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Research Article

Composition Operators from the Weighted Bergman Space to the nth Weighted Spaces on the Unit Disc

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The boundedness of the composition operator from the weighted Bergman space to the recently introduced by the author, the nth weighted space on the unit disc, is characterized. Moreover, the norm of the operator in terms of the inducing function and weights is estimated.

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$, $dm(z)$ the Lebesgue area measure on $\mathbb{D}$, $dm_\alpha(z) = (1-|z|^2)^\alpha dm(z)$, $\alpha > -1$, and $H(\mathbb{D})$ the space of all analytic functions on the unit disc.

The weighted Bergman space $A^p_\alpha(\mathbb{D})$, where $p > 0$ and $\alpha > -1$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\alpha(\mathbb{D})} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^p \left(1-|z|^2\right)^\alpha dm(z) < \infty. \tag{1.1}$$

With this norm, $A^p_\alpha(\mathbb{D})$ is a Banach space when $p \geq 1$, while for $p \in (0,1)$, it is a Fréchet space with the translation invariant metric

$$d(f,g) = \|f-g\|^p_{A^p_\alpha(\mathbb{D})}, \quad f,g \in A^p_\alpha(\mathbb{D}). \tag{1.2}$$

Let $\mu(z)$ be a positive continuous function on a set $X \subset \mathbb{C}$ (*weight*) and $n \in \mathbb{N}_0$ be fixed. The *nth weighted-type space* on X, denoted by $\mathcal{W}_\mu^{(n)}(X)$, consists of all $f \in H(X)$ such that

$$b_{\mathcal{W}_\mu^{(n)}(X)}(f) := \sup_{z\in X} \mu(z)\left|f^{(n)}(z)\right| < \infty. \tag{1.3}$$

For $n = 0$, the space becomes the weighted-type space $H_\mu^\infty(X)$, for $n = 1$ the Bloch-type space $\mathcal{B}_\mu(X)$, and for $n = 2$ the Zygmund-type space $\mathcal{Z}_\mu(X)$.

For $n \in \mathbb{N}$, the quantity $b_{\mathcal{W}_\mu^{(n)}(X)}(f)$ is a seminorm on the nth weighted-type space $\mathcal{W}_\mu^{(n)}(X)$ and a norm on $\mathcal{W}_\mu^{(n)}(X)/\mathbb{P}_{n-1}$, where $\mathbb{P}_{n-1}$ is the set of all polynomials whose degrees are less than or equal to $n-1$. A natural norm on the nth weighted-type space can be introduced as follows:

$$\|f\|_{\mathcal{W}_\mu^{(n)}(X)} = \sum_{j=0}^{n-1}\left|f^{(j)}(a)\right| + b_{\mathcal{W}_\mu^{(n)}(X)}(f), \tag{1.4}$$

where a is an element in X. With this norm, the nth weighted-type space becomes a Banach space.

For $X = \mathbb{D}$ is obtained the space $\mathcal{W}_\mu^{(n)}(\mathbb{D})$, on which a norm is introduced as follows:

$$\|f\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} := \sum_{j=0}^{n-1}\left|f^{(j)}(0)\right| + \sup_{z\in\mathbb{D}} \mu(z)\left|f^{(n)}(z)\right|. \tag{1.5}$$

Some information on Zygmund-type spaces on the unit disc and some operators on them can be found, for example, in [1–6], for the case of the upper half-plane, see [7, 8], while some information in the setting of the unit ball can be found, for example, in [9–13]. This considerable interest in Zygmund-type spaces motivated us to introduce the nth weighted-type space (see [8]).

Assume φ is a holomorphic self-map of $\mathbb{D}$. The composition operator induced by φ is defined on $H(\mathbb{D})$ by

$$(C_\varphi f)(z) = f(\varphi(z)). \tag{1.6}$$

A typical problem is to provide function theoretic characterizations when φ induces bounded or compact composition operators between two given spaces of holomorphic functions. Some classical results on composition and weighted composition operators can be found, for example, in [14], while some recent results can be found in [1, 5, 7, 15–34] (see also related references therein).

Here we characterize the boundedness of the composition operator from the weighted Bergman space to the nth weighted space on the unit disc when $n \in \mathbb{N}$. The case $n = 0$ was previously treated in [16, 22, 24, 31, 35]. Hence we will not consider this case here. See also [36] for some good results on weighted composition operators between weighted-type spaces. The case $n = 1$ was treated, for example, in [26, 32]. For some other results on weighted composition operators which map a space into a weighted or a Bloch-type space, see, for example, [15, 17–21, 23, 25, 33, 34].

Let X and Y be topological vector spaces whose topologies are given by translation-invariant metrics d_X and d_Y, respectively, and $T : X \to Y$ be a linear operator. It is said that T is *metrically bounded* if there exists a positive constant K such that

$$d_Y(Tf, 0) \leq K d_X(f, 0) \tag{1.7}$$

for all $f \in X$. When X and Y are Banach spaces, the metrically boundedness coincides with the usual definition of bounded operators between Banach spaces.

If Y is a Banach space, then the quantity $\|C_\varphi\|_{A^p_\alpha(\mathbb{D})\to Y}$ is defined as follows:

$$\|C_\varphi\|_{A^p_\alpha(\mathbb{D})\to Y} := \sup_{\|f\|_{A^p_\alpha(\mathbb{D})}\leq 1} \|C_\varphi f\|_Y. \tag{1.8}$$

It is easy to see that this quantity is finite if and only if the operator $C_\varphi : A^p_\alpha(\mathbb{D}) \to Y$ is metrically bounded. For the case $p \geq 1$ this is the standard definition of the norm of the operator $C_\varphi : A^p_\alpha(\mathbb{D}) \to Y$, between two Banach spaces. If we say that an operator is bounded, it means that it is metrically bounded.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preccurlyeq b$ means that there is a positive constant C such that $a \leq Cb$. Moreover, if both $a \preccurlyeq b$ and $b \preccurlyeq a$ hold, then one says that $a \asymp b$.

2. Auxiliary Results

Here, we quote several auxiliary results. The first lemma is a direct consequence of a well-known estimate in [37, Proposition 1.4.10]. Hence, we omit its proof.

Lemma 2.1. *Assume $p > 0$, $\alpha > -1$, $n \in \mathbb{N}_0$, and $w \in \mathbb{D}$. Then the function*

$$g_{w,n}(z) = \frac{\left(1 - |w|^2\right)^{n+(\alpha+2)/p}}{(1 - \overline{w}z)^{n+2((\alpha+2)/p)}}, \tag{2.1}$$

belongs to $A^p_\alpha(\mathbb{D})$. Moreover, $\sup_{w\in\mathbb{D}}\|g_{w,n}\|_{A^p_\alpha} < \infty$.

The next lemma is folklore and was essentially proved in [38]. We will sketch a proof of it for the completeness and the benefit of the reader.

Lemma 2.2. *Assume $p > 0$, $\alpha > -1$, $n \in \mathbb{N}_0$, and $z \in \mathbb{D}$. Then there is a positive constant C independent of f such that*

$$\left|f^{(n)}(z)\right| \leq C \frac{\|f\|_{A^p_\alpha(\mathbb{D})}}{(1 - |z|)^{n+(\alpha+2)/p}}. \tag{2.2}$$

Proof. By the subharmonicity of the function $|f^{(n)}(z)|^p$, $p > 0$, applied on the disk:

$$D\left(z, \frac{1-|z|}{2}\right) = \left\{z \in \mathbb{C} \mid |z-w| < \frac{1-|z|}{2}\right\}, \tag{2.3}$$

and since

$$1-|w| \asymp 1-|z|, \quad w \in D\left(z, \frac{1-|z|}{2}\right), \tag{2.4}$$

we have that

$$\left|f^{(n)}(z)\right|^p \le \frac{C}{(1-|z|)^{2+\alpha+pn}} \int_{D(z,(1-|z|)/2)} \left|f^{(n)}(w)\right|^p dm_{\alpha+pn}(w). \tag{2.5}$$

From (2.5) and in light of the following well-known asymptotic relation [38]:

$$\int_{\mathbb{D}} |f(z)|^p \left(1-|z|^2\right)^{\alpha} dm(z) \asymp \sum_{j=0}^{n-1} \left|f^{(j)}(0)\right| + \int_{\mathbb{D}} \left|f^{(n)}(z)\right|^p \left(1-|z|^2\right)^{\alpha+np} dm(z), \tag{2.6}$$

the lemma easily follows. □

Lemma 2.3. *Assume $a > 0$ and*

$$D_n(a) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+1 & \cdots & a+n-1 \\ a(a+1) & (a+1)(a+2) & \cdots & (a+n-1)(a+n) \\ & & \cdots & \\ \prod_{j=0}^{n-2}(a+j) & \prod_{j=0}^{n-2}(a+j+1) & \cdots & \prod_{j=0}^{n-2}(a+j+n-1) \end{vmatrix}. \tag{2.7}$$

Then $D_n = \prod_{j=1}^{n-1} j!$.

Proof. By using elementary transformations, we have

$$D_n(a) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ a & 1 & \cdots & 1 \\ a(a+1) & 2(a+1) & \cdots & 2(a+n-1) \\ & & \cdots & \\ \prod_{j=0}^{n-2}(a+j) & (n-1)\prod_{j=0}^{n-3}(a+j+1) & \cdots & (n-1)\prod_{j=0}^{n-3}(a+j+n-1) \end{vmatrix}, \tag{2.8}$$

from which it follows that

$$D_n(a) = (n-1)!D_{n-1}(a+1), \tag{2.9}$$

which along with the fact $D_2(a+n-2) = 1$ implies the lemma. □

We will also need the classical Faà di Bruno's formula

$$(f \circ \varphi)^{(n)}(z) = \sum \frac{n!}{k_1! \cdots k_n!} f^{(k)}(\varphi(z)) \prod_{j=1}^{n} \left(\frac{\varphi^{(j)}(z)}{j!} \right)^{k_j}, \tag{2.10}$$

where $k = k_1 + k_2 + \cdots + k_n$ and the sum is over all nonnegative integers $k_1, k_2, \ldots, k_n$ satisfying $k_1 + 2k_2 + \cdots + nk_n = n$. For a nice exposition related to this formula see, for example, [39].

By using Bell polynomials $B_{n,k}(x_1, \ldots, x_{n-k+1})$, (2.10) can be written in the following form:

$$(f \circ \varphi)^{(n)}(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z)) B_{n,k}\left(\varphi'(z), \varphi''(z), \ldots, \varphi^{(n-k+1)}(z)\right). \tag{2.11}$$

Remark 2.4. Since $B_{n,0}(x_1, \ldots, x_{n+1}) = 0$ the summation in (2.11) is from 1 to k. Moreover, since $B_{n,1}(x_1, \ldots, x_n) = x_n$ and $B_{n,n}(x_1) = x_1^n$, (2.11) can be written in the following form:

$$\begin{aligned}(f \circ \varphi)^{(n)}(z) = f'(\varphi(z))\varphi^{(n)}(z) &+ \sum_{k=2}^{n-1} f^{(k)}(\varphi(z)) B_{n,k}\left(\varphi'(z), \ldots, \varphi^{(n-k+1)}(z)\right) \\ &+ f^{(n)}(\varphi(z))\left(\varphi'(z)\right)^n.\end{aligned} \tag{2.12}$$

3. Main Result

Here, we formulate and prove our main result.

Theorem 3.1. *Assume $p > 0$, $\alpha > -1$, $n \in \mathbb{N}$, μ is a weight on $\mathbb{D}$ and φ is a holomorphic self-map of $\mathbb{D}$. Then $C_\varphi : A_\alpha^p(\mathbb{D}) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded if and only if*

$$I_k := \sup_{z \in \mathbb{D}} \frac{\mu(z) \left| \sum (n!/(k_1! \cdots k_n!)) \prod_{j=1}^{n} \left(\varphi^{(j)}(z)/j! \right)^{k_j} \right|}{\left(1 - |\varphi(z)|^2\right)^{k+(\alpha+2)/p}} < \infty, \quad k = 1, \ldots, n, \tag{3.1}$$

where for each fixed $k \in \{1, \ldots, n\}$, the sum is over all nonnegative integers $k_1, k_2, \ldots, k_n$ such that $k = k_1 + k_2 + \cdots + k_n$ and $k_1 + 2k_2 + \cdots + nk_n = n$.

Moreover, if the operator $C_\varphi : A_\alpha^p(\mathbb{D}) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})/\mathbb{P}_{n-1}$ is bounded, then

$$\|C_\varphi\|_{A_\alpha^p(\mathbb{D}) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})/\mathbb{P}_{n-1}} \asymp \sum_{k=1}^{n} I_k. \tag{3.2}$$

Remark 3.2. Note that by (2.11) we see that the conditions in (3.1) can be written in the following form:

$$I_k = \sup_{z\in\mathbb{D}} \frac{\mu(z)\left|B_{n,k}\left(\varphi'(z),\varphi''(z),\ldots,\varphi^{(n-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{k+(\alpha+2)/p}} < \infty, \quad k=1,\ldots,n. \tag{3.3}$$

Proof. First assume that conditions in (3.1) hold. By formula (2.10) and Lemma 2.2 we have

$$\begin{aligned}
\|C_\varphi f\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} &= \sum_{j=0}^{n-1}\left|(f\circ\varphi)^{(j)}(0)\right| + \sup_{z\in\mathbb{D}}\mu(z)\left|(C_\varphi f)^{(n)}(z)\right| \\
&= \sum_{j=0}^{n-1}\left|\sum \frac{j!}{l_1!\cdots l_j!} f^{(l)}(\varphi(0)) \prod_{s=1}^{j}\left(\frac{\varphi^{(s)}(0)}{s!}\right)^{l_s}\right| \\
&\quad + \sup_{z\in\mathbb{D}}\mu(z)\left|\sum \frac{n!}{k_1!\cdots k_n!} f^{(k)}(\varphi(z)) \prod_{j=1}^{n}\left(\frac{\varphi^{(j)}(z)}{j!}\right)^{k_j}\right| \\
&\le \sum_{j=0}^{n-1}\sum_{l=0}^{j}\left|f^{(l)}(\varphi(0))\right|\left|\sum \frac{j!}{l_1!\cdots l_j!}\prod_{s=1}^{j}\left(\frac{\varphi^{(s)}(0)}{s!}\right)^{l_s}\right| \\
&\quad + C\|f\|_{A_\alpha^p(\mathbb{D})}\sum_{k=1}^{n}\sup_{z\in\mathbb{D}}\frac{\mu(z)\left|\sum(n!/(k_1!\cdots k_n!))\prod_{j=1}^{n}\left(\varphi^{(j)}(z)/j!\right)^{k_j}\right|}{\left(1-|\varphi(z)|^2\right)^{k+(\alpha+2)/p}}.
\end{aligned} \tag{3.4}$$

From this, (2.2) with $z=\varphi(0)$, and by conditions in (3.1), it follows that the operator $C_\varphi : A_\alpha^p(\mathbb{D}) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded. Moreover, if we consider the space $\mathcal{W}_\mu^{(n)}(\mathbb{D})/\mathbb{P}_{n-1}$, we have that

$$\|C_\varphi\|_{A_\alpha^p(\mathbb{D})\to\mathcal{W}_\mu^{(n)}(\mathbb{D})/\mathbb{P}_{n-1}} \le C\sum_{k=1}^{n}\sup_{z\in\mathbb{D}}\frac{\mu(z)\left|\sum(n!/(k_1!\cdots k_n!))\prod_{j=1}^{n}\left(\varphi^{(j)}(z)/j!\right)^{k_j}\right|}{\left(1-|\varphi(z)|^2\right)^{k+(\alpha+2)/p}}. \tag{3.5}$$

Now assume that the operator $C_\varphi : A_\alpha^p(\mathbb{D}) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ is bounded. For a fixed $w\in\mathbb{D}$, and constants $c_1,\ldots,c_n$, set

$$g_w(z) = \sum_{j=1}^{n}\frac{c_j}{n-2+j+2\left((\alpha+2)/p\right)}\,\frac{\left(1-|w|^2\right)^{n-2+j+(\alpha+2)/p}}{(1-\overline{w}z)^{n-2+j+2((\alpha+2)/p)}}. \tag{3.6}$$

Applying Lemma 2.1 we see that $g_w\in A_\alpha^p(\mathbb{D})$ for every $w\in\mathbb{D}$. Moreover, we have that

$$\sup_{w\in\mathbb{D}}\|g_w\|_{A_\alpha^p(\mathbb{D})} \le C. \tag{3.7}$$

Now we show that for each $l \in \{1, \dots, n\}$, there are constants $c_1, c_2, \dots, c_n$, such that

$$g_w^{(l)}(w) = \frac{\overline{w}^l}{\left(1 - |w|^2\right)^{l+(\alpha+2)/p}}, \qquad g_w^{(m)}(w) = 0, \quad m \in \{1, \dots, n\} \setminus \{l\}. \tag{3.8}$$

Indeed, by differentiating function g_w, for each $l \in \{1, \dots, n\}$, the system in (3.8) becomes

$$\begin{gathered}
c_1 + c_2 + \cdots + c_n = 0, \\
\left(n + 2\frac{\alpha+2}{p}\right)c_1 + \left(n + 1 + 2\frac{\alpha+2}{p}\right)c_2 + \cdots + \left(2n - 1 + 2\frac{\alpha+2}{p}\right)c_n = 0, \\
\vdots \\
\prod_{j=0}^{l-2}\left(n + j + 2\frac{\alpha+2}{p}\right)c_1 + \prod_{j=0}^{l-2}\left(n + 1 + j + 2\frac{\alpha+2}{p}\right)c_2 + \cdots + \prod_{j=0}^{l-2}\left(2n - 1 + j + 2\frac{\alpha+2}{p}\right)c_n = 1, \\
\vdots \\
\prod_{j=0}^{n-2}\left(n + j + 2\frac{\alpha+2}{p}\right)c_1 + \prod_{j=0}^{n-2}\left(n + 1 + j + 2\frac{\alpha+2}{p}\right)c_2 + \cdots + \prod_{j=0}^{n-2}\left(2n - 1 + j + 2\frac{\alpha+2}{p}\right)c_n = 0.
\end{gathered} \tag{3.9}$$

By using Lemma 2.3 with $a = n + 2(2 + \alpha)/p > 0$, we obtain that the determinant of system (3.9) is different from zero from which the claim follows.

Now for each $k \in \{1, \dots, n\}$, we choose the corresponding family of functions which satisfy (3.8) and denote it by $g_{w,k}$.

For each $k \in \{1, \dots, n\}$, the boundedness of the operator $C_\varphi : A_\alpha^p(\mathbb{D}) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})$ along with (2.10) and (3.7) implies that for each $\varphi(w) \neq 0$:

$$\begin{aligned}
&\frac{\mu(w)|\varphi(w)|^k \left|\sum (n!/(k_1! \cdots k_n!)) \prod_{j=1}^n \left(\varphi^{(j)}(w)/j!\right)^{k_j}\right|}{\left(1 - |\varphi(w)|^2\right)^{k+(\alpha+2)/p}} \\
&\quad \le \sup_{w \in \mathbb{D}} \left\| C_\varphi\left(g_{\varphi(w),k}\right)\right\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} \le C\left\|C_\varphi\right\|_{A_\alpha^p(\mathbb{D}) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})},
\end{aligned} \tag{3.10}$$

where (for each fixed $k \in \{1, \dots, n\}$) the sum is over all nonnegative integers $k_1, k_2, \dots, k_n$ such that $k = k_1 + k_2 + \cdots + k_n$ and $k_1 + 2k_2 + \cdots + nk_n = n$.

From (3.10), it follows that for each $k \in \{1, \dots, n\}$,

$$\sup_{|\varphi(z)|>1/2} \frac{\mu(z)\left|\sum (n!/(k_1! \cdots k_n!)) \prod_{j=1}^n \left(\varphi^{(j)}(z)/j!\right)^{k_j}\right|}{\left(1 - |\varphi(z)|^2\right)^{k+(\alpha+2)/p}} \le C\left\|C_\varphi\right\|_{A_\alpha^p(\mathbb{D}) \to \mathcal{W}_\mu^{(n)}(\mathbb{D})}. \tag{3.11}$$

Now we use consecutively the test functions

$$h_k(z) = z^k \in A^p_\alpha(\mathbb{D}), \quad k = 1, \dots, n, \tag{3.12}$$

in order to deal with the case $|\varphi(z)| \le 1/2$. Note that

$$\|h_k\|_{A^p_\alpha(\mathbb{D})} \le 1, \quad \text{for each } k \in \mathbb{N}. \tag{3.13}$$

By applying (2.11) to the function $f(z) = h_1(z)$, we get

$$(h_1 \circ \varphi)^{(n)}(z) = h_1'(\varphi(z)) B_{n,1}\left(\varphi'(z), \dots, \varphi^{(n)}(z)\right) = B_{n,1}\left(\varphi'(z), \dots, \varphi^{(n)}(z)\right), \tag{3.14}$$

which along with the boundedness of the operator $C_\varphi : A^p_\alpha(\mathbb{D}) \to \mathcal{W}^{(n)}_\mu(\mathbb{D})$ and (3.13) implies that

$$\sup_{z\in\mathbb{B}} \mu(z)\left|B_{n,1}\left(\varphi'(z), \dots, \varphi^{(n)}(z)\right)\right| \le \|C_\varphi(z)\|_{\mathcal{W}^{(n)}_\mu(\mathbb{D})} \le \|C_\varphi\|_{A^p_\alpha(\mathbb{D})\to\mathcal{W}^{(n)}_\mu(\mathbb{D})}, \tag{3.15}$$

or equivalently $\varphi \in \mathcal{W}^{(n)}_\mu(\mathbb{D})$ (see Remark 2.4).

Further, by applying formula (2.11) to the function $f(z) = h_2(z)$, we get

$$\begin{aligned}(h_2 \circ \varphi)^{(n)}(z) &= h_2'(\varphi(z)) B_{n,1}\left(\varphi'(z), \dots, \varphi^{(n)}(z)\right) \\ &\quad + h_2''(\varphi(z)) B_{n,2}\left(\varphi'(z), \dots, \varphi^{(n-1)}(z)\right).\end{aligned} \tag{3.16}$$

From the boundedness of $C_\varphi : A^p_\alpha(\mathbb{D}) \to \mathcal{W}^{(n)}_\mu(\mathbb{D})$ and (3.13), we get

$$\sup_{z\in\mathbb{B}} \mu(z)\left|(h_2 \circ \varphi)^{(n)}(z)\right| \le \left\|C_\varphi\left(z^2\right)\right\|_{\mathcal{W}^{(n)}_\mu(\mathbb{D})} \le \|C_\varphi\|_{A^p_\alpha(\mathbb{D})\to\mathcal{W}^{(n)}_\mu(\mathbb{D})}. \tag{3.17}$$

From (3.16) and (3.17), and by using the triangle inequality it follows that

$$\begin{aligned}&2\sup_{z\in\mathbb{B}} \mu(z)\left|B_{n,2}\left(\varphi'(z), \dots, \varphi^{(n-1)}(z)\right)\right| \\ &\quad \le \|C_\varphi\|_{A^p_\alpha(\mathbb{D})\to\mathcal{W}^{(n)}_\mu(\mathbb{D})} + 2\sup_{z\in\mathbb{B}} \mu(z)\left|\varphi(z) B_{n,1}\left(\varphi'(z), \dots, \varphi^{(n)}(z)\right)\right|.\end{aligned} \tag{3.18}$$

Using the fact $\sup_{z\in\mathbb{D}}|\varphi(z)| \le 1$ and applying inequality (3.15) in (3.18) we get

$$\sup_{z\in\mathbb{B}} \mu(z)\left|B_{n,2}\left(\varphi'(z), \dots, \varphi^{(n-1)}(z)\right)\right| \le \frac{3}{2}\|C_\varphi\|_{A^p_\alpha(\mathbb{D})\to\mathcal{W}^{(n)}_\mu(\mathbb{D})}. \tag{3.19}$$

Assume that we have proved the following inequalities:

$$\sup_{z\in\mathbb{B}} \mu(z)\left|B_{n,j}\left(\varphi'(z),\ldots,\varphi^{(n-j+1)}(z)\right)\right| \le C\|C_\varphi\|_{A^p_\alpha(\mathbb{D})\to\mathcal{W}^{(n)}_\mu(\mathbb{D})}, \tag{3.20}$$

for $j\in\{1,\ldots,k-1\}$ and a $k\le n$.

Applying formula (2.11) to the function $f(z)=h_k(z)$, $k\in\{1,\ldots,n\}$, we have that

$$\begin{aligned}(h_k\circ\varphi)^{(n)}(z) &= \sum_{j=1}^{k} h_k^{(j)}(\varphi(z))B_{n,j}\left(\varphi'(z),\ldots,\varphi^{(n-j+1)}(z)\right)\\ &= \sum_{j=1}^{k} k(k-1)\cdots(k-j+1)(\varphi(z))^{k-j}B_{n,j}\left(\varphi'(z),\ldots,\varphi^{(n-j+1)}(z)\right).\end{aligned} \tag{3.21}$$

From this, by using the boundedness of the operator $C_\varphi : A^p_\alpha(\mathbb{D}) \to \mathcal{W}^{(n)}_\mu(\mathbb{D})$, the boundedness of function φ, the triangle inequality, noticing that the coefficient at $B_{n,k}(\varphi'(z),\ldots,\varphi^{(n-k+1)}(z))$ is independent of z (it is equal $k!$), and finally using hypothesis (3.20), we easily obtain

$$\sup_{z\in\mathbb{B}} \mu(z)\left|B_{n,k}\left(\varphi'(z),\ldots,\varphi^{(n-k+1)}(z)\right)\right| \le C\|C_\varphi\|_{A^p_\alpha(\mathbb{D})\to\mathcal{W}^{(n)}_\mu(\mathbb{D})}. \tag{3.22}$$

Hence, by induction, we get that (3.22) holds for each $k\in\{1,\ldots,n\}$.

From (3.22) and bearing in mind Remark 2.4, for each fixed $k\in\{1,\ldots,n\}$, we have that

$$\begin{aligned}&\sup_{|\varphi(z)|\le 1/2} \frac{\mu(z)\left|\sum(n!/(k_1!\cdots k_n!))\prod_{j=1}^{n}\left(\varphi^{(j)}(z)/j!\right)^{k_j}\right|}{\left(1-|\varphi(z)|^2\right)^{k+(\alpha+2)/p}}\\ &\quad\le \sup_{z\in\mathbb{B}} \mu(z)\left|B_{n,k}\left(\varphi'(z),\ldots,\varphi^{(n-k+1)}(z)\right)\right| \le C\|C_\varphi\|_{A^p_\alpha(\mathbb{D})\to\mathcal{W}^{(n)}_\mu(\mathbb{D})},\end{aligned} \tag{3.23}$$

where as usual for a fixed $k\in\{1,\ldots,n\}$, the sum is over all nonnegative integers $k_1,k_2,\ldots,k_n$ such that $k=k_1+k_2+\cdots+k_n$ and $k_1+2k_2+\cdots+nk_n=n$.

Hence from (3.11) and (3.23), we get

$$\sum_{k=1}^{n} I_k \le C\|C_\varphi\|_{A^p_\alpha(\mathbb{D})\to\mathcal{W}^{(n)}_\mu(\mathbb{D})}. \tag{3.24}$$

From (3.5) and (3.24), we obtain asymptotic relation (3.2). □

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