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Research Article

Inequalities for Convex Functions on Simplexes and Their Cones

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The aim of this paper is to present the fundamental inequalities for convex functions on Euclidean spaces. The work is based on the geometry of the simplest convex sets and properties of convex functions. Some obtained inequalities are applied to demonstrate a natural way of generalizing the Hermite-Hadamard inequality.

1. Introduction

Let $\mathcal{X}$ be a real linear (vector) space, $a_i \in \mathcal{X}$ points (vectors), and $\alpha_i \in \mathbb{R}$ coefficients (scalars). The combination

$$c = \sum_{i=1}^{n} \alpha_i a_i \tag{1}$$

belongs to the linear subspace $\mathcal{L} = \operatorname{lin}\{a_i\}$ as the smallest linear space that contains all a_i, and it is called the linear combination. If $\sum_{i=1}^{n} \alpha_i = 1$, the combination in (1) belongs to the affine hull $\mathcal{A} = \operatorname{aff}\{a_i\}$ as the smallest translated linear space that contains all a_i, and it is called the affine combination. If $\sum_{i=1}^{n} \alpha_i = 1$ and all $\alpha_i \in [0,1]$, the combination in (1) belongs to the convex hull $\mathcal{C} = \operatorname{conv}\{a_i\}$ as the smallest convex point set that contains all a_i, and it is called the convex combination. The point c itself is called the center of the observed combination.

A function $f : \mathcal{X} \to \mathbb{R}$ is affine if the equality

$$f(c) = \sum_{i=1}^{n} \alpha_i f(a_i) \tag{2}$$

holds for all affine combinations in (1), and f is convex if the inequality

$$f(c) \le \sum_{i=1}^{n} \alpha_i f(a_i) \tag{3}$$

holds for all convex combinations in (1). The formulas may be referred to for $n = 2$ and then by applying the mathematical induction extended to any positive integer n. The important Jensen's inequality in (3) was just so proven in [1].

Jensen's inequality was extended to special affine combinations and their centers in [2], generalizing Mercer's variant of Jensen's inequality obtained in [3]. The connection between discrete and integral forms was observed in [4]; the integral forms were studied in [5], and the Jensen type inequalities for Q-class functions were investigated in [6]. Different variants and forms can be found in [7, 8].

2. Convex Functions on the Line

Known results are presented in this section, in a way that can be generalized. Convex combinations of the line segment and affine combinations of the convex cone represent the backbone of the work. Theorem 3 is the most important in terms of scope of its content.

If $a, b \in \mathbb{R}$ are different numbers, then every number $x \in \mathbb{R}$ can be uniquely presented as the affine combination

$$x = \alpha a + \beta b, \tag{4}$$

where

$$\alpha = \frac{\begin{vmatrix} x & 1 \\ b & 1 \end{vmatrix}}{\begin{vmatrix} a & 1 \\ b & 1 \end{vmatrix}}, \qquad \beta = -\frac{\begin{vmatrix} x & 1 \\ a & 1 \end{vmatrix}}{\begin{vmatrix} a & 1 \\ b & 1 \end{vmatrix}}. \tag{5}$$

The above binomial combination is convex if and only if the number x belongs to the interval $\operatorname{conv}\{a, b\}$. Given the function $f : \mathbb{R} \to \mathbb{R}$, let $f_{\{a,b\}}^{\text{line}} : \mathbb{R} \to \mathbb{R}$ be the function of the line passing through the points $(a, f(a))$ and $(b, f(b))$ of the graph of f. Using the affinity of $f^{\text{line}}_{\{a,b\}}$, we get

$$f^{\text{line}}_{\{a,b\}}(x) = \alpha f(a) + \beta f(b). \tag{6}$$

Let $\mathscr{C}_a$ be the convex cone (half-line) with the vertex at a spanned by $a - b$ containing binomial affine combinations $x = a + p(a - b) = (1 + p)a - pb$, where $p \geq 0$; that is,

$$\mathscr{C}_a = \{(1 + p)a - pb : p \geq 0\}. \tag{7}$$

Convex cone $\mathscr{C}_b$ is defined similarly. That means that if $a < b$, then $\mathscr{C}_a = (-\infty, a]$ and $\mathscr{C}_b = [b, +\infty)$. The consequence of the representations in (4) and (6) is the well-known characterization of convex functions with one variable, as specified in the following lemma.

Lemma 1. *If $a, b \in \mathbb{R}$ are the line segment endpoints, then every convex function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality*

$$f(x) \leq f^{line}_{\{a,b\}}(x) \quad \textit{for } x \in \operatorname{conv}\{a, b\} \tag{8}$$

and the reverse inequality

$$f(x) \geq f^{line}_{\{a,b\}}(x) \quad \textit{for } x \in \mathscr{C}_a \cup \mathscr{C}_b. \tag{9}$$

Using combinations $x = (1 + p)a - pb$ with $p \geq 0$, the inequality in (9) takes the form

$$f((1 + p)a - pb) \geq (1 + p) f(a) - pf(b). \tag{10}$$

Example 2. Let a, b and $x = (1 + p)a - pb \in \mathscr{C}_a$ be positive real numbers.

Substituting the values of the convex power function $f(x) = x^r$ with the exponent $r \in (-\infty, 0] \cup [1, +\infty)$ in the inequality in (10), we get the inequality

$$[(1 + p)a - pb]^r \geq (1 + p)a^r - pb^r \tag{11}$$

and the reverse inequality by substituting the values of the concave power function with the exponent $r \in [0, 1]$.

Substituting the values of the concave logarithmic function $f(x) = \ln x$ in the inequality in (10) and rearranging them, it follows the inequality

$$a^{1+p}b^{-p} \geq (1 + p)a - pb. \tag{12}$$

Combining the application of Lemma 1 and Jensen's inequality to the convex combinations $x = \sum_{i=1}^{n} \alpha_i a_i$, it follows

$$f(x) \leq \sum_{i=1}^{n} \alpha_i f(a_i) \leq f^{\text{line}}_{\{a,b\}}(x) \tag{13}$$

if all $a_i \in \operatorname{conv}\{a, b\}$ and

$$\sum_{i=1}^{n} \alpha_i f(a_i) \geq f(x) \geq f^{\text{line}}_{\{a,b\}}(x) \tag{14}$$

if all $a_i \in \mathscr{C}_a$ or all $a_i \in \mathscr{C}_b$.

Convex combinations with the common center are considered in the following theorem.

Theorem 3. *Let $a, b \in \mathbb{R}$ be the line segment endpoints. Let $\sum_{i=1}^{n} \alpha_i a_i$ be a convex combination of the points $a_i \in \operatorname{conv}\{a, b\}$, and let $\sum_{j=1}^{m} \beta_j b_j$ be a convex combination of the points $b_j \in \mathscr{C}_a \cup \mathscr{C}_b$.*

If the center equality

$$c = \sum_{i=1}^{n} \alpha_i a_i = \sum_{j=1}^{m} \beta_j b_j \tag{15}$$

is valid, then the inequality

$$f(c) \leq \sum_{i=1}^{n} \alpha_i f(a_i) \leq \sum_{j=1}^{m} \beta_j f(b_j) \tag{16}$$

holds for every convex function $f : \mathbb{R} \to \mathbb{R}$.

Proof. The first inequality in (16) is the Jensen inequality. The last inequality follows from the series of inequalities

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i f(a_i) &\leq \sum_{i=1}^{n} \alpha_i f^{\text{line}}_{\{a,b\}}(a_i) = f^{\text{line}}_{\{a,b\}}\left(\sum_{i=1}^{n} \alpha_i a_i\right) \\ &= f^{\text{line}}_{\{a,b\}}\left(\sum_{j=1}^{m} \beta_j b_j\right) = \sum_{j=1}^{m} \beta_j f^{\text{line}}_{\{a,b\}}(b_j) \\ &\leq \sum_{j=1}^{m} \beta_j f(b_j) \end{aligned} \tag{17}$$

derived by applying the inequality in (8) to a_i and the inequality in (9) to b_j. ☐

The geometric insight to the inequality in (16) presented in Figure 1 shows that the point $P_1(c, f(c))$ is below the point

$$P_2\left(c, \sum_{i=1}^{n} \alpha_i f(a_i)\right) \in \operatorname{conv}\{(a_1, f(a_1)), \ldots, (a_n, f(a_n))\}, \tag{18}$$

and the point P_2 is below the point

$$P_3\left(c, \sum_{j=1}^{m} \beta_j f(b_j)\right) \in \operatorname{conv}\{(b_1, f(b_1)), \ldots, (b_m, f(b_m))\}. \tag{19}$$

Corollary 4. *Let $a, b \in \mathbb{R}$ be the line segment endpoints. Let $\sum_{i=1}^{n} \alpha_i a_i$ be a convex combination of the points $a_i \in \operatorname{conv}\{a, b\}$, and let $\alpha a + \beta b$ be the convex combination such that*

$$c = \sum_{i=1}^{n} \alpha_i a_i = \alpha a + \beta b. \tag{20}$$

Then the inequality

$$f(c) \leq \sum_{i=1}^{n} \alpha_i f(a_i) \leq \alpha f(a) + \beta f(b) \tag{21}$$

holds for every convex function $f : \operatorname{conv}\{a, b\} \to \mathbb{R}$.

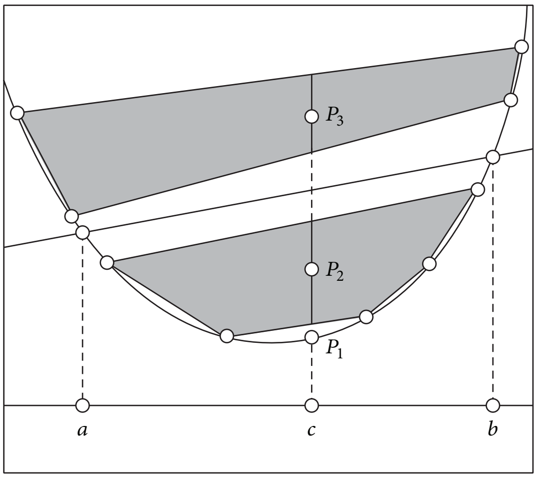

Figure 1: Geometric representation of the inequality in (16).

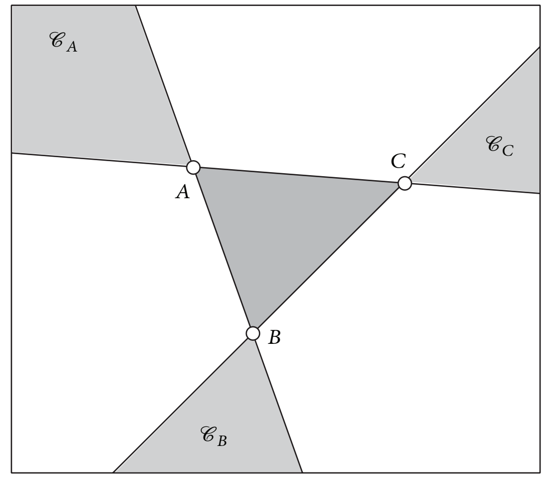

Figure 2: Triangle with cones.

3. Main Results: Convex Functions on the Plane

We assume that $\mathbb{R}^2$ is the real vector space with the standard coordinate addition $(x_1, y_1)+(x_2, y_2) = (x_1+x_2, y_1+y_2)$ and the scalar multiplication $\alpha(x, y) = (\alpha x, \alpha y)$.

If $A(x_A, y_A)$, $B(x_B, y_B)$, and $C(x_C, y_C)$ are the planar points that do not belong to one line, then every point $P(x, y) \in \mathbb{R}^2$ can be presented by the unique affine combination:

$$P = \alpha A + \beta B + \gamma C, \tag{22}$$

where

$$\alpha = \frac{\begin{vmatrix} x & y & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}{\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}, \qquad \beta = -\frac{\begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_C & y_C & 1 \end{vmatrix}}{\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}, \qquad \gamma = \frac{\begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix}}{\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}}. \tag{23}$$

The above trinomial combination is convex if and only if point P belongs to the triangle $\operatorname{conv}\{A, B, C\}$. Given the function $f : \mathbb{R}^2 \to \mathbb{R}$, let $f^{\text{plane}}_{\{A,B,C\}} : \mathbb{R}^2 \to \mathbb{R}$ be the function of the plane passing through the points $(A, f(A))$, $(B, f(B))$, and $(C, f(C))$ of the graph of f. Because of the affinity of $f^{\text{plane}}_{\{A,B,C\}}$, it follows

$$f^{\text{plane}}_{\{A,B,C\}}(P) = \alpha f(A) + \beta f(B) + \gamma f(C). \tag{24}$$

Let $\mathscr{C}_A$ be the convex cone with the vertex at A spanned by the vectors $A - B$ and $A - C$ containing trinomial affine combinations $P = A + p(A - B) + q(A - C) = (1 + p + q)A - pB - qC$, where $p, q \geq 0$; that is,

$$\mathscr{C}_A = \{(1 + p + q) A - pB - qC : p, q \geq 0\}. \tag{25}$$

Cones $\mathscr{C}_B$ and $\mathscr{C}_C$ are defined in the same way, and all tree cones can be seen in Figure 2.

Using the cones, we achieve the generalization of Lemma 1 to convex functions on the plane as follows.

Lemma 5. *If $A, B, C \in \mathbb{R}^2$ are the triangle vertices, then every convex function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the inequality*

$$f(P) \leq f^{\text{plane}}_{\{A,B,C\}}(P) \quad \text{for } P \in \operatorname{conv}\{A, B, C\}, \tag{26}$$

and the reverse inequality

$$f(P) \geq f^{\text{plane}}_{\{A,B,C\}}(P) \quad \text{for } P \in \mathscr{C}_A \cup \mathscr{C}_B \cup \mathscr{C}_C. \tag{27}$$

Proof. If $P \in \operatorname{conv}\{A, B, C\}$, the combination in (22) is convex, and the inequality in (26) follows from the convexity of f and the equality in (24).

If $P \in \mathscr{C}_A \cup \mathscr{C}_B \cup \mathscr{C}_C$, say $P \in \mathscr{C}_A$ and $P \neq A$, we can represent the point P as the binomial affine combination:

$$\begin{aligned} P &= (1 + p + q) A - pB - qC \qquad (28) \\ &= (1 + p + q) A - (p + q)\left(\frac{p}{p+q}B + \frac{q}{p+q}C\right). \qquad (29) \end{aligned}$$

Applying the inequality in (10) to the combination in (29), then using the convexity of f and the affinity of $f^{\text{plane}}_{\{A,B,C\}}$, we get the series of inequalities:

$$\begin{aligned} f(P) &\geq (1 + p + q) f(A) - (p + q) f\left(\frac{p}{p+q}B + \frac{q}{p+q}C\right) \\ &\geq (1 + p + q) f(A) - pf(B) - qf(C) \\ &= f^{\text{plane}}_{\{A,B,C\}}((1 + p + q) A - pB - qC) = f^{\text{plane}}_{\{A,B,C\}}(P) \end{aligned} \tag{30}$$

which includes the inequality in (27). □

The area outside the triangle and outside the cones (the white area in Figure 2) cannot be generally covered with one inequality. Such area does not exist in the previous one-dimensional case.

Applying Lemma 5 and Jensen's inequality to the planar convex combinations $P = \sum_{i=1}^{n} \alpha_i A_i$, we get the inequality

$$f(P) \leq \sum_{i=1}^{n} \alpha_i f(A_i) \leq f^{\text{plane}}_{\{A,B,C\}}(P) \tag{31}$$

if all $A_i \in \text{conv}\{A, B, C\}$ and the inequality

$$\sum_{i=1}^{n} \alpha_i f(A_i) \geq f(P) \geq f_{\{A,B,C\}}^{\text{plane}}(P) \tag{32}$$

if all $A_i \in \mathscr{C}_A$, or all $A_i \in \mathscr{C}_B$, or all $A_i \in \mathscr{C}_C$.

The following are planar convex combinations with the common center.

Theorem 6. *Let $A, B, C \in \mathbb{R}^2$ be the triangle vertices. Let $\sum_{i=1}^{n} \alpha_i A_i$ be a convex combination of the points $A_i \in \text{conv}\{A, B, C\}$, and let $\sum_{j=1}^{m} \beta_j B_j$ be a convex combination of the points $B_j \in \mathscr{C}_A \cup \mathscr{C}_B \cup \mathscr{C}_C$.*

If the center equality

$$P = \sum_{i=1}^{n} \alpha_i A_i = \sum_{j=1}^{m} \beta_j B_j \tag{33}$$

is valid, then the inequality

$$f(P) \leq \sum_{i=1}^{n} \alpha_i f(A_i) \leq \sum_{j=1}^{m} \beta_j f(B_j) \tag{34}$$

holds for every convex function $f : \mathbb{R}^2 \to \mathbb{R}$.

Proof. We employ the proof of Theorem 3 using $f_{\{A,B,C\}}^{\text{plane}}$ instead of $f_{\{a,b\}}^{\text{line}}$. □

If (33) is valid, and if all points A_i and B_j are the triangle vertices, then the equality

$$\sum_{i=1}^{n} \alpha_i f(A_i) = \sum_{j=1}^{m} \beta_j f(B_j) \tag{35}$$

holds. The reverse statement is true if the function f is strictly convex.

Corollary 7. *Let $A, B, C \in \mathbb{R}^2$ be the triangle vertices. Let $\sum_{i=1}^{n} \alpha_i A_i$ be a convex combination of the points $A_i \in \text{conv}\{A, B, C\}$, and let $\alpha A + \beta B + \gamma C$ be the convex combination such that*

$$P = \sum_{i=1}^{n} \alpha_i A_i = \alpha A + \beta B + \gamma C. \tag{36}$$

Then the inequality

$$f(P) \leq \sum_{i=1}^{n} \alpha_i f(A_i) \leq \alpha f(A) + \beta f(B) + \gamma f(C) \tag{37}$$

holds for every convex function $f : \text{conv}\{A, B, C\} \to \mathbb{R}$.

4. Generalization to Higher Dimensions

Let $P_1, \ldots, P_{N+1} \in \mathbb{R}^N$ be the points so that the vectors $P_1 - P_2, \ldots, P_1 - P_{N+1}$ be linearly independent. In this case, the convex hull $\text{conv}\{P_1, \ldots, P_{N+1}\}$ is called the N-simplex in $\mathbb{R}^N$ with the vertices $P_1, \ldots, P_{N+1}$. All of the simplex vertices cannot belong to the same hyperplane in $\mathbb{R}^N$. Every point $P \in \mathbb{R}^N$ can be presented by the unique affine combination:

$$P = \sum_{k=1}^{N+1} \alpha_k P_k, \tag{38}$$

where the coefficients

$$\alpha_k = (-1)^{k+1} \frac{\begin{vmatrix} x_1 & \cdots & x_N & 1 \\ x_{11} & \cdots & x_{1N} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{k-1\,1} & \cdots & x_{k-1\,N} & 1 \\ x_{k+1\,1} & \cdots & x_{k+1\,N} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{N+1\,1} & \cdots & x_{N+1\,N} & 1 \end{vmatrix}}{\begin{vmatrix} x_{11} & \cdots & x_{1N} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{k-1\,1} & \cdots & x_{k-1\,N} & 1 \\ x_{k1} & \cdots & x_{kN} & 1 \\ x_{k+1\,1} & \cdots & x_{k+1\,N} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{N+1\,1} & \cdots & x_{N+1\,N} & 1 \end{vmatrix}} \tag{39}$$

are determined generalizing the coefficients in (23). The combination in (38) is convex if and only if the point P belongs to the N-simplex $\text{conv}\{P_1, \ldots, P_{N+1}\}$.

Given the function $f : \mathbb{R}^N \to \mathbb{R}$, let $f_{\{P_1,\ldots,P_{N+1}\}}^{\text{hyperplane}} : \mathbb{R}^N \to \mathbb{R}$ be the function of the hyperplane (in $\mathbb{R}^{N+1}$) passing through the points $(P_k, f(P_k))$ of the graph of f. Therefore

$$f_{\{P_1,\ldots,P_{N+1}\}}^{\text{hyperplane}}(P) = \sum_{k=1}^{N+1} \alpha_k f(P_k). \tag{40}$$

Let $\mathscr{C}_k$ $(k = 1, \ldots, N+1)$ be the convex cone with the vertex at P_k spanned by the vectors $P_k - P_j$ for $k \neq j = 1, \ldots, N+1$ containing $(N+1)$-membered affine combinations $P = P_k + \sum_{k \neq j=1}^{N+1} p_j (P_k - P_j) = (1 + \sum_{k \neq j=1}^{N+1} p_j) P_k - \sum_{k \neq j=1}^{N+1} p_j P_j$, where all $p_j \geq 0$; that is,

$$\mathscr{C}_k = \left\{ \left(1 + \sum_{k \neq j=1}^{N+1} p_j \right) P_k - \sum_{k \neq j=1}^{N+1} p_j P_j : p_j \geq 0 \right\}. \tag{41}$$

Lemma 8. *If $P_1, \ldots, P_{N+1} \in \mathbb{R}^N$ are the N-simplex vertices, then every convex function $f : \mathbb{R}^N \to \mathbb{R}$ satisfies the inequality*

$$f(P) \leq f_{\{P_1,\ldots,P_{N+1}\}}^{\textit{hyperplane}}(P) \quad \textit{for } P \in \text{conv}\{P_1, \ldots, P_{N+1}\} \tag{42}$$

and the reverse inequality

$$f(P) \geq f_{\{P_1,\ldots,P_{N+1}\}}^{\textit{hyperplane}}(P) \quad \textit{for } P \in \mathscr{C}_1 \cup \cdots \cup \mathscr{C}_{N+1}. \tag{43}$$

Proof. The proof is similar to that of Lemma 5. We sketch the arguments briefly as follows.

To prove (42), we firstly apply Jensen's inequality to the convex combination $P = \sum_{k=1}^{N+1} \alpha_k P_k \in \text{conv}\{P_1, \ldots, P_{N+1}\}$ and then use (40).

To prove (43) for $P \in \mathscr{C}_k$ other than P_k, we firstly implement the inequality in (10) to the binomial affine combination:

$$\begin{aligned} P &= \left(1 + \sum_{k\neq j=1}^{N+1} p_j\right) P_k - \sum_{k\neq j=1}^{N+1} p_j P_j \\ &= (1+p) P_k - pQ_k, \end{aligned} \tag{44}$$

where $p = \sum_{k\neq j=1}^{N+1} p_j$ and $Q_k = \sum_{k\neq j=1}^{N+1} (p_j/p) P_j$ and then apply Jensen's inequality to the convex combination of Q_k, thus obtaining

$$\begin{aligned} f(P) &\geq \left(1 + \sum_{k\neq j=1}^{N+1} p_j\right) f(P_k) - \sum_{k\neq j=1}^{N+1} p_j f(P_j) \\ &= f^{\text{hyperplane}}_{\{P_1,\ldots,P_{N+1}\}}(P) \end{aligned} \tag{45}$$

as the desired inequality. □

Using the inequality in (45) with power and logarithmic functions, the inequalities of Example 2 can be generalized as follows.

Example 9. Let $P_j(x_{j1},\ldots,x_{jN}) \in \mathbb{R}^N$ for $j = 1,\ldots,N+1$ be the N-simplex vertices with all coordinates $x_{ji} > 0$. Let $P(x_1,\ldots,x_N) \in \mathscr{C}_k$ be a point with all coordinates $x_i = (1+p)x_{ki} - \sum_{k\neq j=1}^{N+1} p_j x_{ji} > 0$, where $p = \sum_{k\neq j=1}^{N+1} p_j$.

Including the values of the convex power sum function

$$f(x_1,\ldots,x_N) = \sum_{i=1}^{N} x_i^{r_i} \quad \text{with all } r_i \in (-\infty, 0] \cup [1, +\infty) \tag{46}$$

in the inequality in (45), we get the inequality

$$\begin{aligned} &\sum_{i=1}^{N} \left[(1+p)x_{ki} - \sum_{k\neq j=1}^{N+1} p_j x_{ji}\right]^{r_i} \\ &\quad \geq \sum_{i=1}^{N} \left[(1+p)\, x_{ki}^{r_i} - \sum_{k\neq j=1}^{N+1} p_j x_{ji}^{r_i}\right] \end{aligned} \tag{47}$$

and the reverse inequality by including the values of the concave power sum function with all $r_i \in [0,1]$.

Including the values of the concave logarithmic sum function

$$f(x_1,\ldots,x_N) = \sum_{i=1}^{N} \ln x_i = \ln \prod_{i=1}^{N} x_i \tag{48}$$

in the inequality in (45) and rearranging them, it follows the inequality

$$\prod_{i=1}^{N} \left[x_{ki}^{1+p} \prod_{k\neq j=1}^{N+1} x_{ji}^{-p_j}\right] \geq \prod_{i=1}^{N} \left[(1+p)\, x_{ki} - \sum_{k\neq j=1}^{N+1} p_j x_{ji}\right]. \tag{49}$$

Applying Lemma 8 and Jensen's inequality to the convex combinations $P = \sum_{i=1}^{n} \alpha_i A_i$, we get the inequality

$$f(P) \leq \sum_{i=1}^{n} \alpha_i f(A_i) \leq f^{\text{hyperplane}}_{\{P_1,\ldots,P_{N+1}\}}(P) \tag{50}$$

if all $A_i \in \operatorname{conv}\{P_1,\ldots,P_{N+1}\}$ and the inequality

$$\sum_{i=1}^{n} \alpha_i f(A_i) \geq f(P) \geq f^{\text{hyperplane}}_{\{P_1,\ldots,P_{N+1}\}}(P) \tag{51}$$

if all A_i belong to the same cone $\mathscr{C}_k$.

Relying on Lemma 8, we reach the conclusion written in the next theorem.

Theorem 10. *Let $P_1,\ldots,P_{N+1} \in \mathbb{R}^N$ be the N-simplex vertices. Let $\sum_{i=1}^{n} \alpha_i A_i$ be a convex combination of the points $A_i \in \operatorname{conv}\{P_1,\ldots,P_{N+1}\}$, and let $\sum_{j=1}^{m} \beta_j B_j$ be a convex combination of the points $B_j \in \mathscr{C}_1 \cup \cdots \cup \mathscr{C}_{N+1}$.*

If the center equality

$$P = \sum_{i=1}^{n} \alpha_i A_i = \sum_{j=1}^{m} \beta_j B_j \tag{52}$$

is valid, then the inequality

$$f(P) \leq \sum_{i=1}^{n} \alpha_i f(A_i) \leq \sum_{j=1}^{m} \beta_j f(B_j) \tag{53}$$

holds for every convex function $f : \mathbb{R}^N \to \mathbb{R}$.

Corollary 11. *Let $P_1,\ldots,P_{N+1} \in \mathbb{R}^N$ be the N-simplex vertices. Let $\sum_{i=1}^{n} \alpha_i A_i$ be a convex combination of the points $A_i \in \operatorname{conv}\{P_1,\ldots,P_{N+1}\}$, and let $\sum_{j=1}^{N+1} \beta_j P_j$ be the convex combination such that*

$$P = \sum_{i=1}^{n} \alpha_i A_i = \sum_{j=1}^{N+1} \beta_j P_j. \tag{54}$$

Then the inequality

$$f(P) \leq \sum_{i=1}^{n} \alpha_i f(A_i) \leq \sum_{j=1}^{N+1} \beta_j f(P_j) \tag{55}$$

holds for every convex function $f : \operatorname{conv}\{P_1,\ldots,P_{N+1}\} \to \mathbb{R}$.

5. Application to the Hermite-Hadamard Inequality

Applying the integral method with the convex combinations to the inequalities obtained in the theorems, we get their integral forms. Using the Jensen type inequalities, we briefly demonstrate the generalization of the Hermite-Hadamard inequality (for essentials on this inequality see [9] or [10]).

Let $f : [a,b] \to \mathbb{R}$ be a convex function. Given the positive integer n, we make the partition $[a,b] = \cup_{i=1}^{n} [a_{ni}, b_{ni}]$, where all subsegments have the same length $(b-a)/n$, and the adjacent subsegments have a common endpoint. If we take subsegment centers $x_{ni} = (a_{ni} + b_{ni})/2$, then we have the convex combination equality:

$$c = \sum_{i=1}^{n} \frac{b_{ni} - a_{ni}}{b - a} x_{ni} = \frac{a + b}{2}. \tag{56}$$

Applying the inequality in (21) to the above convex combination, it follows

$$f\left(\frac{a+b}{2}\right) \leq \sum_{i=1}^{n} \frac{b_{ni} - a_{ni}}{b - a} f\left(x_{ni}\right) \leq \frac{f(a) + f(b)}{2}, \tag{57}$$

and letting n to infinity, we obtain the classic Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{[a,b]} f(x)\, dx \leq \frac{f(a) + f(b)}{2}. \tag{58}$$

The transition to the planar case can be done using Corollary 7. Let $f : \triangle \to \mathbb{R}$ be a convex function, where $\triangle = \operatorname{conv}\{A, B, C\}$ is the triangle with vertices A, B, and C. Given the positive integer n, we use the partition $\triangle = \cup_{i=1}^{n^2} \triangle_i$ with congruent subtriangles $\triangle_i = \operatorname{conv}\{A_{ni}, B_{ni}, C_{ni}\}$ having a common edge or endpoint if they are adjacent. So, the area $\operatorname{ar}(\triangle_i)$ of each subtriangle $\triangle_i$ is equal to $\operatorname{ar}(\triangle)/n^2$. If we take subtriangle centers $P_{ni} = (A_{ni} + B_{ni} + C_{ni})/3$, then we have the convex combination equality:

$$P = \sum_{i=1}^{n^2} \frac{\operatorname{ar}\left(\triangle_i\right)}{\operatorname{ar}\left(\triangle\right)} P_{ni} = \frac{A + B + C}{3}. \tag{59}$$

Applying the inequality in (37) to the above convex combination, it follows

$$f\left(\frac{A+B+C}{3}\right) \leq \sum_{i=1}^{n^2} \frac{\operatorname{ar}\left(\triangle_i\right)}{\operatorname{ar}\left(\triangle\right)} P_{ni} \leq \frac{f(A) + f(B) + f(C)}{3}, \tag{60}$$

and letting n to infinity, we obtain the planar Hermite-Hadamard inequality:

$$f\left(\frac{A+B+C}{3}\right) \leq \frac{1}{\operatorname{ar}\left(\triangle\right)} \int_{\triangle} f\left(x, y\right) dx\, dy \leq \frac{f(A) + f(B) + f(C)}{3}. \tag{61}$$

The transition to any dimension N suggests Corollary 11 using the N-simplex $\triangle = \operatorname{conv}\{P_1, \ldots, P_{N+1}\}$. Applying the previous procedure to $\triangle$, we reach the conclusion that every convex function $f : \triangle \to \mathbb{R}$ satisfies the inequality

$$f\left(\frac{1}{N+1} \sum_{k=1}^{N+1} P_k\right) \leq \frac{1}{\operatorname{vol}\left(\triangle\right)} \int_{\triangle} f\left(x_1, \ldots, x_N\right) dx_1 \cdots dx_N \leq \frac{1}{N+1} \sum_{k=1}^{N+1} f\left(P_k\right). \tag{62}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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