

**A COMMENT ON THE DARBOUX TRANSFORMATION****J. H. Caltenco, J. López Bonilla, M. A. Acevedo (Mexico)**

**Abstract.** It is known that the Darboux transformation (DT) allows us to construct isospectral potentials in the frame of the Schrödinger equation. Here we give a simple mathematical deduction for the DT.

**Introduction**

In the one-dimensional stationary case the Schrödinger equation is given by [1, 2]

$$(1) \quad -\frac{d^2}{dx^2}\psi + u(x)\psi = \lambda\psi$$

which is written in natural units taking  $\frac{\hbar}{2m} = 1$ . The values of  $\lambda$  represent the energy spectrum allowed for determinated boundary conditions and corresponding to the standard potential  $u(x)$ . With the very useful Darboux transformation (DT) [3–6] we can generalize any specific standard potential and thus generate new interaction models with the same energy levels. The DT is related to the Sturm–Liouville theory [7–10], and it is easy to see the implicit presence of DT in supersymmetric quantum mechanics [1, 2, 5, 11–15]. We suppose that (1) accepts the particular solution  $\psi_1$  for the eigenvalue  $\lambda_1$

$$(2) \quad -\psi_1'' + u(x)\psi_1 = \lambda_1\psi_1$$

then we employ  $\psi_1$  as "seed function" to construct the DT [3–5, 16]:

$$(3) \quad \phi(x) = \psi' - \sigma_1(x)\psi \quad \sigma_1 = \frac{d}{dx} \ln \psi_1$$

therefore (1) adopts the structure:

$$(4) \quad -\frac{d^2}{dx^2}\phi + U(x)\phi = \lambda\phi$$

with the generalized isospectral potential:

$$(5) \quad U(x) = u(x) - 2\frac{d}{dx}\sigma_1$$

That is, the Schrödinger equation is covariant with respect to DT. Selecting other "seed functions" we can generate many DT-s and thus a great family of generalized potentials with the same energy spectrum.

In the next section we show a simple procedure to motivate (3), (4) and (5), that is, we exhibit how the basic expressions of the DT are born.

### Darboux transformation

If in (1) we introduce the new dependent variable  $y(x) = \psi/\theta(x)$ , where  $\theta$  is an arbitrary function for the time being, then this equation takes the form:

$$(6) \quad y'' + 2\frac{\theta'}{\theta}y' + \left(\lambda - \lambda_1 + \frac{\theta''}{\theta} - \frac{\psi''}{\psi_1}\right)y = 0$$

because from (2) we have that  $u = \lambda_1 + \psi_1''/\psi_1$ . Therefore it is natural the election  $\theta = \psi_1$ , that yields:

$$(7) \quad y = \frac{\psi}{\psi_1}$$

and reduces this equation to the form:

$$(8) \quad y'' + 2\frac{\psi_1'}{\psi_1}y' + (\lambda - \lambda_1)y = 0$$

if the definition of  $y$  written above is applied in deducing each of the equations of (7) and (8). Now we apply  $\frac{d}{dx}$  to (8) and introduce the notation:

$$(9) \quad \eta(x) = \frac{d}{dx}y(x), \quad \sigma_1 = \frac{\psi_1'}{\psi_1}$$

for thus to obtain the equation:

$$(10) \quad \eta'' + 2\sigma_1\eta' + (\lambda - \lambda_1 + 2\sigma_1')\eta = 0$$

Finally, in (10) we make a transformation similar to (7):

$$(11) \quad \eta = \frac{\phi}{\psi_1}$$

Then this equation adopts the structure of (4) with the generalized isospectral potential  $U(x) = \sigma_1^2 - \sigma_1' + \lambda_1 = u - 2\sigma_1'$ , in according with (5). Besides, from (7), (9) and (11) we have that  $\phi = \psi_1\eta = \psi_1y' = \psi_1\frac{d}{dx}(\psi/\psi_1)$ , which reproduces (3) q.e.d.

In the literature on DT there is not an explicit motivation for these important transformations of mathematical physics. Thus, the present Note was dedicated to a simple demonstration of the basic expressions of DT.

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