MINIMAL CONDITIONS ON CLIFFORD SEMIGROUP CONGRUENCES

M. EL-GHALI M. ABDALLAH, L. N. GAB-ALLA, AND SAYED K. M. ELAGAN

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A known result in groups concerning the inheritance of minimal conditions on normal subgroups by subgroups with finite indexes is extended to semilattices of groups $[E(S), S_e, \varphi_{e,f}]$ with identities in which all $\varphi_{e,f}$ are epimorphisms (called q partial groups). Formulation of this result in terms of q congruences is also obtained.

1. Introduction and preliminaries

A *partial group* as defined in [3] is a semigroup S which satisfies the following axioms.

(i) For every $x \in S$, there exists a (necessarily unique) element $e_x \in S$, called the partial identity of x such that $e_x x = x e_x = x$ and if $yx = xy = x$ then $e_x y = y e_x = e_x$.

(ii) For every $x \in S$, there exists a (necessarily unique) element $x^{-1} \in S$, called the partial inverse of x such that $xx^{-1} = x^{-1}x = e_x$ and $e_x x^{-1} = x^{-1} e_x = x^{-1}$.

(iii) The operation $x \mapsto e_x$ is a homomorphism from S into S, that is, $e_{xy} = e_x e_y$ for all $x, y \in S$, and the operation $x \mapsto x^{-1}$ is an antihomomorphism, that is, $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in S$.

Consequently, a partial group is precisely a Clifford semigroup, that is, a regular semigroup with central idempotents, and this is characterized by Clifford structure theorem (see [4, Chapter IV, Theorem 2.1] or [5, Chapter II, Theorem 2]) as a (strong) semilattice of groups.

Thus, in particular, a partial group S may be viewed as a strong semilattice of groups $S = [E(S); S_e, \varphi_{e,f}]$, where S_e is the maximal subgroup of S with identity e ($e \in E(S)$) and for $e \geq f$ in $E(S)$, $\varphi_{e,f}$ is the homomorphism of groups $S_e \to S_f$, $x \mapsto xf$. Here $E(S)$ is the semilattice ($e \geq f$ if and only if $ef = f$) of idempotents (partial identities) in S.

Let S be a partial group. A *subpartial group* of S is a subsemigroup of S closed under the unary operations of S. A subpartial group of S is *wide* (or *full*) if it contains $E(S)$.

A normal subpartial group of S is a wide subpartial group K of S such that $x^{-1}Kx \subset K$ for all $x \in S$. This notion is standard in the literature, and we refer in particular to [2] for the following consequences.

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Let K be normal in S, K_e is a normal subgroup of S_e for every $e \in E(S)$.

$\rho_K = \{(x,y) \in S \times S : e_x = e_y \text{ and } xy^{-1} \in K\}$ is an idempotent-separating congruence on S with $\ker \rho_K = K$.

Conversely, if ρ is an idempotent-separating congruence on S, then $K = \ker \rho$ is a normal subpartial group of S and $\rho_K = \rho$.

Let $\mathbf{N}(S)$ be the set of all normal subpartial groups of S, and let $\mathbf{C^i}(S)$ be the set of all idempotent-separating congruences on S. For $N, M \in \mathbf{N}(S)$, $N \vee M$ is the join $\langle N \bigcup M \rangle$, that is, the smallest normal subpartial group of S containing $N \bigcup M$, and we have $N \vee M = NM = MN$. For ρ, $\sigma \in \mathbf{C^i}(S)$, $\rho \vee \sigma$ is defined similarly, and we have $\rho \vee \sigma = \sigma \circ \rho = \rho \circ \sigma$. Also $\rho_{NM} = \rho_N \circ \rho_M$ for all $N, M \in \mathbf{N}(S)$. Moreover, we have the following theorem.

Theorem 1.1 [2]. *($\mathbf{N}(S)$, $\subset$, $\bigcap$, $\vee$) and ($\mathbf{C^i}(S)$, $\subset$, $\bigcap$, $\vee$) are complete modular lattices and $\mathbf{N}(S) \to \mathbf{C^i}(S)$, $N \mapsto \rho_N$ is a lattice isomorphism.*

For sake of reference, cite from [1] the material required for the present work.

A q partial group is a partial group S with identity 1 such that $\varphi_{1,e} : S_1 \to S_e$ is an epimorphism (i.e., $S_1 e = S_e$) for every $e \in E(S)$. This implies clearly that $\varphi_{e,f} : S_e \to S_f$ is an epimorphism for every $e, f \in E(S)$ with $e \geq f$.

Theorem 1.2 [1]. *Let S be a partial group with identity 1. Every wide subpartial group K of S contains a unique maximal q subpartial group, denoted by $\mathbf{Q}(K)$, of S given by $(\mathbf{Q}(K))_e = \operatorname{Im}$, $\varphi_{1,e}|_{K_1} = K_1 e$ for all $e \in E(S)$. That is, $\mathbf{Q}(K)$ is a strong semilattice $E(S)$ of groups $K_1 e$. Also $\mathbf{Q}(K)$ is non trivial (i.e., $\mathbf{Q}(K) \neq E(S)$) if and only if K_1 is a nontrivial group.*

On the class of partial groups with identities, we have idempotent unary operation $S \mapsto \mathbf{Q}(S)$, where $\mathbf{Q}(S)$, defind as above, is the unique maximal q subpartial group of S.

Lemma 1.3. *Let S be a partial group with identity and let $\{S_i,\, i \in I\}$ be a family of wide subpartial groups of S. Then $\mathbf{Q}(\langle \bigcup_{i \in I} S_i \rangle) = \langle \bigcup_{i \in I} \mathbf{Q}(S_i) \rangle$. In particular, the join of any family of q subpartial groups of S is a q subpartial group of S.*

A normal subpartial group of a partial group S with identity need not be a q subpartial group, even if S is a q partial group (e.g., [1, Example 4.4]). On the other hand, if S is a q partial group and N is normal in S, then $\mathbf{Q}(N)$ is normal in S.

Let S be a q partial group and let $\mathbf{QN}(S)$ be the set of all q normal subpartial groups of S.

Lemma 1.4. *$\mathbf{QN}(S)$ is a complete modular lattice with meet and join defined by*

$$M \wedge N = \mathbf{Q}(M \cap N),$$
$$M \vee N = MN = \langle M \cup N \rangle. \tag{1.1}$$

An idempotent-separating congruence ρ on a partial group S with identity 1 is called a *q congruence* if for all $x, y \in S$, $x \rho y$ implies that $x = sy$ for some $s \in (\ker \rho)_1$, that is, for some $s \in S_1$ with $s \rho 1$.

Lemma 1.5. *Let S be a partial group with identity, and let ρ be an idempotent-separating congruence on S. Then ρ is a q congruence if and only if $K = \ker \rho$ is a q normal subpartial group of S. Equivalently for any subpartial group K of S, K is a q normal subpartial group of S if and only if ρ_K is a q congruence.*

Let S be a partial group with identity 1. The **Q** operation is defined on idempotent-separating congruences as follows:

$$\mathbf{Q}(\rho) = \{(x, y) : x\rho y,\ x = sy(\text{or } x = ys) \text{ for some } s \in S \text{ with } s\rho 1\}. \tag{1.2}$$

Precisely, $\mathbf{Q}(\rho)$ is the unique maximal q congruence on S contained in ρ.

Lemma 1.6. *Let S be a q partial group and let $\rho \in \mathbf{C}^{\mathbf{i}}(S)$. Then*

$$\mathbf{Q}(\rho) = \rho_{\mathbf{Q}(N)}, \quad \text{where } N = \ker \rho. \tag{1.3}$$

Let S be a q partial group and let $\mathbf{QC}^{\mathbf{i}}(S)$ be the set of all q congruences on S.

Theorem 1.7. *With join and meet given by $\rho \vee \sigma = \rho \circ \sigma = \sigma \circ \rho$ and $\rho \wedge \sigma = \mathbf{Q}(\rho \bigcap \sigma)$, $\mathbf{QC}^{\mathbf{i}}(S)$ is a complete modular lattice and the mapping*

$$\mathbf{QN}(S) \longrightarrow \mathbf{QC}^{\mathbf{i}}(S), \quad N \longmapsto \rho_N, \tag{1.4}$$

is a lattice isomorphism.

As observed in [1, Section 1], q partial groups exist naturally as partial mappings from sets to groups. In the present paper, we consider minimal conditions on normal subpartial groups of partial groups with identities and we discuss the situations with which such conditions could be inherited by subpartial groups with finite indexes. Our principal goal is to extend the following known result in groups to appropriate classes of partial groups.

Theorem 1.8 [6, Theorem 3.1.8]. *If a group G satisfies* $\min - n$ *and H is a subgroup of G with finite index, then H satisfies* $\min - n$.

In this theorem, $\min - n$ is the minimal condition on normal subgroups. In other words a group G is said to satisfy $\min - n$ if any nonempty family of normal subgroups has a minimal member, or equivalently there does not exist a proper descending chain $N_1 \supset N_2 \supset \cdots$ of normal subgroups of G. The proof of the above result depends on the notion of normal closures and cores in groups. Here we give the definitions and some properties (see [6, Section I.3]).

If X is a nonempty subset of a group G, the *normal closure* of X in G, denoted by X^G, is the intersection of all normal subgroups of G which contains X. Dually, the *core* of X in G, denoted by X_G, is the join of all normal subgroups of G that contains by X. In other words, X^G is the smallest normal subgroup of G containing X, whereas X_G is the largest one contained in X.

Fact 1.9. $X^G = \langle g^{-1}Xg : g \in G \rangle$.

Fact 1.10. For any subgroup H of G, $H_G = \bigcap_{g \in G} g^{-1}Hg$.

For our work, we introduce in Section 2 *normal closures* and *cores* in partial groups, showing that if S is a partial group and $\varnothing \neq X \subset S$, then X^S is characterized as in Fact 1.9. If either $X \bigcap S_e \neq \varnothing$ for all $e \in E(S)$ or S has an identity 1 and $1 \in X$, that S is a q partial group, and K is a normal subgroup of S_1 (the maximal subgroup of S with identity 1), then K^S is a q subpartial group of S.

We also characterize the *(normal) core* H_S of a wide subpartial group H of a q partial group S in terms of the group-theoretic cores of H_e in S_e ($e \in E(S)$). H_S need not be a q subpartial group of S, whence we introduce the notion of a q *(normal) core* showing that the q core of a wide subpartial group H of a q partial group S is precisely $\mathbf{Q}(H_S)$.

In Section 3, we show that a q partial group S satisfies $\min -qn$ (the minimal condition on q normal subpartial groups) if and only if S_1 satisfies $\min -n$. Introducing the notions of *finite index* and *local finite index* in partial groups (with identities), we show that $\min -qn$ in q partial groups is inherited by q subpartial groups with local finite indexes extending Theorem 1.8.

Whereas for partial groups with identities, we show that $\min -n$ implies $\min -qn$ for wide subpartial groups with finite indexes.

These two results can be formulated in terms of (q) congruences. In particular, the former result may have the version.

In q partial groups, the minimal condition on q congruences is inherited by q subpartial groups with local finite indexes.

2. Normal closures and cores

Given a partial group S and a nonempty subset X of S, we define the *normal closure* of X, denoted by X^S, to be the intersection of all normal subpartial groups of S containing X. Evidently, X^S is the smallest normal subpartial group of S containing X. Analogous to the known characterization of normal closures in groups (see Section 1), we have the following lemma.

Lemma 2.1. *Let S be a partial group and let $\varnothing \neq X \subset S$.*

(i) *If $X \cap S_e \neq \varnothing$ for all $e \in E(S)$, then*

$$X^S = \langle s^{-1}Xs : s \in S\rangle. \tag{2.1}$$

In particular, if H is a wide subpartial group of S, then

$$H^S = \langle s^{-1}Hs : s \in S\rangle. \tag{2.2}$$

(ii) *If S has an identity* 1 *and* $1 \in X$, *then*

$$X^S = \langle s^{-1}Xs : s \in S\rangle. \tag{2.3}$$

Proof. (i) Clearly, $\langle s^{-1}Xs : s \in S\rangle$ is a subpartial group of S containing X, and by the hypothesis, it is also wide. Let $s \in S$ and let $y \in \langle s^{-1}Xs : s \in S\rangle$ be a generator, say $y = s_1^{-1}xs_1$, for some $s_1 \in S$, $x \in X$. Then $s^{-1}ys = s^{-1}s_1^{-1}xs_1s = (s_1s)^{-1}x(s_1s) \in \langle s^{-1}Xs : s \in S\rangle$. If $y_1,\dots,y_n$ are generators in $\langle s^{-1}Xs : s \in S\rangle$, then

$$\begin{aligned} s^{-1}(y_1y_2\cdots y_n)s &= s^{-1}y_1ss^{-1}y_2ss^{-1}\cdots ss^{-1}y_ns \\ &= (s^{-1}y_1s)(s^{-1}y_2s)\cdots(s^{-1}y_ns) \in \langle s^{-1}Xs : s \in S\rangle. \end{aligned} \tag{2.4}$$

Hence, $\langle s^{-1}Xs : s \in S\rangle$ is a normal subpartial group of S containing X. Suppose that K is a normal subpartial group of S and that $X \subset K$. Then for any $s \in S$, $x \in X$, $s^{-1}xs \in K$

(since $x \in K$), and so $\langle s^{-1}Xs : s \in S\rangle \subset K$. Therefore, $\langle s^{-1}Xs : s \in S\rangle$ is the normal closure of X and (i) is proved.

(ii) For any $e \in E(S)$, $e = e^{-1}1e \in \langle s^{-1}Xs : s \in S\rangle$. Thus $\langle s^{-1}Xs : s \in S\rangle$ is a wide subpartial group of S containing X. The result follows as in (i). □

Lemma 2.2. *Let S be a q partial group and let K be a normal subgroup of S_1. Then K^S is a q normal subpartial group of S.*

Proof. Let $H = K^S$. By Lemma 2.1(ii), $H = \langle s^{-1}Ks : s \in S\rangle$. Thus, it is sufficient to show that $H_e \subset H_1 e$ for all $e \in E(S)$ (which implies that $\varphi_{1,e} : H_1 \to H_e$ is an epimorphism for all $e \in E(S)$). Let $e \in E(S)$ be fixed but arbitrary, and let $x \in H_e$. Since K is a subgroup of S_1, then x may be written as a product of generators $x = x_1^{-1}k_1x_1x_2^{-1}k_2x_2\cdots x_n^{-1}k_nx_n$ for some $x_i \in S$ and $k_i \in K$, and $i = 1,\ldots,n$. We have $e_{k_i} = 1$ for all $i \in I$, and so

$$e = e_x = e_{x_1}e_{x_2}\cdots e_{x_n}. \tag{2.5}$$

Since S is a q partial group, we have

$$x_i = s_i e_{x_i} \tag{2.6}$$

for some $s_i \in S_1$, $i = 1, 2, \ldots, n$. Thus

$$x = s_1^{-1}k_1s_1s_2^{-1}k_2s_2\cdots s_n^{-1}k_ns_ne_{x_1}e_{x_2}\cdots e_{x_n}. \tag{2.7}$$

Since K is normal in S_1, $k_i' = s_i^{-1}k_is_i \in K$, $i = 1,2,\ldots,n$, and we have

$$x = k_1'k_2'\cdots k_n'e = ke, \qquad \text{where } k = k_1'k_2'\cdots k_n' \in K, \tag{2.8}$$

and so $x \in Ke$. Clearly, $H_1 = (K^S)_1 = \langle s^{-1}Ks : s \in S_1\rangle = K$. Thus $x \in H_1e$. Therefore $H_e \subset H_1e$. □

The notion of *a core* (or *normal interior*) in groups can be also extended to partial groups. Let S be a partial group and let H be a wide subpartial group of S. The *core (normal interior)* of H in S, denoted by H_S, is the join of all normal subpartial groups of S contained in H. In other words, H_S is the largest normal subpartial group of S contained in H. Cores in q partial groups can be characterized in terms of cores in groups. We have the following theorem.

Theorem 2.3. *Let S be a q partial group and let H be a wide subpartial group of S. Then $(H_S)_e$ is the group-theoretic core of H_e in S_e for every $e \in E(S)$, that is, H_S is a semilattice of groups*

$$H_S = [E(S), K_e, \varphi_{e,f}], \tag{2.9}$$

where K_e is the core of the subgroup H_e in S_e, $e \in E(S)$.

Proof. By hypothesis, $K = \bigcup_{e\in E(S)} K_e$ is a union of disjoint groups indexed by the semilattice $E(S)$. By Fact 1.10, for every $e \in E(S)$, $K_e = \bigcap_{s\in S_e} s^{-1}H_es$. For $e \geq f$ in $E(S)$, define $\varphi_{e,f} : K_e \to K_f$ by $x \mapsto xf$ $(x \in K_e)$. To show that K is a semilattice of groups, it is sufficient to show that $\varphi_{e,f}$ is well defined in the sense that $\varphi_{e,f}(K_e) \subset K_f$ for all $e \geq f$ in $E(S)$.

Thus, let $x \in K_e = \bigcap_{s\in S_e} s^{-1}H_e s$. We must show that $xf \in K_f = \bigcap_{s\in S_f} s^{-1}H_f s$. For this, let s be an arbitrary element of S_f. Since (by hypothesis) $\varphi_{e,f} : S_e \to S_f$ is an epimorphism, there exists $s_1 \in S_e$ such that $s = s_1 f$. Now, $x \in \bigcap_{s\in S_e} s^{-1}H_e s$ implies that $x = s_1^{-1} y s_1$, for some $y \in H_e$. Whence, $xf = (s_1 f)^{-1}(yf)(s_1 f) \in s^{-1}H_f s$. Since s is arbitrary in S_f, we obtain $xf \in \bigcap_{s\in S_f} s^{-1}H_f s = K_f$. Hence K is a (strong) semilattice of groups. It follows that K is a subpartial group of S which is clearly wide. To complete the proof, we have to show that $H_S = K$. Thus it is sufficient to show that K is normal in S, that $K \subset H$, and that K is the largest with respect to this property. That $K \subset H$ follows trivially from the definition of K. To show that K is normal in S, let $x \in S$ and let $y \in K$, say $x \in S_e$, $y \in K_f$, for some $e, f \in E(S)$. Thus, $xy = xyef$, and we have $x^{-1}yx = (xf)^{-1}(yef)(xf)$. Clearly, $xf \in S_{ef}$ and $yef \in K_{ef}$. By the construction of K, K_{ef} is normal in S_{ef}. It follows that $x^{-1}yx \in K_{ef} \subset K$. Thus K is a normal subpartial group of S. Finally, let N be a normal subpartial group of S such that $N \subset H$. We have for every $e \in E(S)$, N_e is normal in S_e and $N_e \subset H_e$. Since K_e is the core of H_e in S_e, we obtain $N_e \subset K_e$. Therefore $N \subset K$. □

Let S be a partial group with 1 and let H be a wide subpartial group of S. We define the *q core of H in S* to be the join of all q normal subpartial groups of S contained in H.

Theorem 2.4. *If S is a q partial group and H is a wide subpartial group of S, then the q core of H in S is $\mathbf{Q}(H_S)$, where H_S is the core of H in S.*

Proof. We have $H_S = \langle N : N \lhd S, N \subset H\rangle$, let $\mathbf{N}_H$ be the family of all normal subpartial groups N of S contained in H. Then $H_S = \langle N : N \in \mathbf{N}_H\rangle$. Let $q\mathbf{N}_H$ be the family of all q normal subpartial groups of S contained in H, we have $K \in q\mathbf{N}_H$ if and only if $K = \mathbf{Q}(N)$, for some $N \in \mathbf{N}_H$, that is, $q\mathbf{N}_H \subset \{\mathbf{Q}(N) : N \in \mathbf{N}_H\}$. On the other hand, if $N \in \mathbf{N}_H$, then, since $\mathbf{Q}(N)$ is normal in S [1], we have $\mathbf{Q}(N) \in q\mathbf{N}_H$. Thus, $\{\mathbf{Q}(N) : N \in \mathbf{N}_H\} \subset q\mathbf{N}_H$. Then, $q\mathbf{N}_H = \{\mathbf{Q}(N) : N \in \mathbf{N}_H\}$. By definition and [1, Lemma 4.2], we have the q core of H in S which is

$$\begin{aligned}\langle K : K \in q\mathbf{N}_H\rangle &= \langle \mathbf{Q}(N) : N \in \mathbf{N}_H\rangle \\ &= \mathbf{Q}(\langle N : N \in \mathbf{N}_H\rangle) = \mathbf{Q}(H_S).\end{aligned} \tag{2.10}$$

□

3. Minimal conditions on q congruences

We use the machinary that has been developed so far to extend a result in groups concerning the inheritance of minimal condition on normal subgroups of a given group to subgroups with finite indexes, and obtain analogous results for partial groups and q partial groups.

Recall that a partially ordered set $(P, \leq)$ is said to satisfy the *minimal condition* if any nonempty subset of P contains a minimal element. This is equivalent to saying that P satisfies the *descending chain condition*: there does not exist an infinite properly descending chain $x_1 > x_2 > \cdots$ in P.

In particular, a group G satisfies $\min -n$ if $(\mathbf{N}(G), \subset)$ satisfies the minimal condition, where $\mathbf{N}(G)$ is the set of all normal subgroups of G. Analogously, we say that a partial group S satisfies $\min -n$ if $(\mathbf{N}(S), \subset)$ satisfies the minimal condition. Let S be a partial group with identity 1, we say that S satisfies $\min -qn$ if $(\mathbf{QN}(G), \subset)$ satisfies the minimal condition, or equivalently if there does not exist an infinite properly descending chain $K_1 \supset K_2 \supset \cdots$ in $\mathbf{QN}(S)$. Here $\mathbf{N}(S)$ and $\mathbf{QN}(S)$ are defined as in Section 1.

Lemma 3.1. *Let S be a partial group with identity* 1.
(i) *If S satisfies* $\min - n$*, then S_1 satisfies* $\min - n$.
(ii) *If S_1 satisfies* $\min - n$*, then S satisfies* $\min - qn$.

Proof. (i) Suppose that S *satisfies* $\min - n$ but S_1 does not. There exists an infinite properly descending chain

$$N_1 \supset N_2 \supset \cdots \quad \text{in } \mathbf{N}(S_1). \tag{3.1}$$

For each $i = 1, 2, \ldots$, N_i^S (the normal closure of N_i in S) is a normal subpartial group of S and we have a descending chain

$$N_1^S \supset N_2^S \supset \cdots \quad \text{in } \mathbf{N}(S). \tag{3.2}$$

By $\min - n$ of S, we obtain

$$N_j^S = N_{j+1}^S \quad \text{for some } j. \tag{3.3}$$

Since $N_j \supset N_{j+1}$ and $N_j \neq N_{j+1}$, there exists some element $s \in N_j$ with $s \notin N_{j+1}$. We have $s \in N_j^S = N_{j+1}^S$. Thus, s may be written as an expansion,

$$s = y_1^{-1} x_1 y_1 y_2^{-1} x_2 y_2 \cdots y_n^{-1} x_n y_n, \tag{3.4}$$

with $y_i \in S$, $x_i \in N_{j+1}$, $i = 1, 2, \ldots, n$. Since $e_s = 1$, we must have $e_{y_i} = 1$ for all $i = 1, 2, \ldots, n$, that is, $y_i \in S_1$ for all $i = 1, 2, \ldots, n$. Now, N_{j+1} being a normal subgroup of S_1 implies that $y_i^{-1} x_i y_i \in N_{j+1}$, for all $i = 1, 2, \ldots, n$. Therefore, $s \in N_{j+1}$, a contradiction.

(ii) Suppose that S_1 *satisfies* $\min - n$ but S does not satisfy $\min - qn$. There exists an infinite properly descending chain

$$K_1 \supset K_2 \supset \cdots \quad \text{in } \mathbf{QN}(S). \tag{3.5}$$

For each $i = 1, 2, \ldots, (K_i)_1$ is a normal subgroup of S_1, where $(K_i)_1$ is the maximal subgroup of K_i with identity 1. Thus, the above chain induces a descending chain

$$(K_1)_1 \supset (K_2)_1 \supset \cdots \quad \text{in } \mathbf{N}(S_1). \tag{3.6}$$

Since S_1 satisfies $\min - n$, we have for some j

$$(K_j)_1 = (K_{j+1})_1. \tag{3.7}$$

By assumption, $K_j \supset K_{j+1}$ and $K_j \neq K_{j+1}$. Thus we must have, for some $e \in E(S)$,

$$(K_j)_e \supset (K_{j+1})_e, \qquad (K_j)_e \neq (K_{j+1})_e. \tag{3.8}$$

There exists $x \in (K_j)_e$ such that $x \notin (K_{j+1})_e$. Since K_j is a q partial group, we have $x = se$ for some $s \in (K_j)_1 = (K_{j+1})_1$. Now, $s \in (K_{j+1})_1$, which implies that $se \in (K_{j+1})_e$ and so $x \in (K_{j+1})_e$, a contradiction. □

Using Lemmas 3.1(ii), 2.2, and proceeding as in the proof of Lemma 3.1(i), we obtain the following lemma.

Lemma 3.2. *A q partial group S satisfies* $\min-qn$ *if and only if S_1 satisfies* $\min-n$.

Let S be an arbitrary partial group. A wide subpartial group H of S is said to have a finite index in S *if $S = HT$, for some finite subset T of S.*

Lemma 3.3. *Let S be a partial group and let H be a wide subpartial group of S. If H has a finite index in S, then H_e has a finite index in S_e for every $e \in E(S)$.*

Proof. By assumption, $S = HT$, for some finite subset T of S. Let $e \in E(S)$ be fixed but arbitrary. For each $x \in S_e$, we have $x \in S_e \subset S = HT$. Thus $x = ht_x$, for some $h \in H$ and $t_x \in T$. We have, $e = e_x = e_h e_{t_x}$, and so $e \leq e_h$ and $e \leq e_{t_x}$, that is, $ee_h = e$ and $ee_{t_x} = e$. Now, $x = ht_x = (he)t_x \in H_e t_x = H_e(et_x)$. Setting ${}_xT = \{et_x : x \in S_e\}$, then clearly ${}_xT$ is a finite subset of S_e and $S_e \subset H_{ex}T$. Also, $H_{ex}T \subset S_e S_e = S_e$. Therefore, $S_e = H_{ex}T$, which proves that H_e has a finite index in S_e. □

Let S be a partial group with identity 1 and let H be a wide subpartial group of S. We say that H has *a local finite index in* S if H_1 has a finite index in S_1. Clearly, by Lemma 3.3, for a wide subpartial group H of a partial group S with identity, we have a finite index in S which implies a local finite index in S.

For q partial groups, the implication of Lemma 3.3 may be refined as follows.

Lemma 3.4. *Let S be a q partial group and let H be a wide subpartial group of S. If H has a local finite index in S, then H_e has a finite index in S_e for every $e \in E(S)$.*

Proof. We have $S_1 = H_1 T$, for some finite subset $T \subset S_1$. Let $e \in E(S)$, since S is a q partial group, then $S_e = S_1 e$. Thus

$$S_e = H_1 Te = (H_1 e)(Te) = (H_1 e)_e T, \tag{3.9}$$

where ${}_eT$ is the finite set $Te \subset S_e$. Since H is wide, $H_1 e \subset H$, in particular, $H_1 e \subset H_e$. Therefore,

$$S_e = (H_1 e)_e T \subset H_e\, {}_eT \subset S_e. \tag{3.10}$$

Thus, $S_e = H_e\, {}_eT$ and the result follows. □

Now we give our main result which is an extension of [6, Theorem 3.1.8].

Theorem 3.5. *If a q partial group S satisfies* $\min-qn$ *and H is a q subpartial group of S with local finite index, then H satisfies* $\min-qn$.

Proof. By definition (of local finite index), H_1 has a finite index in S_1, and by Lemma 3.2, S_1 satisfies $\min-n$. Since H_1 is a subgroup of S_1,Wilson theorem [6, Theorem 3.1.8] implies that H_1 satisfies $\min-n$. Again by Lemma 3.2, we have H satisfying $\min-qn$. □

For partial groups with identities, we have the following version of Theorem 3.5.

Theorem 3.6. *If a partial group S with identity satisfying* $\min-n$ *and H is a wide subpartial group of S with finite index, then H satisfies* $\min-qn$.

The proof follows similarly by applying Lemmas 3.1(i), 3.3, [6, Theorem 3.1.8], and Lemma 3.1(ii).

In view of Theorems 1.7 and 1.1, the above two Theorems (3.5 and 3.6) can be formulated in terms of q congruences and idempotent-separating congruences, respectively. For instance, by Theorems 1.7 and 3.5, we have the following corollary.

Corollary 3.7. *If a q partial group S satisfies the minimal condition on q congruences and H is a q subpartial group of S with local finite index, then H satisfies the minimal condition on q congruences.*

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M. El-Ghali M. Abdallah: Department of Mathematics, Faculty of Science, Menoufiya University, Shebin El-kom 32511, Egypt
E-mail address: mohamed_elghaly@yahoo.com

L. N. Gab-Alla: Department of Mathematics, Faculty of Science, Menoufiya University, Shebin El-kom 32511, Egypt
E-mail address: layla_nashed2006@yahoo.com

Sayed K. M. Elagan: Department of Mathematics, Faculty of Science, Menoufiya University, Shebin El-kom 32511, Egypt
E-mail address: sayed_khalil2000@yahoo.com

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