Applied Mathematical Sciences, Vol. 7, 2013, no. 128, 6381 - 6391
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2013.310597

A Mixed-type Distribution for Inventory Management

Marisa Cenci

Department of Business Studies
University of Rome III, Roma, Italy
marisa.cenci@uniroma3.it

Andrea Gheno

Department of Business Studies
University of Rome III, Roma, Italy
andrea.gheno@uniroma3.it

Abstract

This paper is concerned with the determination of optimal base-stock levels under the hypothesis of stochastic demand characterized by a mixed-type distribution. The advantages arising from the use of a mixed-type distribution are examined from an analytical point of view considering a distribution with a single discontinuity point. Some implications are discussed and a numerical analysis concerning two different non-negative distributions is performed.

Keywords: Inventory management; Base-stock policy; Newsvendor model; Mixed-type distribution.

1 Introduction

The inventory management problem under the hypothesis of stochastic demand has been widely studied in the literature because of its great theoretical interest as well as practical relevance. When making orders inventory managers have to take into account two sources of risk: the risk of ordering too little (understocking risk) and the risk of ordering too much (overstocking risk). When overstocking in respect to the actual demand the firm will incur in losses whilst in case of understocking the firm will lose profits.

A very general and elegant approach related to this problem is represented by the Newsvendor model where the objective is to find the order quantity for a given product that maximizes the expected profit of the firm in a single period characterized by a stochastic demand. As reported in Porteus [7] the model can be used as a building block for more complex inventory management problems.
In order to describe the Newsvendor model we use the following notation:

D non-negative random variable representing the stochastic demand
μ mean value of D
σ standard deviation of D
x base-stock level
c product unit cost of purchase
p product unit selling price
r product unit salvage value (where $r < c < p$)

If the linearity property holds for multiple units, i.e. there are no economies of scale, and if the demand probability distribution, $F(x) = \Pr(D \le x)$, is known and continuous, the Newsvendor model has an analytical solution x^* for the optimal base-stock level that maximizes the expected profit:

$$x^* = F^{-1}\left(\frac{p-c}{p-r}\right). \tag{1}$$

In practice the hypothesis of full knowledge of the demand probability distribution is rarely satisfied, hence the analysis of distribution-free inventory management problems has soon started developing. Representative papers of this strand of research are Gallego and Moon [2], Moon and Gallego [4], Moon and Choi [5] and Scarf [8]. However practitioners very often use a simplified model based on the implicit hypothesis that the demand distribution is Normal with known mean and standard deviation. In fact, as pointed out in Gallego et al. [1], even in presence of inventory management situations with significant demand uncertainty, as measured by the coefficient of variation $cv = \sigma/\mu$, inventory managers use base-stock levels of the form:

$$\bar{z} = \mu + z\sigma = \mu\left(1 + z \cdot cv\right) \tag{2}$$

where z is the Standard Normal distribution critical value corresponding to the service level α required in order to have $\Pr\left(D \le \bar{z}\right) = \alpha$. For example if $\alpha = 99\%$ and $cv = 1$, then the resulting base-stock level $\bar{z} = \mu\left(1 + 2{,}58 \cdot 1\right) = 3{,}58\mu$, i.e. over 3,5 times the expected demand, carries a considerable overstocking risk. In fact under the Normal distribution base-stock levels increase linearly respect to the standard deviation, resulting in excessively large orders. Furthermore, the assumption of a demand characterized by a Normal distribution introduces a distortion due to the fact that under such hypothesis the underlying random variable can take also negative values with positive probability.

Given these issues in several papers it is assumed that the demand has a non-negative distribution such as the Lognormal and the Gamma (see for instance Keaton [3], Namit and Chen [6], Strijbosch and Heuts [9], Tadikamalla [10], Tyworth and Ganeshan [11] and Tyworth et al. [12]). However non-negative distributions are not well suited to represent the demand: they may have the disadvantage to assign a null probability to the event that demand takes a null value, as it is often the case. In order to overcome such problem in this paper we propose for the demand a mixed-type distribution with a single discontinuity point. Moreover we find a condition under which the mixed-type distribution considered is able to reduce effectively the overstocking risk.

The remainder of this paper is organized as follows. In Section 2 an alternative method for optimal base-stock determination based on mixed distributions is proposed. Numerical results for Gamma and Lognormal distributions are reported and commented in Section 3. The conclusions are drawn in Section 4.

2 Optimal base-stock level under a mixed-type distribution

This paper proposes the use of a mixed-type distribution with a single discontinuity for the demand. In particular it is considered a distribution that assigns a non-zero probability to the minimum value that the demand can take. If we assume that such minimum value $D_{\min}$ is zero, the probability distribution function is null for $D < 0$, has a jump equal to $\Pr\left(D = 0\right)$ for $D = 0$, and is continuous for $D > 0$. In Fig. 1 is depicted an example of mixed-distribution with $\Pr\left(D = 0\right) = 0{,}2$.

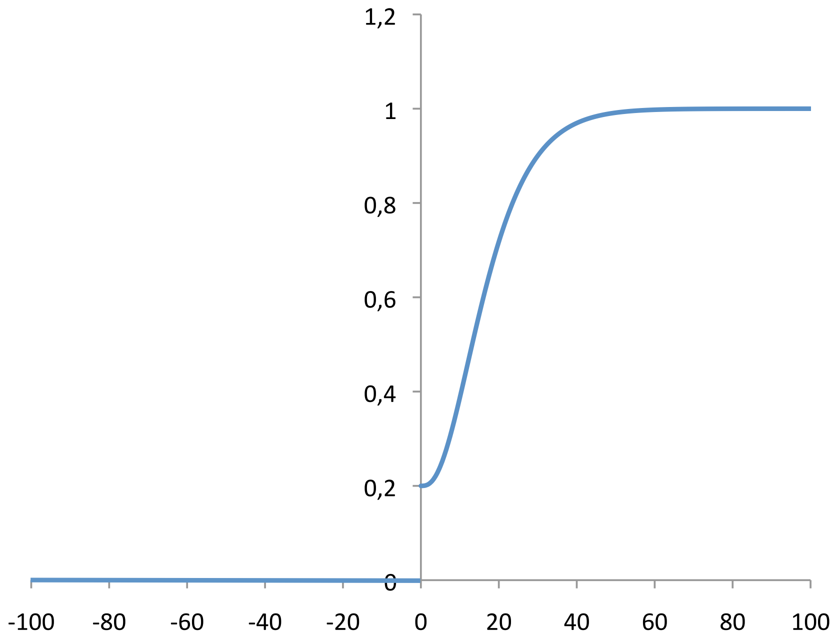

Figure 1. Example of mixed-distribution with $\Pr(D=0)=0,2$.

If we pose $F(x^-)=\Pr\left[D\in(-\infty,x)\right]$ it follows that:

$$\begin{cases} F(x)-F(x^-)=0 & \forall x>0 \\ F(x)-F(x^-)>0 & x=0 \end{cases} \tag{3}$$

The probability distribution function can thus be decomposed into two addends, a discrete component (jump related) and a continuous component:

$$F_M(x)=\Pr(D\le x)=\Pr(D=0)\Pr(D\le x|D=0)+\Pr(D>0)\Pr(D\le x|D>0) \tag{4}$$

Demand values greater than zero are represented by the demand continuous component (henceforth denoted with D_C) while values equal to zero are represented by the discrete component that is characterized by the following probability distribution function:

$$\Pr(D\le x|D=0)=F_D(x)=\begin{cases} 0 & x<0 \\ 1 & x\ge 0 \end{cases} \tag{5}$$

The corresponding probability distribution function for the continuous component is:

$$\Pr\left(D \le x \middle| D > 0\right) = F_C\left(x\right) = \begin{cases} 0 & x \le 0 \\ \int_0^x f\left(\xi\right) d\xi & x > 0 \end{cases} \tag{6}$$

where $f(\bullet)$ is the probability density function of a positive random variable. Moreover since D can take non-negative values only, it results that:

$$\Pr(D > 0) = 1 - \Pr(D = 0). \tag{7}$$

Posing $\Pr(D = 0) = P_o$, it can be inferred that:

$$\mu = (1 - P_0)\mu_C \tag{8}$$

and

$$\sigma^2 = (1 - P_0)\sigma_C^2 + P_0(1 - P_0)\mu_C^2 \tag{9}$$

where μ_C and σ_C^2 respectively are the mean and the variance of D_C.

Remark 1. Adopting the notation introduced in Eq. (6), we can express Eq. (4) as $F_M(x) = P_0 + (1 - P_0)F_C(x)$.

Remark 2. If the minimum demand value is strictly positive (i.e. there is always positive demand) the discontinuity point moves from $D = 0$ to $D = D_{\min}$. In this case by substituting D with $Y = D - D_{\min}$ into Eqs. (4)–(9) we obtain the corresponding expressions in the case of strictly positive minimum demand value.

2.1 Optimal Base-Stock Level

The optimal base-stock level can be determined as the minimum level of inventory $\bar{x}$ that is necessary in order to satisfy demand with a specified probability α, i.e. the level $\bar{x}$ such that $\Pr(D \le \bar{x}) = \alpha$ where $\Pr(D \le \bar{x})$ is defined as in Eq. (4).

Proposition 1. If D has the following mixed-type probability distribution:

$$F_M(x) = P_0 + (1 - P_0) F_C(x) \tag{10}$$

and $\bar{x}$ is the minimum demand value such that $F(\bar{x}) = \alpha$, with $0 < \alpha < 1$, then $\bar{x}$ is an increasing function of α and a decreasing function of P_0.

Proof. From Eq. (4) it can be inferred that $\alpha = P_0 + (1 - P_0) F_C(\bar{x})$, hence:

$$\bar{x} = F_C^{-1}\left(\frac{\alpha - P_0}{1 - P_0}\right). \tag{11}$$

Since $F_C(x)$ is an increasing function of x, $F_C^{-1}(x)$ is an increasing function of x too and therefore the signs of its partial derivatives respect to α and P_0 depend only on the signs of the partial derivatives of its argument. Given that:

$$\frac{\partial}{\partial \alpha}\left(\frac{\alpha - P_0}{1 - P_0}\right) = \frac{1}{1 - P_0} > 0, \tag{12}$$

and

$$\frac{\partial}{\partial P_0}\left(\frac{\alpha - P_0}{1 - P_0}\right) = \frac{\alpha - 1}{(1 - P_0)^2} < 0 \tag{13}$$

it follows that Proposition 1 holds.

2.2 Overstocking risk mitigation

Inventory managers do not commonly use mixed-type distributions but rather refer to continuous distributions with first and second order moments matching those of the sample available, in spite of the fact that some continuous distributions adopted can take negative values or are not defined in zero. To this extent the use of mixed-type distributions can be preferred over the use of continuous distributions especially if it leads to an optimal base-stock level lower than that obtained with the continuous distribution. Therefore it is interesting to investigate the conditions that ensure a reduction of the overstocking risk for significant levels of α assuming that the continuous component of the mixed-type distribution and the continuous distribution used belong to the same family.

Proposition 2. Let $F_M(x) = P_0 + (1-P_0)F_C(x)$ and $F(x)$ respectively be a mixed-type distribution and the corresponding continuous distribution. For $x \to \infty$ the mixed-type distribution dominates the continuous distribution if $\sigma_C^2 < \sigma^2$, where σ_C^2 and σ^2 respectively are the variance of D_C and the variance of D.

Proof. The value of the limit $\lim_{x\to\infty} \frac{1-F_M(x)}{1-F(x)} = \lim_{x\to\infty}(1-P_0)\frac{1-F_C(x)}{1-F(x)}$ depends on the tail heaviness of the mixed-type distribution and of the corresponding continuous distribution. Hence if $\sigma_C^2 < \sigma^2$ the limit is null and the mixed-type distribution dominates the continuous distribution.

If $\sigma_C^2 < \sigma^2$, given that $\mu < \mu_C$, the distributions either cross at two points or do not cross.

Posing:

$$\alpha^* = \begin{cases} \max\left\{F(x) \text{ such that } F_M(x) = F(x)\right\} & \text{if distributions cross} \\ 0 \quad \text{if distributions do not cross} \end{cases} \tag{14}$$

the following corollary holds.

Corollary 1. If $1-P_0 > \sigma_C^2/\mu_C^2$, for $\alpha > \alpha^*$ it follows that $\overline{x} < \overline{\overline{x}}$, where $\overline{\overline{x}} = F^{-1}(\alpha)$ is the optimal base-stock level of the continuous distribution.

Proof. From Eq. (9) it immediately follows that if $1-P_0 > \sigma_C^2/\mu_C^2$ then $\sigma_C^2 < \sigma^2$. Therefore for $\alpha > \alpha^*$ the mixed-type distribution dominates the continuous distribution, hence the thesis of Corollary 1 follows.

Remark 3. For a service level $\alpha > \alpha^*$, the sufficient condition introduced in Corollary 1 allows by using a mixed-type distribution to determine an optimal base-stock level able to reduce the risk of overstocking respect to using a continuous distribution. Alternatively the condition may be expressed as $cv_C^2 < 1-P_0$ where $cv_C = \sigma_C/\mu_C$.

Remark 4. If the mixed-type distribution and the corresponding continuous distribution cross only once, the sufficient condition of Corollary 1 does not hold and for $\alpha > \alpha^*$ it follows that $\overline{x} > \overline{\overline{x}}$.

3 Numerical analysis

In this section we provide some evidence of the overstocking risk mitigation if mixed-type distributions are used for the demand process. In particular we take into consideration two commonly used non-negative continuous distributions: the Gamma distribution and the Lognormal distribution. We focus on the optimal base-stock levels variation by comparing optimal base-stock levels obtained in our framework with those obtained by practitioners using the corresponding continuous distributions.

As measure of variation we adopt $\Delta = \left(\bar{\bar{x}} - \bar{x}\right)/\bar{x}$ where $\bar{\bar{x}}$ and $\bar{x}$ respectively are the optimal base-stock level of the continuous distribution and the optimal base-stock level of the mixed-type distribution. For positive values, the greater Δ, the more advantageous the use of mixed-type distribution is in terms of overstocking risk mitigation. A sensitivity analysis is also performed respect to the service level α, the probability P_0 that D takes a null value, and the coefficient of variation of the continuous component of the mixed distribution cv_C. For each level of cv_C the indifference service level value α^* is also computed. The results for the Gamma distribution and the Lognormal distribution are respectively reported in Tables 1 and 2.

From Tables 1 and 2 the following conclusions can be drawn:

1. For any given level of cv_C and P_0, the higher the service level required α, the higher the optimal base-stock level reduction.
2. For any given level of α and P_0, the higher the cv_C level, the smaller the optimal base-stock level reduction.
3. For any given level of P_0, the higher the cv_C level, the higher the indifference service level value α^*.

Table 1. Gamma distribution: optimal base-stock level variation Δ.

$P_0 = 0,1$				
cv_c	0,25	0,5	0,75	1
$\alpha = 0,90$	7,55%	1,50%	-0,35%	-0,94%
$\alpha = 0,95$	12,61%	4,07%	1,27%	*0,20%*
$\alpha = 0,99$	21,77%	8,43%	3,94%	*2,05%*
$\alpha^* =$	0,7727	0,8597	0,9127	0,9429

$P_0 = 0,2$				
cv_c	0,25	0,5	0,75	1
$\alpha = 0,90$	9,52%	1,30%	-1,46%	-2,32%
$\alpha = 0,95$	18,65%	6,29%	1,76%	-0,04%
$\alpha = 0,99$	35,68%	14,93%	7,14%	*3,73%*
$\alpha^* =$	0,8158	0,8829	0,9256	0,9508

$P_0 = 0,3$				
cv_c	0,25	0,5	0,75	1
$\alpha = 0,90$	8,33%	-0,35%	-3,35%	-4,19%
$\alpha = 0,95$	21,04%	6,95%	1,46%	-0,77%
$\alpha = 0,99$	45,31%	19,82%	9,62%	*5,00%*
$\alpha^* =$	0,8503	0,9030	0,9374	0,9581

$P_0 = 0,4$				
	0,25	0,5	0,75	1
$\alpha = 0,90$	4,65%	-3,39%	-6,08%	-6,60%
$\alpha = 0,95$	20,56%	6,12%	0,29%	-2,05%
$\alpha = 0,99$	51,64%	23,23%	11,33%	*5,81%*
$\alpha^* =$	0,8794	0,9208	0,9482	0,9650

Table 1 (continues). Gamma distribution: optimal base-stock level variation Δ .

$P_0 = 0,5$				
cv_c	0,25	0,5	0,75	1
$\alpha = 0,90$	-1,41%	-7,93%	-9,77%	-9,64%
$\alpha = 0,95$	17,37%	3,71%	-1,87%	-4,02%
$\alpha = 0,99$	54,94%	25,09%	*12,16%*	*6,05%*
$\alpha^* =$	0,9049	0,9369	0,9583	0,9716

$P_0 = 0,6$				
cv_c	0,25	0,5	0,75	1
$\alpha = 0,90$	9,52%	1,30%	-1,46%	-2,32%
$\alpha = 0,95$	18,65%	6,29%	1,76%	-0,04%
$\alpha = 0,99$	35,68%	14,93%	7,14%	*3,73%*
$\alpha^* =$	0,8158	0,8829	0,9256	0,9508

$P_0 = 0,7$				
cv_c	0,25	0,5	0,75	1
$\alpha = 0,90$	-22,00%	-22,78%	-20,89%	-17,73%
$\alpha = 0,95$	1,26%	-7,45%	-10,51%	-11,01%
$\alpha = 0,99$	50,91%	22,85%	*10,11%*	*4,00%*
$\alpha^* =$	0,9480	0,9651	0,9765	0,9838

$P_0 = 0,8$				
	0,25	0,5	0,75	1
$\alpha = 0,90$	-38,42%	-34,16%	-27,90%	-20,13%
$\alpha = 0,95$	-14,35%	-18,31%	-18,52%	-16,96%
$\alpha = 0,99$	40,62%	*16,69%*	*5,72%*	*0,55%*
$\alpha^* =$	0,9668	0,9776	0,9848	0,9894

Table 2. Lognormal distribution: optimal base-stock level variation Δ .

$P_0 = 0,1$				
cv_c	0,25	0,5	0,75	1
$\alpha = 0,90$	6,56%	-0,01%	-2,14%	-2,87%
$\alpha = 0,95$	12,73%	3,26%	-0,07%	-1,45%
$\alpha = 0,99$	25,45%	9,99%	4,32%	*1,72%*
$\alpha^* =$	0,8063	0,9002	0,9512	0,9800

$P_0 = 0,2$				
cv_c	0,25	0,5	0,75	1
$\alpha = 0,90$	6,69%	-2,17%	-5,32%	-6,36%
$\alpha = 0,95$	17,79%	4,02%	-1,37%	-3,68%
$\alpha = 0,99$	42,28%	17,42%	7,39%	*2,64%*
$\alpha^* =$	0,8509	0,9213	0,9608	0,9803

$P_0 = 0,3$				
cv_c	0,25	0,5	0,75	1
$\alpha = 0,90$	3,05%	-6,15%	-9,50%	-10,48%
$\alpha = 0,95$	18,15%	2,57%	-3,91%	-6,72%
$\alpha = 0,99$	53,53%	22,41%	9,09%	*2,62%*
$\alpha^* =$	0,8851	0,9386	0,9691	0,9844

$P_0 = 0,4$				
	0,25	0,5	0,75	1
$\alpha = 0,90$	-3,50%	-11,78%	-14,68%	-15,25%
$\alpha = 0,95$	14,64%	-1,03%	-7,78%	-10,67%
$\alpha = 0,99$	59,74%	24,80%	9,20%	*1,48%*
$\alpha^* =$	0,9127	0,9532	0,9763	0,9879

$P_0 = 0,5$				
cv_c	0,25	0,5	0,75	1
$\alpha = 0,90$	-12,57%	-18,99%	-20,86%	-20,67%
$\alpha = 0,95$	7,55%	-6,82%	-13,07%	-15,63%
$\alpha = 0,99$	60,70%	24,23%	*7,41%*	-1,05%
$\alpha^* =$	0,9356	0,9655	0,9824	0,9910

$P_0 = 0,6$				
cv_c	0,25	0,5	0,75	1
$\alpha = 0,90$	-23,95%	-27,75%	-28,04%	-26,67%
$\alpha = 0,95$	-3,14%	-14,96%	-19,98%	-21,76%
$\alpha = 0,99$	55,63%	20,08%	*3,23%*	-5,34%
$\alpha^* =$	0,9546	0,9758	0,9877	0,9937

$P_0 = 0,7$				
cv_c	0,25	0,5	0,75	1
$\alpha = 0,90$	-37,55%	-38,03%	-36,07%	-32,93%
$\alpha = 0,95$	-17,67%	-25,77%	-28,77%	-29,24%
$\alpha = 0,99$	43,13%	*11,28%*	-4,17%	-12,07%
$\alpha^* =$	0,9705	0,9844	0,9920	0,9959

$P_0 = 0,8$				
	0,25	0,5	0,75	1
$\alpha = 0,90$	-53,14%	-49,30%	-43,82%	-37,43%
$\alpha = 0,95$	-36,53%	-39,72%	-39,73%	-38,11%
$\alpha = 0,99$	20,59%	-4,10%	-16,27%	-22,39%
$\alpha^* =$	0,9835	0,9914	0,9957	0,9978

The results also show that even if the sufficient condition of Corollary 1 is not satisfied, i.e. if $cv_C^2 > 1 - P_0$, it may be possible by using a mixed-type distribution to determine an optimal base-stock level able to reduce the overstocking risk respect to using a continuous distribution (values in italics in Table 1 and Table 2).

4 Final remarks

This paper suggests the use of a mixed-type distribution to determine the optimal base-stock level in inventory management problems. In particular the proposed framework assumes a demand probability distribution with a jump corresponding to the sample minimum value. A sufficient condition for overstocking risk mitigation is also derived and a numerical analysis is performed. Such analysis shows that a significant reduction of the overstocking risk may be obtained when using the mixed-type distribution as opposed to using the corresponding continuous component for Lognormal and Gamma distributions.

References

[1] G. Gallego, K. Katircioglu and B. Ramachandran, Inventory management under highly uncertain demand, *Operations Research Letters* **35** (2007), 281–289.

[2] G. Gallego and I. Moon, The distribution free newsboy problem: Review and extensions, *Journal of Operational research Society* **44** (1993), 825–834.

[3] M. Keaton, Using the gamma distribution to model demand when lead time is random, *Journal of Business Logistics* **16** (1995), 107–131.

[4] I. Moon and G. Gallego, Distribution free procedures for some inventory models, *Journal of Operational Research Society* **45** (1994), 651–658.

[5] I. Moon and S. Choi, The distribution free continuous review inventory system with a service level constraint, *Computers and Industrial Engineering* **27** (1994), 209–212.

[6] K. Namit and J. Chen, Solutions to the (Q,r) inventory model for gamma lead time demand, *International Journal of Physical Distribution and Logistics Management* **29 (**1999), 138–151.

[7] E. Porteus, Stochastic inventory theory. In: D. Heyman and M. Sobel (Eds.), *Handbook in Operations Research and Management Science 2*, Elsevier North-Holland, Amsterdam, 1990.

[8] H. Scarf, (1958). A min-max solution to an inventory problem. In: K. Arrow, S. Karlin and H. Scarf (Eds.), *Studies in Mathematical Theory of Inventory and Production*, Stanford University Press, Stanford, 1958.

[9] L. Strijbosch, and R. Heuts, Modelling (s,Q) inventory systems: Parametric versus non-parametric approximations for the lead time demand distribution. *European Journal of Operational Research* **63 (**1992), 86–101.

[10] P. Tadikamalla, A comparison of several approximations to the lead time demand distribution, *Omega* **12** (1984), 575–581.

[11] J. Tyworth, and R. Ganeshan, A note on solutions to the (Q,r) inventory model for gamma lead-time demand, *International Journal of Physical Distribution and Logistics Management* **30** (2000), 534–539.

[12] J. Tyworth, Y. Guo and R. Ganeshan, Inventory control under gamma demand and random lead time, *Journal of Business Logistics* **17** (1996), 291–304.

Received: October 25, 2013