision surrogate f_{kj} for design j associated with designs $k=1,\ldots,5$. For example, design j is efficient, if $f_{kj}\leq f_{ki}$ for $k=1,2,\ldots,5$ and $i=1,\ldots,5$ and $f_{kj}<f_{ki}$ for at least one $k=1,\ldots,5$ and $i=1,\ldots,5$. Design 5 is the only efficient design and hence it is the least-risk design based on 20,000 simulation runs. Design 5 is then the preferred design.

4.1.1 Preference Rule. If there is no single least-risk or efficient design, then a preference rule can be introduced by extending the preference rule used in the deterministic case. Define an ideal vector f^{ideal} with components

$$f_k^{\text{ideal}} = \min_{j=1,\ldots,5} f_{kj}, \quad k=1,\ldots,5$$

The preferred design is the efficient design that solves

$$\min_{j=1,\ldots,5} \|f_{\cdot j} - f^{\text{ideal}}\| \tag{8}$$

that is the design whose decision surrogate is the closest to f^{ideal}, measured in terms of a norm (e.g., the ℓ_2-norm). In the absence of a least-risk design, this preference rule, one of many possibilities, offers a way for the designer to shape the risk profile of the ultimate choice.

4.1.2 Reduction to the Deterministic Case. Note that when "dominance" is reduced to the deterministic case, $Y_i=\mu_i\geq0$ for each design $i=1,2,\ldots,5$. The distribution of Y_i treated as a trivial random variable is given by $P(Y_i\neq\mu_i)=0$ and $P(Y_i=\mu_i)=1$. To check for dominance between two designs, the comparison of the decision surrogate between them reduces to the comparison between their performances (or means).

Consider two designs, j and i, and assume that $\mu_j=Y_j<Y_i=\mu_i$. Then, by Definition 4.1, $0\mu_i=f_{ij}<f_{ii}=1\mu_i$. Consider also all other combinations of the magnitude of the means μ_i, μ_j, and μ_k.

If $\mu_k \le \mu_j < \mu_i$ and since $Y_j = \mu_j$, we obtain $P(Y_j \ge \mu_k) = 1 = P(\mu_i \ge \mu_k)$ and, again by Definition 4.1, $f_{kj} = f_{ki} = \mu_k$. Based on Definition 4.1, we examine the other cases:

If $\mu_j \le \mu_k < \mu_i$ then $P(Y_i \ge \mu_k) = 1$ and $f_{kj} = P(Y_j \ge \mu_k)\mu_k \le \mu_k = f_{ki}$.
If $\mu_j < \mu_k \le \mu_i$ then $P(Y_j \ge \mu_k) = 0 < P(\mu_i \ge \mu_k) = 1$ and $f_{kj} = 0 < \mu_k = f_{ki}$.
If $\mu_j < \mu_i \le \mu_k$ then $P(Y_j \ge \mu_k) = 0$ and $f_{kj} = 0 \le f_{ki}$.
If $\mu_j < \mu_i < \mu_k$ then $P(Y_j \ge \mu_k) = 0 = P(Y_i \ge \mu_k)$ and $f_{kj} = f_{ki} = 0$.

Based on all the cases, $f_{kj} \le f_{ki}$ for all k and $f_{ij} < f_{ii}$, which means that if $\mu_j = Y_j < Y_i = \mu_i$, design j dominates design i.

Conversely, assume that design j dominates design i and outperforms design i with respect to criterion k, i.e., $f_{kj} < f_{ki}$. Then, by Definition 4.1, $P(Y_j \ge \mu_k) < P(Y_i \ge \mu_k)$, which yields $Y_j < Y_i$.

Therefore, when the system is deterministic, design j dominates design i if and only if $Y_j < Y_i$. In effect, the new rule implied by Definitions 4.1 and 4.2 is consistent with the usual idea of dominance for the deterministic case.

Note that this case serves as a foundation for our approach in the sense that the approach stems from the stochastic case with no epistemic uncertainty, as presented in this subsection, but is meant to be used for stochastic systems with epistemic uncertainty.

4.2 Design With Finitely Many Values of Epistemic Uncertainty. To build up the stochastic case, assume a finite number of uncertainty values. For simplicity, assume that b can be either $b^1 = (1.5, 1.5)$ or $b^2 = (1.5, 0.5)$. We are to choose a design without knowing the value of b. Let f_{kj}^1 and f_{kj}^2 denote the surrogate for b^1 and b^2, respectively. The matrix of surrogates can be formed as follows:

$$\begin{bmatrix} f_{11}^1 & f_{12}^1 & f_{13}^1 & f_{14}^1 & f_{15}^1 \\ f_{21}^1 & f_{22}^1 & f_{23}^1 & f_{24}^1 & f_{25}^1 \\ f_{31}^1 & f_{32}^1 & f_{33}^1 & f_{34}^1 & f_{35}^1 \\ f_{41}^1 & f_{42}^1 & f_{43}^1 & f_{44}^1 & f_{45}^1 \\ f_{51}^1 & f_{52}^1 & f_{53}^1 & f_{54}^1 & f_{55}^1 \\ f_{11}^2 & f_{12}^2 & f_{13}^2 & f_{14}^2 & f_{15}^2 \\ f_{21}^2 & f_{22}^2 & f_{23}^2 & f_{24}^2 & f_{25}^2 \\ f_{31}^2 & f_{32}^2 & f_{33}^2 & f_{34}^2 & f_{35}^2 \\ f_{41}^2 & f_{42}^2 & f_{43}^2 & f_{44}^2 & f_{45}^2 \\ f_{51}^2 & f_{52}^2 & f_{53}^2 & f_{54}^2 & f_{55}^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4549 & 0.4087 & 0.2660 & 0.1231 & 0.0883 \\ 0.4605 & 0.4113 & 0.2678 & 0.1240 & 0.0890 \\ 0.7275 & 0.5925 & 0.3835 & 0.1848 & 0.1276 \\ 0.9372 & 0.8288 & 0.5897 & 0.3311 & 0.2475 \\ 0.9535 & 0.8631 & 0.6351 & 0.3714 & 0.2980 \\ 0.4558 & 0.4482 & 0.2882 & 0.0811 & 0.0577 \\ 0.4042 & 0.4145 & 0.2698 & 0.0751 & 0.0537 \\ 0.6914 & 0.6098 & 0.3872 & 0.1165 & 0.0801 \\ 0.9712 & 0.9311 & 0.7275 & 0.3272 & 0.3341 \\ 0.9718 & 0.9334 & 0.7319 & 0.3313 & 0.3396 \end{bmatrix}$$

Definition 4.2 is extended to the case with finitely many uncertainty values. For the example case, design j is said to dominate design i if $f_{kj}^1 \le f_{ki}^1$ and $f_{kj}^2 \le f_{ki}^2$ for all k with strict inequality for at least one k, $k = 1, \ldots, 10$. Designs 4 and 5 are efficient.

4.2.1 Preference Rule. Since two designs are efficient, a preference can be introduced by extending the rule applied in the case of one uncertainty value. Define an ideal vector with the components

$$f_k^{\text{ideal}} = \begin{cases} \min_{1 \le j \le 5} f_{kj}^1, & 1 \le k \le 5 \\ \min_{1 \le j \le 5} f_{(k-5)j}^2, & 6 \le k \le 10 \end{cases}$$

and solve problem (8). Using the ℓ_2-norm, design 4 has the distance of 0.1429 to f^{ideal} and design 5 has the distance of 0.0108 to f^{ideal}. Hence, design 5 is preferred. Remember that this is only one way for making the final decision.

The two uncertainty value case can be extended to any small number of values. The computational burden becomes too heavy for implementation when the number of uncertainty values becomes large.

4.3 Design With Intervals of Epistemic Uncertainty Values. When epistemic uncertainty is modeled with an interval of values, the design criterion $Y(b_1, b_2)$, $b \in \mathbb{B}$, representing the performance of the three-spring system, becomes a random field over the uncertainty space. The second order statistics for the random performance as a function of the uncertainties requires the expected performance, in the example, a function of two variables, and the covariance of the performance, a function of four variables. Our computations involving system performance are simplified if we shift our attention to $Y = H_1(b_1, b_2) - H_2(b_1, b_2)$ instead of $|H_1(b_1, b_2) - H_2(b_1, b_2)|$. The design objective is that Y should be close to zero, i.e., having a small mean and variance. The second order statistics of $Y(b_1, b_2)$, $b \in \mathbb{B}$ are the expected performance

$$\bar{\mu}(b_1, b_2) = E[Y(b_1, b_2)]$$

the covariance kernel

$$\bar{R}(b_1^i, b_2^j, b_1^k, b_2^\ell) = E[(Y(b_1^i, b_2^j) - \bar{\mu}(b_1^i, b_2^j))(Y(b_1^k, b_2^\ell) - \bar{\mu}(b_1^k, b_2^\ell))]$$

and the variance

$$\bar{\sigma}^2(b_1, b_2) = \text{var}[Y(b_1, b_2)] = \bar{R}(b_1, b_2, b_1, b_2)$$

Let $\bar{\mu}_i(b_1, b_2)$, $\bar{R}_i(b_1^i, b_2^j, b_1^k, b_2^\ell)$, and $\bar{\sigma}_i^2(b_1, b_2)$ denote the expected performance, covariance kernel, and variance, respectively, for design $i, i = 1, \ldots, 5$. Figure 5 depicts the surfaces of expected performance and variance for all five feasible designs. Notice the difference in scales for the means and variances.

We turn to the question of incorporating the variances of designs 4 and 5 in the decision process. Since our design preference is based on simulations of the three-spring system, we are limited to estimates of the second order statistics for discrete values of the uncertainties. We will not discuss convergence but the smoothness of the functions used in the decision methodology strongly suggests that the estimates converge and hence the preference is robust. Assume that $\{b_1^i\}_{i=1}^m$ and $\{b_2^j\}_{j=1}^n$ are partitions of $[b_1^l, b_1^u] = [1.5, 4.0]$ and $[b_2^l, b_2^u] = [0.25, 1.5]$, respectively. A complete second order description of the uncertain stochastic system requires the covariance of the response field. Even a modest discretization of the uncertainty space, say, $b_1^i, b_1^k \in \{1.5, 1.75, \ldots, 4.0\}$ and $b_2^j, b_2^\ell \in \{0.25, 0.375, \ldots, 1.5\}$, would require simulating the system with $121^2 = 14{,}641$ different pairs of friction coefficient values. This is a very heavy computational burden.

For our example the situation is much simpler, i.e., we assume that $\bar{R}(b_1^i, b_2^j, b_1^k, b_2^\ell) = 0$ if $(b_1^i, b_2^j) \ne (b_1^k, b_2^\ell)$, and so the estimation of the discrete $\bar{R}$ requires that we only consider the "diagonal" of $\bar{R}$, which is the variance $\bar{\sigma}^2$. We will continue with a discussion of the decision process and will later offer some computational evidence that this simpler situation for the example does hold.

Since we have simulated the system for $b = (b_1^i, b_2^j)$, where $i, j = 1, 2, \ldots, 11$ and all five designs, we could apply the decision methods of the last subsection. However, because of the size of

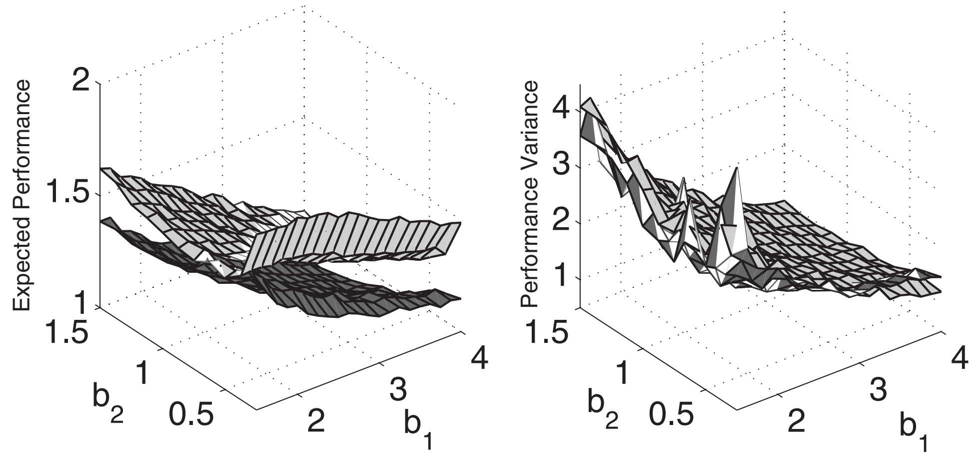

Fig. 5 Expected performance surfaces $\bar{\mu}_i(b_1,b_2)$ (left) and variance surfaces $\bar{\sigma}_i^2(b_1,b_2)$ (right) for design 4 (light gray) and design 5 (dark gray)

the decision surrogates, the interpretation of risk and the justification of the preferred choice become hazy. Further, we would have trouble making a convergence argument, i.e., that the preferred choice does not change with finer partitions of [1.5,4.0] and [0.25,1.5]. We turn to an alternate approach that avoids these difficulties.

4.3.1 A Decision Surrogate. Our objective is to convert the random field Y into a normal field whose statistics will carry information about $\bar{\mu}$ and $\bar{\sigma}^2$. Let $\bar{Y}(b),\ \ b\in\mathbb{B}$ be a random field with mean zero defined as

$$\bar{Y}(b) = H_1(b) - H_2(b) - E[H_1(b) - H_2(b)]$$

Using our assumption that

$$\bar{R}(b_1^i,b_2^j,b_1^k,b_2^\ell) = E[\bar{Y}(b_1^i,b_2^j)\bar{Y}(b_1^k,b_2^\ell)] = 0$$

when $(b_1^i,b_2^j)\neq(b_1^k,b_2^\ell)$, we have the property of statistical independence for $\bar{Y}$. Define a random field $Z(b),\ \ b\in\mathbb{B}$

$$Z(b_1,b_2) = \int_{1.5}^{b_1}\int_{0.25}^{b_2}\bar{Y}(u,v)(du)^{1/2}(dv)^{1/2}$$

then, by the central limit theorem, Z is a normal field with mean zero [27]. We now calculate the variance of Z

$$\sigma^2(b_1,b_2) = E[Z(b_1,b_2)Z(b_1,b_2)] = \int_{1.5}^{b_1}\int_{0.25}^{b_2}E[\bar{Y}^2(u,v)]dudv$$

$$= \int_{1.5}^{b_1}\int_{0.25}^{b_2}\bar{\sigma}^2(u,v)dudv$$

Note that we do not need to simulate Z in order to compute its variance σ^2 because we can compute it from the variance of Y. However, field Z has zero mean and does not carry any information about $\bar{\mu}$. We therefore define μ by applying to $\bar{\mu}$ the transformation

$$\mu(b_1,b_2) = \int_{1.5}^{b_1}\int_{0.25}^{b_2}E[\bar{Y}(u,v)]dudv = \int_{1.5}^{b_1}\int_{0.25}^{b_2}\bar{\mu}(u,v)dudv$$

and construct field $\mu+Z$. This field is random and determined by μ and σ. Note also that the distribution of the random variable $\mu(b_1,b_2)+Z(b_1,b_2)$ is determined by $\mu(b_1,b_2)$ and $\sigma(b_1,b_2)$ but the distribution of the random variable $Y(b_1,b_2)$ is not determined by $\bar{\mu}(b_1,b_2)$ and $\bar{\sigma}(b_1,b_2)$. Typically, the surfaces given explicitly by μ and σ are smoother than those given by $\bar{\mu}$ and $\bar{\sigma}$.

Since $\mu+Z$ is normal, we use a decision surrogate

$$f(b_1,b_2) = \mu(b_1,b_2) + \alpha\sigma(b_1,b_2)$$

with a fixed value α, $\alpha\geq 0$. The expression $\mu_j+\alpha\sigma_j$ for $\alpha>0$ calculated for design j has the probabilistic interpretation that, if, say, $\alpha=1$, approximately 16% of the time the values of μ_j+Z_j will exceed $\mu_j+\sigma_j$. Thus, we can use $\mu_j+\alpha\sigma_j$ as the *decision criterion* for identifying the designs balancing expected performance and risk, i.e., the designs with "minimum" $\mu_j+\sigma_j$. The fixed α reflects the decision maker's tolerance for risk. In effect, we make risk dependent on the uncertainties $(b_1,b_2)\in\mathbb{B}$ and define it as follows:

DEFINITION 4.3. *Risk associated with design i is the probability of an undesirable outcome when design i is used and is calculated as*

$$\text{risk}_i(b_1,b_2) = \text{Prob}(\mu_i(b_1,b_2) + Z_i(b_1,b_2) > \mu_i(b_1,b_2) + \alpha\sigma_i(b_1,b_2)) \tag{9}$$

for a fixed value α, $\alpha\geq 0$.

The advantage of using this definition is that, since the risk tolerance α is the same for all designs, the DM can concentrate on the magnitude of the decision criterion, resulting from that level of probability and choose the design that yields the smallest magnitude. In this case, $\mu(b_1,b_2)+\alpha\sigma(b_1,b_2)$ is minimized. For $\alpha<0$, similar statements can be made and $\mu(b_1,b_2)+\alpha\sigma(b_1,b_2)$ is maximized.

We can now proceed in the same manner as for the other cases. We define domination between designs according to Definition 3.1 and introduce a preference rule. Let $f_i(b_1,b_2)$ denotes the decision surrogate for design i.

DEFINITION 4.4. *Design i is said to dominate design j provided for all* $(b_1,b_2)\in\mathbb{B}: f_i(b_1,b_2)\leq f_j(b_1,b_2)$ *with at least one strict inequality*.

Let an ideal surrogate surface be defined as

$$f^{\text{ideal}}(b_1^k,b_2^\ell) = \min_{j=1,\ldots,5}\ \mu_j(b_1^k,b_2^\ell) + \alpha\sigma_j(b_1^k,b_2^\ell)$$

for $k=1,2,\ldots,m$ and $\ell=1,2,\ldots,n$. We find the preferred design whose nondominated surface is the closest to the ideal surrogate surface by solving

$$\min_{i=4,5}\{\|f_i(b) - f_i^{\text{ideal}}(b)\|_2, b\in\mathbb{B}\} \tag{10}$$

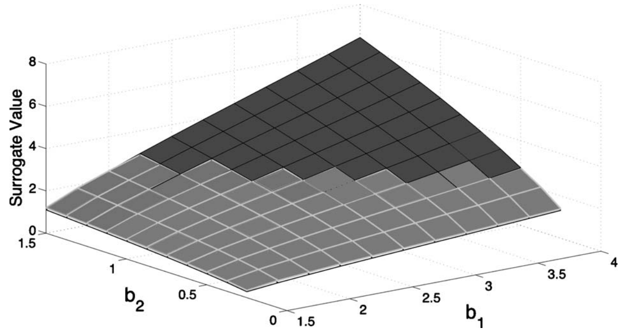

Fig. 6 Decision surrogate surfaces $\mu_i(b_1^k,b_2^\ell)+\alpha\sigma_i(b_1^k,b_2^\ell)$, $k=1,\ldots,m$ and $\ell=1,\ldots,n$ for designs 4 (gray) and 5 (black)

Designs 4 and 5 are efficient and their surrogate surfaces are depicted in Fig. 6. Solving problem (8), we obtain the optimal value of 1.8052 for design 4 and 0.5882 for design 5. Design 5 is again the preferred design.

The designer might now pass the preferred design (2.5,0.5) to the higher-level DM. Recall, however, that the latter does not know or possibly understand the three-spring model and so the two masses are meaningless. Furthermore, the uncertainties are also meaningless, so that the ordered pair of functions of the uncertainties, $(\mu_5(b_1,b_2),\sigma_5(b_1,b_2))$, is meaningless too. However, it seems very reasonable for the lower-level designer to pass up the pair $(\mu_5(b_1^*,b_2^*),\sigma_5(b_1^*,b_2^*))$, where (b_1^*,b_2^*) produces the maximum value of the function $\mu_3(b_1,b_2)+\alpha\sigma_3(b_1,b_2)$, which is *the estimate of the worst case performance of the preferred design*. This estimate is a unique feature of our approach and may be used by the upper-level DM as the measure of the balance of the system while treating its performance as a function of the passed down input F_1.

4.3.2 Experimental Evidence of Statistical Independence. We provide experimental evidence that the assumption

$$\bar{R}(b_1^i,b_2^j,b_1^k,b_2^\ell)=0$$

for $(b_1^i,b_2^j)\neq(b_1^k,b_2^\ell)$ is reasonable. Because of the burden of a complete computation, we will rely on sampling and examine a sample distribution of $\bar{R}(b_1^i,b_2^j,b_1^k,b_2^\ell)=E[\bar{Y}(b_1^i,b_2^j)\bar{Y}(b_1^k,b_2^\ell)]$ for $(b_1^i,b_2^j)\neq(b_1^k,b_2^\ell)$. For random choices of (b_1^i,b_2^j) and (b_1^k,b_2^ℓ), $(b_1^i,b_2^j)\neq(b_1^k,b_2^\ell)$, we observe that $\bar{Y}(b_1^i,b_2^j)$ and $\bar{Y}(b_1^k,b_2^\ell)$ are uncorrelated. For instance, the correlation coefficient of $\bar{Y}(b_1^6,b_2^{10})$ and $\bar{Y}(b_1^{11},b_2^7)$ is -0.0050. For a random sample $\mathcal{S}$ of four-tuples $(b_1^i,b_2^j,b_1^k,b_2^\ell)$ of size 121, chosen without replacement and $(b_1^i,b_2^j)\neq(b_1^k,b_2^\ell)$, the sample mean of $\bar{R}(\mathcal{S})$ is 2.4220×10^{-4} and the sample variance is 7.8929×10^{-6}. Figure 7 depicts a histogram of a sample distribution of $\bar{R}(b_1^i,b_2^j,b_1^k,b_2^\ell)$.

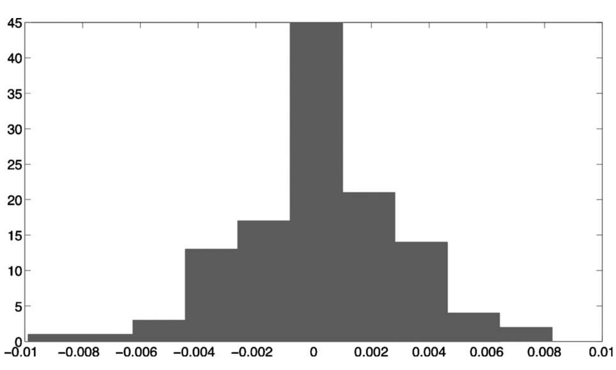

Fig. 7 Histogram of a sample distribution of $\bar{R}(b_1^i,b_2^j,b_1^k,b_2^\ell)$

We conclude that assuming $\bar{R}(b_1^i,b_2^j,b_1^k,b_2^\ell)=0$ for $(b_1^i,b_2^j)\neq(b_1^k,b_2^\ell)$ is a reasonable modeling approximation.

5 Conclusion

The study shows unique features of this approach in which risk-based design decisions are made under both aleatory and epistemic uncertainties without assuming a distribution for epistemic uncertainty. In engineering literature, our risk is associated with aleatory uncertainty. Uncertainty in our sense is associated with epistemic uncertainty. Our departure from more standard engineering approaches is in modeling system performance with uncertainties (epistemic uncertainties) as independent variables.

The three-spring example illustrates our approach to decision making under uncertainty and risk. It consists of three phases: modeling, computational, and decision phases. In the modeling phase, a decision model of the design problem is developed based on the associated engineering context. This phase requires an experienced and knowledgeable designer who is able to formulate the design problem according to our modeling requirements. The computational phase requires the development of computational models of random fields. The resulting decision models for problems with two uncertainties use the decision criterion $\mu(b_1,b_2)+\sigma(b_1,b_2)$, balancing expected payoff and risk. Finally, the decision phase requires the integration of designer's preferences in order to select a preferred design.

Our approach naturally addresses *randomness* and *uncertainty* and enables designers to rapidly and easily assess a large number of design alternatives under uncertainty and risk. The approach identifies a preferred design alternative over all uncertain conditions, balancing expected performance and risk, and is equipped with a variety of preference rules to effectively support the *decision-making process*. The epistemic uncertainties in the three-spring problem are uncorrelated. However, this does not have to be the case in general. In Ref. [35], a set of operator algebra rules is developed to handle cases with correlated uncertainties while an illustrative example with correlated uncertainties is presented in Ref. [36].

Particular sets of uncertainty values can be thought of as scenarios. The methodology promotes strategic thinking, i.e., consideration of multiple scenarios, even a continuum of scenarios. In the context of multicriteria methodologies, single-criterion problems with multiple scenarios are the same as multicriteria problems with a single scenario [26]. The introduction of performance as a random function of the uncertainties allows us to make use of the many tools of abstract analysis.

The independence condition introduced in the interval case is a property of the physics-based three-spring model. Other implications of this important property remain to be explored, including simplified representations of the all important covariance kernel R.

While not discussed in this paper, our methodologies based on multicriteria methods satisfy Savage's first four postulates guaranteeing rational decisions. This is important in light of Hazelrigg's powerful criticism of current design selection methods [37]. Further, the use of multicriteria methods is an important consideration for decisions in the presence of uncertainty as illustrated by a discussion of Ellsberg's famous urn paradox in Ref. [38].

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References

[1] Helton, J. C., and Oberkampf, W. L., 2004, "Special Issue: Alternative Repre-

sentations of Epistemic Uncertainty," Reliab. Eng. Syst. Saf., **85**, pp. 1–10.
[2] Chen, W., Garimella, R., and Michelena, N., 2001, "Robust Design for Improved Vehicle Handling Under a Range of Maneuver Conditions," Eng. Optimiz., **33**(3), pp. 303–326.
[3] Dai, Z., Scott, M. J., and Mourelatos, Z. P., 2004, "Improving Robust Design With Preference Aggregation Methods," Proceedings of the Society of Automotive Engineers World Congress, Paper No. 2004-01-1140.
[4] Du, X., and Chen, W., 2000, "Towards a Better Understanding of Modeling Feasibility Robustness in Engineering Design," ASME J. Mech. Des., **122**(4), pp. 385–394.
[5] Youn, B. D., Cho, K. K., and Park, Y. H., 2003, "Hybrid Analysis Method for Reliability-Based Design Optimization," ASME J. Mech. Des., **125**(2), pp. 221–232.
[6] Du, X., and Chen, W., 2004, "Sequential Optimization and Reliability Assessment Method for Efficient Probabilistic Design," ASME J. Mech. Des., **126**(2), pp. 225–233.
[7] Youn, B. B., Choi, K. K., Yang, R.-J., and Gu, L., 2004, "Reliability-Based Design Optimization for Crashworthiness of Vehicle Side Impact," Struct. Multidiscip. Optim., **26**(3–4), pp. 272–283.
[8] Sinha, K., 2007, "Reliability-Based Multiobjective Optimization for Automotive Crashworthiness and Occupant Safety," Struct. Multidiscip. Optim., **33**, pp. 255–268.
[9] Kim, C., and Choi, K. K., 2008, "Reliability-Based Design Optimization Using Response Surface Method With Prediction Interval Estimation," ASME J. Mech. Des., **130**, p. 121401.
[10] Du, X., Sudjianto, A., and Huang, B., 2005, "Reliability-Based Design With the Mixture of Random and Interval Variables," ASME J. Mech. Des., **127**, pp. 1068–1076.
[11] Mourelatos, Z. P., and Zhou, J., 2008, "A Sequential Algorithm for Possibility-Based Design Optimization," ASME J. Mech. Des., **130**, p. 011001.
[12] Huang, H. Z., and Zhang, X., 2009, "Design Optimization With Discrete and Continuous Variables of Aleatory and Epistemic Uncertainty," ASME J. Mech. Des., **131**, p. 031006.
[13] Du, X., 2008, "Unified Uncertainty Analysis by the First Order Reliability Method," ASME J. Mech. Des., **130**(9), p. 091401.
[14] Pons, D. J., and Raine, J. K., 2003, "Relative Effectiveness of Mechanisms for Simulating Uncertainty in Quantitative Systems," Proc. Inst. Mech. Eng., Part B, **217**(4), pp. 531–540.
[15] Farid, S. S., Washbrook, J., and Titchener-Hooker, N. J., 2005, "Decision-Support Tool for Assessing Biomanufacturing Strategies Under Uncertainty: Stainless Steel Versus Disposable Equipment for Clinical Trial Material Preparation," Biotechnol. Prog., **21**(2), pp. 486–497.
[16] Rusli, E., Drews, T. O., Ma, D. L., Alkire, R. C., and Braatz, R. D., 2006, "Robust Nonlinear Feedback-Feedforward Control of a Coupled Kinetic Monte Carlo-Finite Difference Simulation," J. Process Control, **16**(4), pp. 409–417.
[17] Apley, D., Liu, J., and Chen, W., 2006, "Understanding the Effects of Model Uncertainty in Robust Design With Computer Experiments," ASME J. Mech. Des., **128**(4), pp. 945–958.
[18] Martin, J., and Simpson, T., 2006, "A Methodology to Manage System-Level Uncertainty During Conceptual Design," ASME J. Mech. Des., **128**(4), pp. 959–968.
[19] Fu, Y., 2006, "A New Study on Optimisation Under Uncertainty for Vehicle Design," Int. J. Mater. Prod. Technol., **25**(1–3), pp. 27–41.
[20] Du, X., and Chen, W., 2002, "Efficient Uncertainty Analysis Methods for Multidisciplinary Robust Design," AIAA J., **40**(3), pp. 545–552.
[21] Du, X., and Chen, W., 2005, "Collaborative Reliability Analysis Under the Framework of Multidisciplinary Systems Design," Optim. Eng., **6**(1), pp. 63–84.
[22] Gu, X., Renaud, J. E., and Penniger, C. L., 2006, "Implicit Uncertainty Propagation for Robust Collaborative Optimization," ASME J. Mech. Des., **128**(4), pp. 1001–1013.
[23] Birge, J. R., and Louveaux, F., 1997, *Introduction to Stochastic Programming*, Springer-Verlag, Berlin.
[24] Liu, B., 1999, *Uncertain Programming*, Wiley, New York.
[25] Mulvey, J. M., Vanderbei, R. J., and Zenios, S. A., 1995, "Robust Optimization of Large-Scale Systems," Oper. Res., **43**(2), pp. 264–281.
[26] Kouvelis, P., and Yu, G., 1996, *Robust Discrete Optimization and Its Applications*, Kluwer Academic, Dordrecht.
[27] Samson, S., Reneke, J. A., and Wiecek, M. M., 2009, "A Review of Different Perspectives on Uncertainty and Risk and an Alternate Modeling Paradigm," Reliab. Eng. Syst. Saf., **94**(2), pp. 558–567.
[28] Rockafellar, R. T., 2007, "Coherent Approaches to Risk in Optimization Under Uncertainty," Tutorials in Operation Research, INFORMS 2007.
[29] Knight, F. H., 1921, *Risk, Uncertainty, and Profit*, Houghton Mifflin, Boston, MA.
[30] Reneke, J. A., and Wiecek, M. M., 2005, "Vehicle Design Decomposition Under Uncertainty and Risk," Proceedings of the Sixth World Congresses of Structural and Multidisciplinary Optimization, Rio de Janeiro, Brazil.
[31] Reneke, J. A., and Wiecek, M. M., 2005, "Complex Systems Engineering Design Under Uncertainty and Risk," Department of Mathematical Sciences, Clemson University, Technical Report No. TR200509RW.
[32] Samson, S., Thoomu, S., Fadel, G., and Reneke, J. A., 2009, "Reliable Design Optimization Under Aleatory and Epistemic Uncertainty," Proceedings of the ASME Design Engineering Technical Conference and Design Automation Conference.
[33] Higham, D. J., 2001, "An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations," SIAM Rev., **43**, pp. 525–546.
[34] Ehrgott, M., 2005, *Multicriteria Optimization*, 2nd ed., Springer, Berlin, Germany.
[35] Reneke, J. A., and Samson, S., 2008, "Models and Risk Analysis of Uncertain Complex Systems," International Journal of Pure and Applied Mathematics, **44**(4), pp. 537–561.
[36] Reneke, J. A., and Wiecek, M. M., 2002, "Performance-Based Multicriteria Decision Making for Complex Systems: An Example," Proceedings of the Sixth World Multiconference on Systemics, Cybernetics and Informatics SCI, Information Systems Development III, Vol. 18, pp. 292–298.
[37] Hazelrigg, G. A., 2003, "Validation of Engineering Design Alternative Selection Methods," Eng. Optimiz., **35**(2), pp. 103–120.
[38] Samson, S., and Reneke, J., "A Multicriteria Approach Resolving Ellsberg's Paradox With Comments on Criteria Aggregation," International Journal of Pure and Applied Mathematics (to be published).