

QUADRUPLE COMMON FIXED POINT THEOREMS IN G_b -METRIC SPACES

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Abstract: In this paper, we prove a quadruple common fixed point theorem in G_b -metric space and some results are also given as corollaries. The results obtained are verified with the help of examples.

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Key Words: quadruple fixed point, quadruple coincidence point, quadruple common fixed point, G -metric space, b -metric space, G_b -metric space

1. Introduction

In 1987, Guo and Lakshmikantham (see [1]) introduced the notion of coupled fixed point. In 2006, Bhaskar and Lakshmikantham (see [2]) proved a fixed point theorem for a mixed monotone mapping in a metric space endowed with a partial ordering, using a weak contractivity type of assumption. As an application, they also proved the existence and uniqueness of the solution for a periodic boundary value problem. Later in 2009, Lakshmikantham and Ćirić (see [3]) introduced the concept of mixed g -monotone mappings and proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric space, which also generalized the fixed point theorems due to Bhaskar and Lakshmikantham (see

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[2]). In 2011, V. Berinde and M. Borcut (see [4]) introduced the concept of tripled fixed point for nonlinear mappings in partially ordered complete metric spaces and obtained existence and uniqueness theorems for contractive type mappings. Later, E. Karapnar and N. V. Luong introduced the concept of quadruple fixed point and proved some quadruple fixed point results (see [5]).

The concept of b -metric space was introduced by Bakhtin (see [6]) and later used by Czerwik (see [7]). For more results on b -metric space (see [8], [9], etc.).

Mustafa and Sims introduced the concept of G -metric spaces as a generalization of metric spaces (see [10]). Later A. Aghajani, M. Abbas and J.R. Roshan introduced the concept of G_b -metric spaces using the concepts of G -metric and b -metric (see [11]). For more results on coupled, triple and quadruple see research papers in (see [1-5, 9, 2-17]).

In this paper we will prove a quadruple common fixed point theorem in G_b -metric space and some results as corollaries. In order to start our work we need the following preliminaries.

Definition 1. (see [11]) Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \rightarrow R^+$ satisfies:

$$(G_b1) \quad G(x, y, z) = 0 \text{ if } x = y = z,$$

$$(G_b2) \quad 0 < G(x, y, z) \text{ for all } x, y \in X \text{ with } x \neq y,$$

$$(G_b3) \quad G(x, x, y) < G(x, y, z) \text{ for all } x, y, z \in X \text{ with } y \neq z,$$

$$(G_b4) \quad G(x, y, z) = G(p\{x, y, z\}), \text{ where } p \text{ is a permutation of } x, y, z \text{ (symmetry)},$$

$$(G_b5) \quad G(x, y, z) \leq s(G(x, a, a) + G(a, y, z)) \text{ for all } x, y, z, a \in X \text{ (rectangular inequality)}.$$

Then G is called a generalized b -metric and the pair (X, G) is called a generalized b -metric space or G_b -metric space. It should be noted that each G -metric space is a G_b -metric space with $s = 1$.

Example 2. (see [12]) Let (X, G) be a G -metric space and $G_*(x, y, z) = G(x, y, z)^p$, where $p > 1$ is a real number. Then G is a G_b -metric with $s = 2^{p-1}$.

For more examples on G_b -metric spaces (see [11], [12], [13]).

Definition 3. (see [11]) A G_b -metric space is said to be symmetric if

$$G(x, y, y) = G(y, x, x) \text{ for all } x, y \in X.$$

Definition 4. (see [11]) Let (X, G) be a G_b -metric space. Then, for $x_0 \in X, r > 0$, the G_b -ball with centre x_0 and radius r is

$$B_G(x_0, r) = \{y \in X | G(x_0, y, y) < r\}.$$

Definition 5. (see [11]) Let X be a G_b -metric space and let $d_G(x, y) = G(x, y, y) + G(y, x, x)$. Then, d_G defines a b -metric on X , which is called the b -metric associated with G .

Proposition 6. (see [11]) Let X be a G_b -metric space. For any $x_0 \in X$ and $r > 0$, if $y \in B_G(x_0, r)$ then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(x_0, r)$.
From the above proposition the family of all G_b -balls

$$\Lambda = \{B_G(x, r) | x \in X, r > 0\}$$

is a base of a topology $\tau(G)$ on X , which is called the G_b -metric topology.

Definition 7. (see [11]) Let X be a G_b -metric space. A sequence $\{x_n\}$ in X is said to be:

1. G_b -Cauchy sequence if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m, n, l \geq n_0, G(x_n, x_m, x_l) < \varepsilon$;
2. G_b -convergent to a point $x \in X$ if, for all $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m, n \geq n_0, G(x_n, x_m, x) < \varepsilon$.

Definition 8. (see [11]) A G_b -metric space X is called complete if every G_b -Cauchy sequence is G_b -convergent in X .

Definition 9. (see [11]) Let (X, G) and (X', G') be G_b -metric spaces and let $f : X \rightarrow X'$ be a mapping. Then f is said to be continuous at a point $a \in X$ if and only if every $\varepsilon > 0$ there is $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is continuous on X if and only if it is continuous at all $a \in X$.

Definition 10. (see [5]) Let X be nonempty set. An element $(x, y, z, w) \in X^4$ is called a quadruple fixed point of a mapping $F : X^4 \rightarrow X$ if

$$\begin{aligned} F(x, y, z, w) &= x, \\ F(y, z, w, x) &= y, \\ F(z, w, x, y) &= z, \\ \text{and } F(w, x, y, z) &= w. \end{aligned}$$

Definition 11. (see [14]) Let X be a nonempty set. An element $(x, y, z, w) \in X^4$ is called a quadruple coincidence point of mappings $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ if

$$\begin{aligned} F(x, y, z, w) &= gx, \\ F(y, z, w, x) &= gy, \\ F(z, w, x, y) &= gz, \\ \text{and } F(w, x, y, z) &= gw. \end{aligned}$$

Definition 12. (see [14]) Let X be a nonempty set. An element $(x, y, z, w) \in X^4$ is called a quadruple common fixed point of mappings $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ if

$$\begin{aligned} F(x, y, z, w) &= gx = x, \\ F(y, z, w, x) &= gy = y, \\ F(z, w, x, y) &= gz = z, \\ \text{and } F(w, x, y, z) &= gw = w. \end{aligned}$$

Definition 13. (see [14]) Let X be a nonempty set. Then we say that the mappings $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ are commutative if for all $x, y, z, w \in X$

$$g(F(x, y, z, w)) = F(gx, gy, gz, gw).$$

2. Results

Let Φ denote the class of all function $\phi : [0, \infty) \rightarrow [0, \infty)$ such that ϕ is increasing, continuous, $\phi(t) < \frac{t}{4}$ for all $t > 0$ and $\phi(0) = 0$. It is easy to see that for every $\phi \in \Phi$, we can choose a $0 < k < \frac{1}{4}$ such that $\phi(t) \leq kt$.

Lemma 14. (see [15]) Let (X, G) be a G_b -metric space with $s \geq 1$, and suppose that $\{x_n\}$ is G_b -convergent to x . Then we have

$$\frac{1}{s}G(x, y, y) \leq \liminf_{n \rightarrow \infty} G(x_n, y, y) \leq \limsup_{n \rightarrow \infty} G(x_n, y, y) \leq sG(x, y, y)$$

In particular, if $x = y$, then we have $\lim_{n \rightarrow \infty} G(x_n, y, y) = 0$.

Lemma 15. Let (X, G) be a G_b -metric space and let $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ be two mappings such that

$$G(F(x, y, z, w), F(u, v, r, t), F(a, b, c, d)) \leq \phi(G(gx, gu, ga) + G(gy, gv, gb))$$

$$+ G(gz, gr, gc) + G(gw, gt, gd))$$

zz for some $\phi \in \Phi$ and for all $x, y, z, w, u, v, r, t, a, b, c, d \in X$. Assume that (x, y, z, w) is a quadruple coincidence point of the mappings F and g . Then

$$\begin{aligned} gx &= gy = gz = gw \\ &= F(x, y, z, w) = F(y, z, w, x) = F(z, w, x, y) = F(w, x, y, z). \end{aligned}$$

Proof. Since (x, y, z, w) is a quadruple coincidence point of F and g , we have

$$\begin{aligned} gx &= F(x, y, z, w), \\ gy &= F(y, z, w, x), \\ gz &= F(z, w, x, y), \\ \text{and } gw &= F(w, x, y, z). \end{aligned}$$

In order to prove the lemma, it will be sufficed to show that

$$gx = gy = gz = gw.$$

On the contrary assume that any two of gx, gy, gz and gw are different, say $gx \neq gy$.

Now,

$$\begin{aligned} G(gx, gy, gy) &= G(F(x, y, z, w), F(y, z, w, x), F(y, z, w, x)) \\ &\leq \phi(G(gx, gy, gy) + G(gy, gz, gz) \\ &\quad + G(gz, gw, gw) + G(gw, gx, gx)). \end{aligned}$$

Similarly,

$$\begin{aligned} G(gy, gz, gz) &\leq \phi(G(gy, gz, gz) + G(gz, gw, gw) \\ &\quad + G(gw, gx, gx) + G(gx, gy, gy)), \end{aligned}$$

$$\begin{aligned} G(gz, gw, gw) &\leq \phi(G(gz, gw, gw) + G(gw, gx, gx) \\ &\quad + G(gx, gy, gy) + G(gy, gz, gz)), \end{aligned}$$

and

$$\begin{aligned} G(gw, gx, gx) &\leq \phi(G(gw, gx, gx) + G(gx, gy, gy) \\ &\quad + G(gy, gz, gz) + G(gz, gw, gw)). \end{aligned}$$

Therefore,

$$\begin{aligned} G(gx, gy, gy) + G(gy, gz, gz) + G(gz, gw, gw) + G(gw, gx, gx) \\ \leq 4(\phi(G(gx, gy, gy) + G(gy, gz, gz) \\ + G(gz, gw, gw) + G(gw, gx, gx))). \end{aligned}$$

As $\phi(t) < \frac{t}{4}$, we have

$$\begin{aligned} G(gx, gy, gy) + G(gy, gz, gz) + G(gz, gw, gw) + G(gw, gx, gx) \\ < G(gx, gy, gy) + G(gy, gz, gz) + G(gz, gw, gw) \\ + G(gw, gx, gx) \end{aligned}$$

which is a contradiction. So, $gx = gy = gz = gw$ and hence

$$gx = gy = gz = gw = F(x, y, z, w) = F(y, z, w, x) = F(z, w, x, y) = F(w, x, y, z).$$

□

Theorem 16. Let (X, G) be a complete G_b -metric space. Let $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ be two mappings such that

$$\begin{aligned} G(F(x, y, z, w), F(u, v, r, t), F(a, b, c, d)) \\ \leq \frac{1}{s^2} \phi(G(gx, gu, ga) + G(gy, gv, gb) + G(gz, gr, gc) + (gw, gt, gd)) \end{aligned}$$

for some $\phi \in \Phi$ and for all $x, y, z, w, u, v, r, t, a, b, c, d \in X$. Assume that F and g satisfy the following conditions:

1. $F(X^4) \subseteq g(X)$,
2. $g(X)$ is complete and
3. g is continuous and commutes with F .

Then F and g have a unique quadruple common fixed point, and which is of the form (x, x, x, x) i.e. there is a unique $x \in X$ such that $gx = F(x, x, x, x) = x$.

Proof. Let $x_0, y_0, z_0, w_0 \in X$. Since $F(X^4) \subseteq g(X)$. We can choose $x_1, y_1, z_1, w_1 \in X$ such that $gx_1 = F(x_0, y_0, z_0, w_0)$, $gy_1 = F(y_0, z_0, w_0, x_0)$, $gz_1 = F(z_0, w_0, x_0, y_0)$ and $gw_1 = F(w_0, x_0, y_0, z_0)$. Continuing this process,

we can construct four sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ in X such that $gx_{n+1} = F(x_n, y_n, z_n, w_n)$, $gy_{n+1} = F(y_n, z_n, w_n, x_n)$, $gz_{n+1} = F(z_n, w_n, x_n, y_n)$ and $gw_{n+1} = F(w_n, x_n, y_n, z_n)$. For $n \in N \cup \{0\}$, we have

$$\begin{aligned} G(gx_{n-1}, gx_n, gx_n) &= G(F(x_{n-2}, y_{n-2}, z_{n-2}, w_{n-2}), F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}), \\ &\quad F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1})) \\ &\leq \frac{1}{s^2} \phi(G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G(gy_{n-2}, gy_{n-1}, gy_{n-1}) \\ &\quad + G(gz_{n-2}, gz_{n-1}, gz_{n-1}) + G(gw_{n-2}, gw_{n-1}, gw_{n-1})), \end{aligned}$$

Similarly, we have

$$\begin{aligned} G(gy_{n-1}, gy_n, gy_n) &\leq \frac{1}{s^2} \phi(G(gy_{n-2}, gy_{n-1}, gy_{n-1}) + G(gz_{n-2}, gz_{n-1}, gz_{n-1}) \\ &\quad + G(gw_{n-2}, gw_{n-1}, gw_{n-1}) + G(gx_{n-2}, gx_{n-1}, gx_{n-1})), \end{aligned}$$

$$\begin{aligned} G(gz_{n-1}, gz_n, gz_n) &\leq \frac{1}{s^2} \phi(G(gz_{n-2}, gz_{n-1}, gz_{n-1}) + G(gw_{n-2}, gw_{n-1}, gw_{n-1}) \\ &\quad + G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G(gy_{n-2}, gy_{n-1}, gy_{n-1})), \end{aligned}$$

and

$$\begin{aligned} G(gw_{n-1}, gw_n, gw_n) &\leq \frac{1}{s^2} \phi(G(gw_{n-2}, gw_{n-1}, gw_{n-1}) + G(gx_{n-2}, gx_{n-1}, gx_{n-1}) \\ &\quad + G(gy_{n-2}, gy_{n-1}, gy_{n-1}) + G(gz_{n-2}, gz_{n-1}, gz_{n-1})). \end{aligned}$$

Hence, we have

$$\begin{aligned} p_n &:= G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n) \\ &\quad + G(gz_{n-1}, gz_n, gz_n) + G(gw_{n-1}, gw_n, gw_n) \\ &\leq \frac{4}{s^2} \phi(G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G(gy_{n-2}, gy_{n-1}, gy_{n-1}) \\ &\quad + G(gz_{n-2}, gz_{n-1}, gz_{n-1}) + G(gw_{n-2}, gw_{n-1}, gw_{n-1})) \\ &\leq \frac{4}{s^2} \phi(p_{n-1}) \end{aligned}$$

for all $n \in N$.

Thus we get a k , $0 < k < \frac{1}{4}$ such that

$$p_n \leq \frac{4}{s^2} \phi(p_{n-1}) \leq \frac{4k}{s^2} p_{n-1} \leq \frac{4k}{s} p_{n-1} = qp_{n-1}$$

for $q = \frac{4k}{s}$. Hence we have

$$p_n \leq \frac{4k}{s} p_{n-1} \leq \dots \leq \left(\frac{4k}{s}\right)^n p_0.$$

Let $m, n \in N$ with $m > n$. By (G_b5) of definition of G_b -metric spaces, we have

$$\begin{aligned} & G(gx_{n-1}, gx_m, gx_m) + G(gy_{n-1}, gy_m, gy_m) + G(gz_{n-1}, gz_m, gz_m) \\ & + G(gw_{n-1}, gw_m, gw_m) \\ \leq & s[G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n) + G(gz_{n-1}, gz_n, gz_n) \\ & + G(gw_{n-1}, gw_n, gw_n)] \\ & + s[G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m) + G(gz_n, gz_m, gz_m) \\ & + G(gw_n, gw_m, gw_m)] \\ \leq & \dots \leq sp_n + s^2p_{n+1} + s^3p_{n+2} + \dots + s^{m-n}p_{m-1} + s^{m-n}p_m \\ \leq & sq^n p_0 + s^2q^{n+1}p_0 + \dots + s^{m-n}q^{m-1}p_0 + s^{m-n}q^m p_0 \\ \leq & sq^n p_0(1 + sq + s^2q^2 + \dots) \\ \leq & \frac{sq^n p_0}{1 - sq} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

since $sq = 4k < 1$.

Thus, $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$ and $\{gw_n\}$ are G_b -Cauchy in $g(X)$. Since $g(X)$ is complete, we get $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$ and $\{gw_n\}$ are G_b -convergent to some x, y, z and w in X respectively. Since g is continuous, $\{ggx_n\}$, $\{ggy_n\}$, $\{ggz_n\}$ and $\{ggw_n\}$ are G_b -convergent to gx, gy, gz and gw respectively. Also, since g and F commute, we have

$$\begin{aligned} ggx_{n+1} &= gF(x_n, y_n, z_n, w_n) = F(gx_n, gy_n, gz_n, gw_n), \\ ggy_{n+1} &= gF(y_n, z_n, w_n, x_n) = F(gy_n, gz_n, gw_n, gx_n), \\ ggz_{n+1} &= gF(z_n, w_n, x_n, y_n) = F(gz_n, gw_n, gx_n, gy_n), \\ \text{and} \quad ggw_{n+1} &= gF(w_n, x_n, y_n, z_n) = F(gw_n, gx_n, gy_n, gz_n). \end{aligned}$$

Thus,

$$\begin{aligned} & G(ggx_{n+1}, F(x, y, z, w), F(x, y, z, w)) = G(F(gx_n, gy_n, gz_n, gw_n), \\ & F(x, y, z, w), F(x, y, z, w)) \\ \leq & \frac{1}{s^2} \phi(G(ggx_n, gx, gx) + G(ggy_n, gy, gy) \\ & + G(ggz_n, gz, gz) + G(ggw_n, gw, gw)). \end{aligned}$$

Letting $n \rightarrow \infty$, and using Lemma 14, we have

$$\begin{aligned} \frac{1}{s}G(gx, F(x, y, z, w), F(x, y, z, w)) &\leq \limsup_{n \rightarrow \infty} G(F(gx_n, gy_n, gz_n, gw_n), \\ &\quad F(x, y, z, w), F(x, y, z, w)) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{s^2}\phi(G(ggx_n, gx, gx) \\ &\quad + G(ggy_n, gy, gy) \\ &\quad + G(ggz_n, gz, gz) + G(ggw_n, gw, gw)) \\ &\leq \frac{1}{s^2}\phi(s(G(gx, gx, gx) + G(gy, gy, gy) \\ &\quad + G(gz, gz, gz) + G(gw, gw, gw))) = 0. \end{aligned}$$

Hence,

$$gx = F(x, y, z, w).$$

Similarly, we can show that

$$\begin{aligned} gy &= F(y, z, w, x), \\ gz &= F(z, w, x, y), \\ gw &= F(w, x, y, z). \end{aligned}$$

By Lemma 15, we have

$$\begin{aligned} gx = gy = gz = gw \\ = F(x, y, z, w) = F(y, z, w, x) = F(z, w, x, y) = F(w, x, y, z). \end{aligned}$$

Thus, using Lemma 14, we have

$$\begin{aligned} \frac{1}{s}G(x, gx, gx) &\leq \limsup_{n \rightarrow \infty} G(gx_{n+1}, gx, gx) \\ &= \limsup_{n \rightarrow \infty} G(F(gx_n, gy_n, gz_n, gw_n), F(x, y, z, w), F(x, y, z, w)) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{s^2}\phi(G(gx_n, gx, gx) + G(gy_n, gy, gy) \\ &\quad + G(gz_n, gz, gz) + G(gw_n, gw, gw)) \\ &\leq \frac{1}{s^2}\phi(s(G(x, gx, gx) + G(y, gy, gy) + G(z, gz, gz) \\ &\quad + G(w, gw, gw))). \end{aligned}$$

Hence

$$G(x, gx, gx) \leq \frac{1}{s}\phi(s(G(x, gx, gx) + G(y, gy, gy) + G(z, gz, gz) + G(w, gw, gw))),$$

$$G(y, gy, gy) \leq \frac{1}{s} \phi(s(G(y, gy, gy) + G(z, gz, gz) + G(w, gw, gw) + G(x, gx, gx))),$$

$$G(z, gz, gz) \leq \frac{1}{s} \phi(s(G(z, gz, gz) + G(w, gw, gw) + G(x, gx, gx) + G(y, gy, gy))),$$

and

$$G(w, gw, gw) \leq \frac{1}{s} \phi(s(G(w, gw, gw) + G(x, gx, gx) + G(y, gy, gy) + G(z, gz, gz))).$$

Thus,

$$\begin{aligned} &G(x, gx, gx) + G(y, gy, gy) + G(z, gz, gz) + G(w, gw, gw) \\ &\leq \frac{4}{s} \phi(s(G(x, gx, gx) + G(y, gy, gy) + G(z, gz, gz) + G(w, gw, gw))) \\ &\leq 4k(G(x, gx, gx) + G(y, gy, gy) + G(z, gz, gz) + G(w, gw, gw)). \end{aligned}$$

Since $4k < 1$, the last inequality will be satisfied only if $G(x, gx, gx) = 0$, $G(y, gy, gy) = 0$, $G(z, gz, gz) = 0$ and $G(w, gw, gw) = 0$.

Hence, $x = gx$, $y = gy$, $z = gz$ and $w = gw$. Thus we get

$$gx = F(x, x, x, x) = x.$$

To prove the uniqueness, let $x^* \in X$ with $x^* \neq x$ such that

$$gx^* = F(x^*, x^*, x^*, x^*) = x^*.$$

Thus,

$$\begin{aligned} G(x, x^*, x^*) &= G(F(x, x, x, x), F(x^*, x^*, x^*, x^*), F(x^*, x^*, x^*, x^*)) \\ &\leq \frac{1}{s^2} \phi(4G(gx, gx^*, gx^*)) \\ &< \frac{1}{s^2} 4kG(gx, gx^*, gx^*) \\ &\leq 4kG(x, x^*, x^*). \end{aligned}$$

Since $4k < 1$, we get

$$G(x, x^*, x^*) < G(x, x^*, x^*)$$

which is a contradiction.

Therefore

$$x^* = x.$$

Hence, F and g have a unique quadruple common fixed point. □

Corollary 17. *Let (X, G) be a G_b -metric space. Let $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ be two mappings such that*

$$\begin{aligned}
 G(F(x, y, z, w), F(u, v, r, t), F(u, v, r, t)) &\leq \frac{k}{s^2}(G(gx, gu, gu) \\
 &+ G(gy, gv, gv) + G(gz, gr, gr) \\
 &+ G(gw, gt, gt))
 \end{aligned}$$

for all $x, y, z, w, u, v, r, t \in X$. Assume that F and g satisfy the following conditions:

1. $F(X^4) \subseteq g(X)$,
2. $g(X)$ is complete and
3. g is continuous and commutes with F .

If $k \in (0, \frac{1}{4})$, then F and g have a unique quadruple common fixed point, and which is of the form (x, x, x, x) i.e. there is a unique $x \in X$ such that

$$gx = F(x, x, x, x) = x.$$

Proof. The result follows from Theorem 16 by taking $u = a, v = b, r = c, t = d$ and $\phi(t) = kt$. □

Corollary 18. *Let (X, G) be a complete G_b -metric space. Let $F : X^4 \rightarrow X$ be a mapping such that*

$$\begin{aligned}
 G(F(x, y, z, w), F(u, v, r, t), F(u, v, r, t)) &\leq \frac{k}{s^2}(G(x, u, u) \\
 &+ G(y, v, v) + G(z, r, r) \\
 &+ G(w, t, t))
 \end{aligned}$$

for all $x, y, z, w, u, v, r, t \in X$. If $k \in (0, \frac{1}{4})$, then F has a unique quadruple fixed point which is of the form (x, x, x, x) i.e. there is a unique $x \in X$ such that

$$F(x, x, x, x) = x.$$

Proof. The result follows from corollary 17 by taking $g = I$, where $I : X \rightarrow X$ such that $I(x) = x$, for all $x \in X$. □

3. Examples

Example 19. Let $X = [0, 1]$. Define $G : X^3 \rightarrow R^+$ by

$$G(x, y, z) = (|x - y| + |x - z| + |y - z|)^2$$

for all $x, y, z \in X$. Then (X, G) is a complete G_b -metric space with $s = 2$ (see [12]).

Define a map $F : X^4 \rightarrow X$ by

$$F(x, y, z, w) = \frac{x}{256} + \frac{y}{512} + \frac{z}{1024} + \frac{w}{2048}$$

for all $x, y, z, w \in X$.

Also, define $g : X \rightarrow X$ by $g(x) = \frac{x}{8}$, $\phi(t) = \frac{t}{8}$ for $t \in R^+$.

Now,

$$\begin{aligned} & G(F(x, y, z, w), F(u, v, r, t), F(a, b, c, d)) \\ &= (|F(x, y, z, w) - F(u, v, r, t)| + |F(x, y, z, w) - F(a, b, c, d)| \\ &\quad + |F(u, v, r, t) - F(a, b, c, d)|)^2 \\ &= \left(\left| \frac{x}{256} + \frac{y}{512} + \frac{z}{1024} + \frac{w}{2048} - \frac{u}{256} - \frac{v}{512} - \frac{r}{1024} - \frac{t}{2048} \right| \right. \\ &\quad \left. + \left| \frac{x}{256} + \frac{y}{512} + \frac{z}{1024} + \frac{w}{2048} - \frac{a}{256} - \frac{b}{512} - \frac{c}{1024} - \frac{d}{2048} \right| \right. \\ &\quad \left. + \left| \frac{u}{256} + \frac{v}{512} + \frac{r}{1024} + \frac{t}{2048} - \frac{a}{256} - \frac{b}{512} - \frac{c}{1024} - \frac{d}{2048} \right| \right)^2 \\ &\leq \left[\frac{1}{256} (|x - u| + |x - a| + |u - a|) + \frac{1}{512} (|y - v| + |y - b| + |v - b|) \right. \\ &\quad \left. + \frac{1}{1024} (|z - r| + |z - c| + |r - c|) + \frac{1}{2048} (|w - t| + |w - d| + |t - d|) \right]^2 \\ &= \left[\frac{1}{32} \left(\left| \frac{x}{8} - \frac{u}{8} \right| + \left| \frac{x}{8} - \frac{a}{8} \right| + \left| \frac{u}{8} - \frac{a}{8} \right| \right) \right. \\ &\quad \left. + \frac{1}{64} \left(\left| \frac{y}{8} - \frac{v}{8} \right| + \left| \frac{y}{8} - \frac{b}{8} \right| + \left| \frac{v}{8} - \frac{b}{8} \right| \right) \right. \\ &\quad \left. + \frac{1}{128} \left(\left| \frac{z}{8} - \frac{r}{8} \right| + \left| \frac{z}{8} - \frac{c}{8} \right| + \left| \frac{r}{8} - \frac{c}{8} \right| \right) \right. \\ &\quad \left. + \frac{1}{256} \left(\left| \frac{w}{8} - \frac{t}{8} \right| + \left| \frac{w}{8} - \frac{d}{8} \right| + \left| \frac{t}{8} - \frac{d}{8} \right| \right) \right]^2 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{4}{32^2}(|gx - gu| + |gx - ga| + |gu - ga|)^2 \\
 &\quad + \frac{4}{64^2}(|gy - gv| + |gy - gb| + |gv - gb|)^2 \\
 &\quad + \frac{4}{128^2}(|gz - gr| + |gz - gc| + |gr - gc|)^2 \\
 &\quad + \frac{1}{256^2}(|gw - gt| + |gw - gd| + |gt - gd|)^2 \\
 &= \frac{4}{32^2}G(gx, gu, ga) + \frac{4}{64^2}G(gy, gv, gb) + \frac{4}{128^2}G(gz, gr, gc) + \frac{4}{256^2}G(gw, gt, gd) \\
 &\leq \frac{4}{32^2}G(gx, gu, ga) + \frac{4}{32^2}G(gy, gv, gb) + \frac{4}{32^2}G(gz, gr, gc) + \frac{4}{32^2}G(gw, gt, gd) \\
 &= \frac{4}{32^2}[G(gx, gu, ga) + G(gy, gv, gb) + G(gz, gr, gc) + G(gw, gt, gd)] \\
 &\leq \frac{1}{32}[G(gx, gu, ga) + G(gy, gv, gb) + G(gz, gr, gc) + G(gw, gt, gd)] \\
 &= \frac{1}{4} \frac{G(gx, gu, ga) + G(gy, gv, gb) + G(gz, gr, gc) + G(gw, gt, gd)}{8} \\
 &= \frac{1}{2^2}\phi(G(gx, gu, ga) + G(gy, gv, gb) + G(gz, gr, gc) + G(gw, gt, gd))
 \end{aligned}$$

holds for all $x, y, z, w, u, v, r, t, a, b, c, d \in X$. It is easy to see that F and g satisfy all the hypothesis of Theorem 15. Thus F and g have a unique quadruple common fixed point. Here, $F(0, 0, 0, 0) = g(0) = 0$.

Example 20. Let X and G be same as in above example.

Define a map $F : X^4 \rightarrow X$ by

$$F(x, y, z, w) = \frac{1}{32}x^2 + \frac{1}{32}y^2 + \frac{1}{32}z^2 + \frac{1}{32}w^2 + \frac{1}{8}$$

for all $x, y, z, w \in X$. Then, $F(X^4) = [\frac{1}{8}, \frac{1}{4}]$.

Also,

$$\begin{aligned}
 &G(F(x, y, z, w), F(u, v, r, t), F(u, v, r, t)) \\
 &= (2|F(x, y, z, w), F(u, v, r, t)|)^2 \\
 &= (\frac{1}{16}|x^2 - u^2 + y^2 - v^2 + z^2 - r^2 + w^2 - t^2|)^2 \\
 &\leq \frac{1}{256}(|x^2 - u^2| + |y^2 - v^2| + |z^2 - r^2| + |w^2 - t^2|)^2 \\
 &\leq \frac{1}{64}(|x^2 - u^2|^2 + |y^2 - v^2|^2 + |z^2 - r^2|^2 + |w^2 - t^2|^2) \\
 &\leq \frac{1}{64}(4|x - u|^2 + 4|y - v|^2 + 4|z - r|^2 + 4|w - t|^2)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{64}(G(x, u, u) + G(y, v, v) + G(z, r, r) + G(w, t, t)) \\
&= \frac{\frac{1}{16}}{2^2}(G(x, u, u) + G(y, v, v) + G(z, r, r) + G(w, t, t))
\end{aligned}$$

Then by corollary 17, F has a unique quadruple fixed point. Here $x = 4 - \sqrt{15}$ is the unique point in X such that $F(x, x, x, x) = x$.

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