Analytical Approximations in Modeling Contacting Rough Surfaces

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A critical examination of the analytical solution presented in the classic paper of Greenwood and Williamson (1966), (GW) on the statistical modeling of nominally flat contacting rough surfaces is undertaken in this study. It is found that using the GW simple exponential distribution to approximate the usually Gaussian height distribution of the asperities is inadequate for most practical cases. Some other exponential type approximations are suggested, which approximate the Gaussian distribution more accurately, and still enable closed form solutions for the real area of contact, the contact load, and the number of contacting asperities. The best-modified exponential approximation is then used in the case of elastic-plastic contacts of Chang et al. (1987) (CEB model), to obtain closed-form solutions, which favorably compare with the numerical results using the Gaussian distribution.

1 Introduction

Since the late 1950's, there has been a great interest in surface topography, and its role in surface interactions and friction. When two surfaces are brought together, contact occurs at discrete contact spots due to surface roughness of both surfaces. Deformation occurs in the contacting region, and it can be elastic, plastic, or elastic-plastic, depending on the nominal pressure, surface roughness, and material properties. Many researchers have proposed models of surface roughness. A common assumption/methodology that all these models share, is the representation of surface asperities by simple geometrical shapes with a probability distribution for the different asperity parameters involved.

The milestone of this approach was set forth in the classic paper of Greenwood and Williamson (1966), that introduced a basic elastic contact model (GW model). It assumes that each asperity summit has a spherical shape whose height above a reference plane has a normal (Gaussian) probability density function. It also assumes that the summits have a uniform radius of curvature. The basic GW asperity model has been extended to include other contact geometries, e.g., curved surfaces (Greenwood and Tripp, 1967), more complex geometries, e.g., non uniform radii of curvature of asperity peaks (Whitehouse and Archard, 1970), and anisotropic surfaces (Bush et al., 1979). McCool (1986) compared the basic GW model with other more general isotropic and anisotropic models and found that the simpler GW model, despite its simplistic form, gives good results, thus justifying its use. The above analyses assume elastic asperity deformations only, which are restricted to small values of the plasticity index ψ. Chang et al. (1987) (CEB model) extended the work of GW, to include elastic-plastic deformations of the asperities. More recent work on elastic-plastic contacts based on numerical models, can be found in Tian and Bhushan (1996), and Lee and Ren (1996).

GW also suggested, based on experimental evidence, that for many engineering surfaces, the height distribution of the asperities tends to be Gaussian. Even in the cases where the asperity heights are not Gaussian, the uppermost peaks form a reasonable approximation to a Gaussian distribution. Even though the Gaussian approximation for the distribution of asperity heights received widespread acceptance, a difficulty associated with it is that there is no analytical solution for the parameters of interest at the interface, i.e., contact load, P, real area of contact, A, and number of contacting asperities, N.

Another suggestion, also originated by GW, is that a simple exponential distribution (see Eq. (6)), is a "fair approximation to the uppermost 25 percent of the asperities of most surfaces." By utilizing an exponential type distribution, Eqs. (6) or (7), the relevant contact equations can be solved analytically, thus providing a direct relationship (exact proportionality) between the contact load, the real area of contact, and the number of contacting asperities. In the original work of GW, the results of separation versus load, area of contact versus load, and mean pressure versus load were also presented for a Gaussian distribution (see Figs. 2 and 3 in GW), and the suggestion that "these results approximate closely to those for the exponential distribution" was made. As is shown in this work, this is not quite true and a modified exponential approximation is suggested that better approximates the results of the Gaussian distribution.

Following the pioneering work of GW, numerous other researchers employed the exponential distribution of asperity heights (usually in addition to the Gaussian distribution) in obtaining simplified closed form solutions. For example, Hess and Soom (1992, 1993) and Hess and Wagh (1994) utilized the exponential distribution of asperity heights in their work on dynamic friction modeling. Etsion and Front (1994) used the same exponential distribution in their work in static sealing modeling.

A different approach has been followed by some other researchers, where empirically fitted power laws are used instead of statistical distributions to characterize an interface, see for example Bhushan (1984), Martins et al. (1990), and Lee and Ren (1996). Of particular interest is the work of Bhushan (1984), where using a least squares fit method, power laws were fitted to the Gaussian results of GW, to obtain simplified equations for the contact load, real area of contact, and number of contacting asperities. These fits will be compared later with the modified exponential relationships suggested in this work.

2 Relevant Equations

2.1 Elastic Contact Model. From GW, for any given probability density function (pdf) of asperity heights, ϕ, and

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assuming elastic deformations only, then, the expected contact load, P_e is given by

$$P_e(d^*) = \frac{4}{3}\beta E A_n\left(\frac{\sigma}{R}\right)^{1/2} F_{3/2}(d^*), \tag{1}$$

the real area of contact, A_e, is given by

$$A_e(d^*) = \pi\beta A_n F_1(d^*), \tag{2}$$

and the number of contacting asperities, N, is given by

$$N(d^*) = \eta A_n F_0(d^*) \tag{3}$$

where F_n is the following integral

$$F_n(d^*) = \int_{d^*}^{\infty} (z^* - d^*)^n \phi^*(z^*) dz^* \tag{4}$$

β is a roughness parameter given by

$$\beta = \eta R \sigma \tag{5}$$

E is the composite elastic modulus, and d^* the dimensionless mean separation based on asperity heights, dimensionalized with respect to σ (see nomenclature).

Assuming that the pdf is a simple exponential given by

$$\phi_e^*(z^*) = e^{-z^*} \tag{6}$$

then, the above contact equations, Eqs. (1)–(3) have a closed form solution that was first presented by GW (1966). Similarly, if the ϕ^* is approximated by the following modified exponential function

$$\phi^*(z^*) = c e^{-\lambda z^*} \tag{7}$$

where c and λ are constant coefficients, then the contact equations, Eqs. (1)–(3) also have a closed form solution as follows

$$P_e(d^*) = \frac{c\sqrt{\pi}\beta E A_n}{\lambda^{5/2}}\left(\frac{\sigma}{R}\right)^{1/2} e^{-\lambda d^*} \tag{8}$$

$$A_e(d^*) = \frac{c\pi\beta A_n}{\lambda^2} e^{-\lambda d^*} \tag{9}$$

$$N(d^*) = \frac{c\eta A_n}{\lambda} e^{-\lambda d^*} \tag{10}$$

Equations (8)–(10) are of a more general form, and reduce to the GW exponential results (using Eq. (6)) by letting $c = \lambda = 1$ in Eqs. (8)–(10).

2.2 Elastic-Plastic Contact Model. The GW contact asperity model assumes that all the contacting asperities deform elastically, which is only true for limited cases, e.g., extremely smooth or hard surfaces under very low loads. A more realistic contact model is the CEB elastic-plastic model (Chang et al., 1987), which is an extension of the GW model. The dimensionless contact load, $P^* = P/A_n E$ for an elastic-plastic contact of rough surfaces is given by Chang et al. (1987) as

$$P^*(d^*) = \beta\left[\frac{4}{3}\left(\frac{\sigma}{R}\right)^{1/2}\int_{d^*}^{d^*+\omega_c^*} (z^* - d^*)^{3/2}\phi^*(z^*)dz^* + \frac{\pi K H}{E}\int_{d^*+\omega_c^*}^{\infty} [2(z^* - d^*) - \omega_c^*]\phi^*(z^*)dz^*\right] \tag{11}$$

where, the first and second integrals are the contributions of the elastically and plastically deformed asperities, respectively. H is the hardness of the softer material. Note that $(z^* - d^*)$ is the normalized local interference, $\omega^* = \omega/\sigma$ of a contacting asperity; ω_c is the critical interference of an asperity at the inception of plastic deformation given by

$$\omega_c^* = \frac{\omega_c}{\sigma} = \left(\frac{\pi K H}{2E}\right)^2 \frac{R}{\sigma} \tag{12}$$

and is another form of the plasticity index, ψ, defined by GW, i.e.,

$$\psi = \frac{2E}{\pi K H}\left(\frac{\sigma}{R}\right)^{1/2} \tag{13}$$

Also, following CEB, the total real area of contact is the sum of the real areas of contact from the elastically and plastically deformed asperities, as follows

$$A^*(d^*) = A_e^* + A_p^* = \pi\beta\left[\int_{d^*}^{d^*+\omega_c^*} (z^* - d^*)\phi^*(z^*)dz^* + \int_{d^*+\omega_c^*}^{\infty} [2(z^* - d^*) - \omega_c^*]\phi^*(z^*)dz^*\right] \tag{14}$$

Nomenclature

- A = real area of contact, $A_e + A_p$
- A^* = A/A_n
- A_e = elastic real area of contact
- A_p = plastic real area of contact
- A_n = nominal contact area
- c = constant coefficient in ϕ^*, Eq. (7)
- d = mean separation based on asperity heights
- d^* = d/σ
- E = composite elastic modulus for the two contacting surfaces
- F_n = integral equation, Eq. (4)
- H = hardness of softer material
- K = maximum contact pressure factor
- n = coefficient in Eq. (4)
- N = number of contacting asperities
- N^* = $N/\eta A_n$
- N_e = number of elastic contacting asperities
- N_e^* = $N_e/\eta A_n$
- N_p = number of plastic contacting asperities
- N_p^* = $N_p/\eta A_n$
- P = total contact load, $P_e + P_p$
- P^* = dimensionless contact load, $P/A_n E$
- P_e = elastic contact load
- P_e^* = $P_e/A_n E$
- P_p = plastic contact load
- P_p^* = $P_p/A_n E$
- p_e = mean real elastic pressure, P_e/A_e
- R = radius of curvature of asperity summits
- z = height of asperity measured from the mean of asperity heights
- z^* = z/σ
- β = roughness parameter, $\eta R \sigma$
- η = areal density of asperities
- λ = constant coefficient in ϕ^*, Eq. (7)
- ν = Poisson's ratio
- σ = standard deviation of asperity heights
- ϕ = distribution function (pdf) of asperity heights
- ϕ^* = ϕ/σ
- ϕ_1^* = modified exponential function, Eq. (25)
- ϕ_2^* = modified exponential function, Eq. (26)
- ϕ_e^* = simple exponential pdf, Eq. (6)
- ϕ_g^* = Gaussian pdf, Eq. (24)
- ψ = GW plasticity index, Eq. (13)
- ω = local interference, $z - d$
- ω^* = ω/σ
- ω_c = critical interference at the inception of plastic deformation, Eq. (12)
- ω_c^* = ω_c/σ

and again the first and second integrals are the contributions of the elastically and plastically deformed asperities, respectively.

Similarly, the total number of contacting asperities is the sum of the elastically and plastically deformed asperities, which can be written in the following dimensionless form

$$N^*(d^*) = \frac{N}{\eta A_n} = N_e^* + N_p^*$$
$$= \int_{d^*}^{d^*+\omega_c^*} \phi^*(z^*)dz^* + \int_{d^*+\omega_c^*}^{\infty} \phi^*(z^*)dz^* \quad (15)$$

Assuming that ϕ^* is approximated by the modified exponential function given by Eq. (7), then out of the 6 integrals in Eqs. (11), (14), and (15), 5 of them have simple closed form solutions. The only integral that poses some difficulties is the elastic contact load integral (first integral in Eq. (11)), but it can be expressed either in a standard hypergeometric mathematical function (error function), or it can be expanded in a power series (Etsion and Front, 1994). In this work, P_e^* will be expressed in terms of the standard error function, since its value is readily available from any standard reference. The closed form solution for the dimensionless elastic load, P_e^*, plastic load, P_p^*, elastic real area of contact, A_e^*, plastic real area of contact, A_p^*, number of elastically deformed asperities, N_e^*, and the number of plastically deformed asperities, N_p^* are given below, respectively

$$P_e^*(d^*) = \frac{4c\beta(\sigma/R)^{1/2}}{3\lambda^{5/2}} \left(\frac{3\sqrt{\pi}}{4} \text{ erf } (\sqrt{\lambda\omega_c^*}) - \frac{(\lambda\omega_c^*)^{3/2} + \frac{3}{2}\sqrt{\lambda\omega_c^*}}{e^{\lambda\omega_c^*}} \right) e^{-\lambda d^*} \quad (16)$$

$$P_p^*(d^*) = \frac{c\pi\beta KH}{E\lambda^2}(2 + \lambda\omega_c^*)e^{-\lambda(d^*+\omega_c^*)} \quad (17)$$

$$A_e^*(d^*) = \frac{c\pi\beta}{\lambda^2}[1 - (1 + \lambda\omega_c^*)e^{-\lambda\omega_c^*}]e^{-\lambda d^*} \quad (18)$$

$$A_p^*(d^*) = \frac{c\pi\beta}{\lambda^2}(2 + \lambda\omega_c^*)e^{-\lambda(d^*+\omega_c^*)} \quad (19)$$

$$N_e^*(d^*) = \frac{c}{\lambda}[1 - e^{-\lambda\omega_c^*}]e^{-\lambda d^*} \quad (20)$$

$$N_p^*(d^*) = \frac{c}{\lambda} e^{-\lambda(d^*+\omega_c^*)} \quad (21)$$

where erf $((\lambda\omega_{cr})^{1/2})$ is the integral of the Gaussian distribution

$$\text{erf } (\sqrt{\lambda\omega_{cr}}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\lambda\omega_{cr}}} e^{-t^2} dt \quad (22)$$

and is widely tabulated. Also, note that the total number of contacting asperities, obtained by summing up Eqs. (20) and (21) simplifies to

$$N^*(d^*) = \frac{c}{\lambda} e^{-\lambda d^*} \quad (23)$$

For the case of the simple exponential approximation (Eq. (6)), set $\lambda = c = 1$ in Eqs. (16)–(21) to obtain the dimensionless elastic load, plastic load, elastic real area of contact, and plastic real area of contact, respectively.

3 Approximations to the Gaussian Distribution

Since the introduction of the simple exponential model as an alternative to the Gaussian distribution, some other papers, primarily by Greenwood, were indirectly indicating that a simple exponential as given by Eq. (6), may not be adequate. For example, in 1977 in a review paper by Greenwood and Williamson (1977) on the developments in the theory of surface roughness, the modified exponential function, Eq. (7), was mentioned as an alternative to the simple exponential, Eq. (6). However, no further discussion or analysis of the issue was presented in that paper. More recently, Greenwood (1992) in another review paper on the contact of rough surfaces introduces the exponential approximation as given by Eq. (7), but without the coefficient c (i.e., c is assumed to be equal to 1). Even though Greenwood and coworkers at some point indirectly indicated that an exponential type distribution with both a linear and exponential coefficients may be needed, instead of a simple exponential, Eq. (6), no one investigated the issue any further.

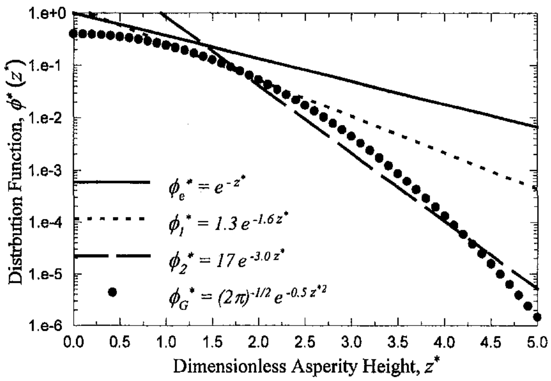

Fig. 1 Approximate functions of asperity heights as compared to the Gaussian distribution, ϕ_G^*

Assuming that the Gaussian distribution represents a certain random process (e.g., asperity heights) fairly well, we will find the coefficients needed in the modified exponential function, Eq. (7), in order to approximate the Gaussian distribution in the range of practical interest. This will be done with a least squares fit method, directly on the probability density function. Then, the parameters of interest, i.e., contact load, Eq. (1), real area of contact, Eq. (2), and number of contacting asperities, Eq. (3), will be calculated using both the exact Gaussian (solved numerically) and the modified exponential approximations (solved analytically), as well as the simple exponential distribution, and compare them.

Before presenting the results, a word on the choice of the exponential type distributions should be said. From a statistics point of view, one may question such a choice, in order to approximate the Gaussian distribution, since there are other types of distributions, e.g., the Gamma distribution, that give a better approximation. This is less desirable since the Gamma distribution, like the Gaussian, does not allow closed form solution of the contact equations.

4 Results and Discussion

Figure 1 shows the Gaussian distribution,

$$\phi_G^*(z^*) = \frac{1}{\sqrt{2\pi}} e^{-(z^{*2}/2)} \quad (24)$$

for $z^* > 0$ in a logarithmic scale. Also, shown is the simple exponential, Eq. (6), and two different least squares fits to the Gaussian distribution based on the modified exponential approximation, Eq. (7), with the following coefficients:

$$\phi_1^*(z^*) = 1.3e^{-1.6z^*} \quad (25)$$

$$\phi_2^*(z^*) = 17e^{-3.0z^*} \quad (26)$$

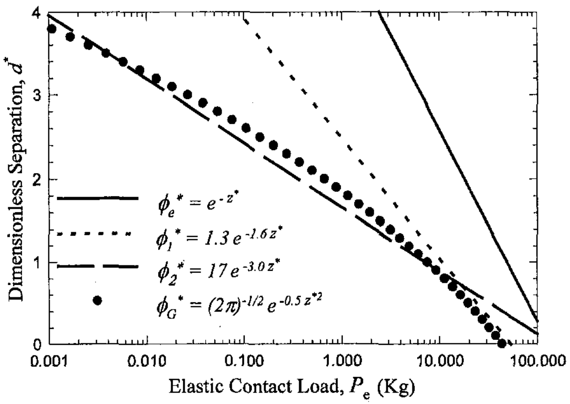

Fig. 2 Dimensionless separation, d^* versus elastic contact load, P_e [Eq. (1)], for 4 different functions of asperity heights as indicated [compare with Fig. 2(*a*) in Greenwood and Williamson, (1966) for ϕ_G^*]

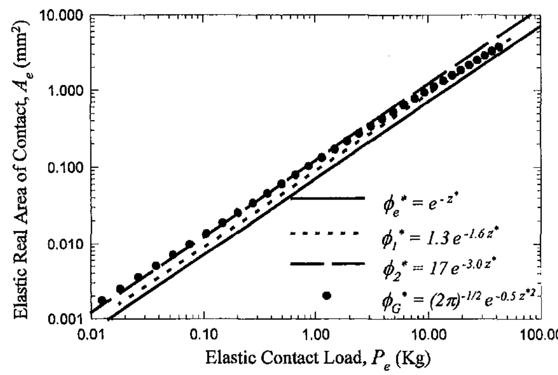

Fig. 3 Elastic real area of contact, A_e versus elastic contact load, P_e [Eqs. (1), (2)], for 4 different functions of asperity heights as indicated [compare with Fig. 2(*b*) in Greenwood and Williamson, (1966) for ϕ_G^*]

As can be seen from the figure, the simple exponential is the worst fit to the Gaussian distribution, especially at large z^*. In general, ϕ_2^* is a better fit in a larger range, i.e., $1 \leq z^* \leq 5$. Nevertheless, in cases where a better fit is desired at extremely low z^*, then ϕ_1^* is more appropriate, say for $0 \leq z^* \leq 2$.

4.1 Elastic Contact. Next, a comparison of the different approximations has been made, as to how well they predict the contact load and the real area of contact for the parameters used in the GW paper, see Fig. 2 in GW (1966). These parameters are given below, and are of the same units as originally used by GW, in order to allow exact reproduction of the GW results:

$$\eta = 300/\text{mm}^2; \quad R\sigma = 10^{-4}\ \text{mm}^2; \quad E(\sigma/R)^{1/2} = 25\ \text{Kg/mm}^2$$

Figure 2 is the same figure as Fig. 2(*a*) in GW, for the Gaussian distribution, ϕ_G^*, along with the exponential and the two modified exponentials, for a nominal area of contact of $A_n = 1\ \text{cm}^2$. The different plots in Fig. 2 are obtained by plotting Eqs. (1) with (4) and (24) for ϕ_G^*, Eq. (8) with $c = 1.3$, $\lambda = 1.6$ for ϕ_1^*, $c = 17$, $\lambda = 3$ for ϕ_2^*, and $c = \lambda = 1$ for ϕ_e^*. As expected, in this load range of 5 orders of magnitude, corresponding to a normal mean separation of 0 to 4σ, ϕ_2^* gives the overall best fit. In the high load range, say $P > 10$ Kg, ϕ_1^* is better, as expected from the better approximation to the Gaussian distribution at low z^* (see Fig. 1). Contrary to the statement made by GW that "the results approximate closely to those for the exponential distribution," the exponential distribution overestimates the normal separation at a given contact load. More specifically, at large $d^* = 3.6$, then $P_e = 2.5$ g, 3.0 g, 168 g, and 3.6 Kg, using ϕ_G^*, ϕ_2^*, ϕ_1^*, and ϕ_e^*, respectively. Also, at low dimensionless separations, say $d^* = 0.4$, then $P_e = 20$ Kg, 32 Kg, 24 Kg, and 81 Kg, using ϕ_G^*, ϕ_2^*, ϕ_1^*, and ϕ_e^*, respectively. That is, the exponential distribution gives the largest deviation from the Gaussian results at all separations.

Figure 3 shows the real area of contact versus the contact load for all distributions, for $A_n = 1\ \text{cm}^2$, as compared to Fig. 2(*b*) in GW. The plots for the contact load are obtained as described for Fig. 2, and the real area of contact is obtained from Eqs. (2) with (4) and (24) for ϕ_G^*, Eq. (9) with $c = 1.3$, $\lambda = 1.6$ for ϕ_1^*, $c = 17$, $\lambda = 3$ for ϕ_2^*, and $c = \lambda = 1$ for ϕ_e^*. As with the contact load, the best correlation with the Gaussian results is with ϕ_2^*, whereas ϕ_e^* gives the largest error. In particular, ϕ_e^* underestimates the real area of contact by a factor of $(\lambda)^{1/2}$ (see Eq. (28)) at any load.

As far as the mean real pressure, p_e versus load is concerned, and depicted in Fig. 4 (see also Fig. 3 in GW for the Gaussian distribution), it changes from around 6.5 Kg/mm² at $P = 10^{-3}$ Kg, to around 11.5 Kg/mm² at $P = 40$ Kg, for the Gaussian distribution, but it is constant for the exponential type distributions, with values of 8.1, 11.2 and 14.1 Kg/mm² for ϕ_2^*, ϕ_1^*, and ϕ_e^*, respectively. As will be derived in the next section, these values correspond to $E(\sigma/R)^{1/2}/(\lambda\pi)^{1/2}$, and ϕ_2^* gives the average for ϕ_G^*, (average of $p_e = 8.1$ Kg/mm², using ϕ_G^*) whereas ϕ_1^* and ϕ_e^* overestimate the mean pressure.

4.2 Comparisons With Bhushan (1984). As was mentioned in the Introduction, Bhushan (1984), also realized the importance of having simplified relations for the contact equations. In his case, these simple relationships were obtained by curve fitting the Gaussian numerical results (elastic deformations only) of the contact load, real area of contact, and number of contacting asperities, using power laws. The mean real pressure, $p_e = P_e/A_e$, is given by Bhushan (1984) in the following dimensionless form

$$\frac{p_e}{E\sqrt{\sigma/R}} = 0.42\left(\frac{P_e/A_n}{\beta E\sqrt{\sigma/R}}\right)^{0.04} \cong 0.32 \tag{27}$$

In order to obtain a similar relationship for the mean real pressure, in the case of the modified exponential function, Eq. (7), we use Eq. (9) and Eq. (8), and rearrange terms as follows

$$p_e = \frac{P_e}{A_e} = \frac{1}{\sqrt{\lambda\pi}} E\sqrt{\sigma/R} \tag{28}$$

For ϕ_2^*, $\lambda = 3.0$, and the dimensionless mean real pressure becomes

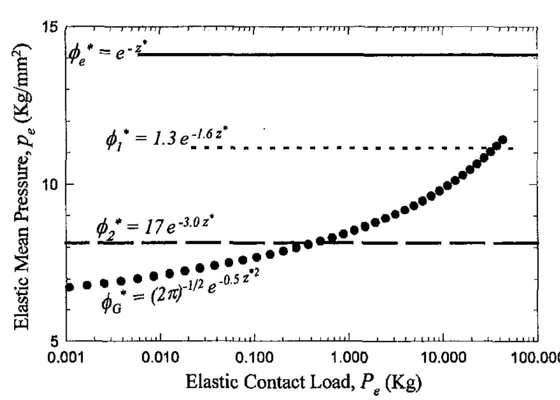

Fig. 4 Effect of the different functions of asperity heights, ϕ^* on the elastic mean pressure, $p_e = P_e/A_e$, [compare with Fig. (3) in Greenwood and Williamson, (1966) for ϕ_G^*]

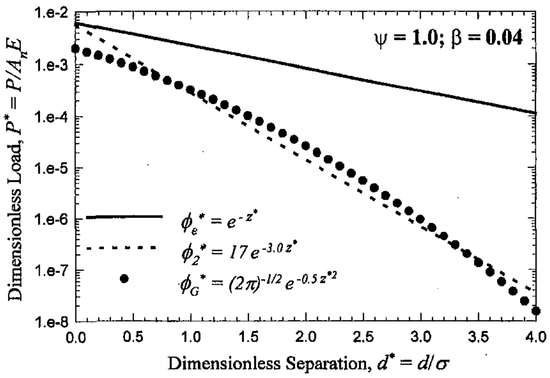

Fig. 5 Total Dimensionless contact load, P^* versus dimensionless separation, d^*, for 3 different functions of asperity heights as indicated, $\psi = 1.0$; $\sigma/R = 3.9 \times 10^{-3}$; $\beta = 0.04$

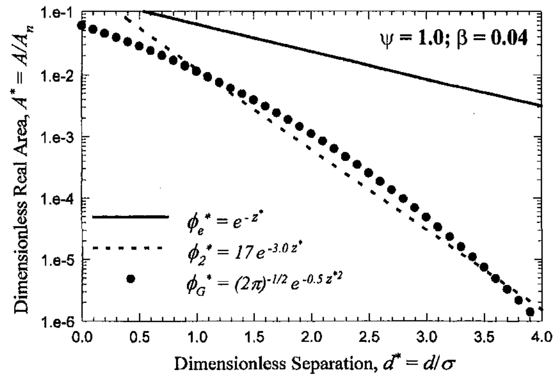

Fig. 6 Total dimensionless real area of contact, A^* versus dimensionless separation, d^*, for 3 different functions of asperity heights as indicated, $\psi = 1.0$; $\sigma/R = 3.9 \times 10^{-3}$; $\beta = 0.04$

$$\frac{p_e}{E\sqrt{\sigma/R}} = \frac{1}{\sqrt{\lambda\pi}} = 0.326 \tag{29}$$

which is very close to the value of 0.32 obtained by Bhushan (1984).

Similarly, Bhushan (1984) curve fitted for the real area of contact as a function of the apparent pressure:

$$\frac{A_e}{A_n\beta} = 2.40\left(\frac{P_e/A_n}{\beta E\sqrt{\sigma/R}}\right)^{0.96} \cong 3.2\left(\frac{P_e/A_n}{\beta E\sqrt{\sigma/R}}\right) \tag{30}$$

A similar relationship is obtained for the case of the modified exponential function, by solving Eq. (8) for $e^{-\lambda d^*}$, and substituting it into Eq. (9), to get

$$\frac{A_e}{A_n\beta} = \sqrt{\pi\lambda}\left(\frac{P_e/A_n}{\beta E\sqrt{\sigma/R}}\right) = 3.07\left(\frac{P_e/A_n}{\beta E\sqrt{\sigma/R}}\right), \quad \text{for} \quad \lambda = 3.0 \tag{31}$$

Bhushan's fits are on the numerical results for the contact load and real area of contact, whereas in this work, the fitting is performed initially on ϕ^* (and not the specific results), and then the contact equations are solved analytically. Therefore, the approach in this work is straightforward and more practical.

4.3 Elastic-Plastic Contact. It is of practical importance to also compare the elastic-plastic results, using the Gaussian distribution (Chang et al., 1986), with the exponential distribution (Etsion and Front, 1994), and the modified exponential function suggested in this work. This will further demonstrate the practical significance of this work. The modified exponential ϕ_2^*, which gives the best correlation with the Gaussian ϕ_G^* is chosen for the comparisons. The results will be presented in a dimensionless form, as given by Eqs. (16)–(21). The following roughness and material parameters which specify the severity of the contact were selected,

$$\sigma/R = 3.9 \times 10^{-3}; \quad \beta = 0.04; \quad \psi = 1.0; \quad \nu = 0.3$$

which correspond to both elastic and plastic deformation of the asperities.

Figure 5 shows the total dimensionless load, P^*, versus the dimensionless separation, d^*, for the numerical results for ϕ_G^* using Eqs. (11) and (24), the analytical results for ϕ_e^* using Eqs. (16) and (17) with $\lambda = c = 1$, and finally the results for ϕ_2^* using Eqs. (16) and (17) with $\lambda = 3$ and $c = 17$. As with the elastic results presented earlier (Fig. 2), there is a very good agreement between the numerical results using ϕ_G^* and the analytical results using ϕ_2^*. On the other hand, the simple exponential distribution ϕ_e^* overestimates the contact load at a certain d^*. For example, for $0.5 \le d^* \le 4.0$, which corresponds to most practical situations (change of 5 orders of magnitude for P^*), the average error in P^* calculated analytically using ϕ_2^* is about 20 percent, compared to the numerical results using ϕ_G^*. On the other hand, the simple exponential distribution overestimates P^* by as little as 3 times at $d^* = 0.5$, and as much as 4 orders of magnitude at $d^* = 4.0$.

Similar results are obtained for the total real area of contact (see Eqs. (14), (18), and (19)), as shown in Fig. 6. As with the contact load, the analytical real area of contact, A^*, obtained using ϕ_2^* contains an average error of about 20 percent as compared to the numerical results using ϕ_G^*, over a range of 4 orders of magnitude for A^*. In the same range of A^*, the estimates using ϕ_e^* are 1 to 3 orders of magnitude larger than the numerical results.

Finally, the estimates for the total number of contacting asperities, N^* (Eq. (15)), expressed analytically by Eq. (23) for ϕ_e^*, and ϕ_2^*, and the numerical results using ϕ_G^*, are depicted in Fig. 7. Note that the calculation of N^*, using ϕ_G^*, can also be expressed in the form of the well known and tabulated error function, see Eq. (22), as follows

$$N^*(d^*) = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{d^*}{\sqrt{2}}\right)\right] \tag{32}$$

The estimation of N^*, as compared to the estimates of P^* and A^*, using ϕ_2^*, contains the largest deviation from the exact (numerical) results using ϕ_G^*. Nevertheless, in the range of $1 \le d^* \le 4$, the average error of the estimates using ϕ_2^* is about

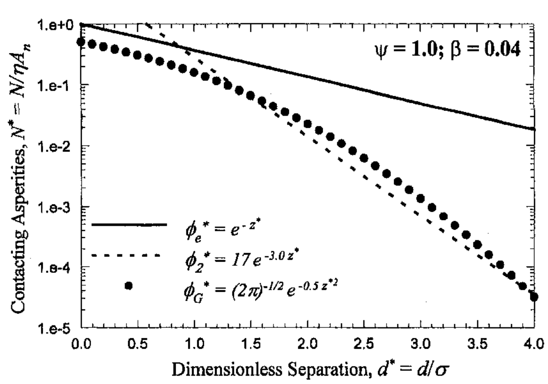

Fig. 7 Total dimensionless number of contacting asperities, N^* versus dimensionless separation, d^*, for 3 different functions of asperity heights as indicated, $\psi = 1.0$; $\sigma/R = 3.9 \times 10^{-3}$; $\beta = 0.04$

25 percent of the values obtained using ϕ_G^*. On the other hand, in the same dimensionless separation range, corresponding to 4 orders of magnitude for N^*, ϕ_e^* overestimates the numerical results by as little as a factor of 3 at low d^* and as much as 3 orders of magnitude at large d^*.

The significance of the above results is that they unambiguously confirm that a simple exponential distribution of asperity heights is inadequate to capture the results of a Gaussian distribution of asperity heights. On the other hand, an improved fit to the Gaussian distribution as given by the simple modified exponential function of Eq. (7), which still has the advantage of providing closed form solutions for the contact equations, approximates the numerical results of the Gaussian distribution quite accurately.

5 Conclusions

The GW assumption that the Gaussian distribution of asperity heights can be approximated by a simple exponential distribution was critically examined. It is found that it overestimates the contact load, real area of contact, and number of contacting asperities, at all practical dimensionless separations. On the other hand, when a modified exponential function was fitted to the Gaussian distribution, and used to obtain simple expressions for the contact equations, the results compare favorably with the Gaussian numerical results. More specifically, in the case of the elastic-plastic expressions for the dimensionless contact load, depicted in Fig. 5, over a load range of 5 orders of magnitude, the simple exponential distribution overestimates the dimensionless contact load by as little as 3 times at small separations (corresponding to extremely high external loads), to as much as 4 orders of magnitude at very light external loads (large separations). The largest error in the modified exponential is at $d^* = 0$, where it is about the same as that of the simple exponential. At all other d^*, the average error in estimating the load is about 20 percent compared to the numerical results.

Therefore, by curve fitting the Gaussian distribution with an exponential function, simple analytical expressions were derived for the contact load, real area of contact, and number of contacting asperities for both elastic contacts (GW), as well as for elastic-plastic contacts of asperities (CEB). These expressions compare favorably with the numerical Gaussian results, and can be used in cases when analytical solutions for the contact parameters are needed.

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