A property of the elementary symmetric functions on the frequencies of sinusoidal signals

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Abstract

In this paper, a relation between the elementary symmetric functions on the frequencies of multi-sine wave signal and its multiple integrals is proposed. In particular, such relation is useful to obtain a closed-form expression for the frequencies estimation. The approach used herein is based on the algebraic derivative method in the frequency domain, which allows to yield exact formula in terms of multiple integrals of the signal when placed in the time domain. Moreover, it allows to free oneself from the hypothesis of uniform sampling. Two different ways to approach the estimation are advised, the first is based on least-squares estimation, while the second one is based on the solution of a linear system of dimension equal to the number of sinusoidal components involved. For an easy time realization of such formula, a time-varying filter is proposed. Due to use of multiple integrals of the signal, the resulting parameters estimation is accurate in the face of large measurement noise. To corroborate the theoretical analysis and to investigate the performance of the developed algorithm, computer simulated and laboratory experiments data records are processed.

Key words: Elementary symmetric functions, multi-sine wave signal, frequency estimation, least-squares

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1 Introduction

The process of estimating the frequencies of multi-sine wave signals, from a finite number of noisy discrete-time measurements, is an important task from both the theoretical and practical point of view. Such problem has been the focus of research for quite some time and still is an active research area to date [1]-[12], since it is used in a wide range of applications in many fields such as control theory, relaying protection, intelligent instrumentation of power systems [1], [7], [9], signal processing [12], digital communications, distribution automation, biomedical engineering [13], radar applications, radio frequency, instrumentation and measurement, to name just a few. There is a vast amount of literature regarding the estimation procedures as well as the theoretical behavior of the different estimators; in [14] (and the references therein) a list of several algorithms is reported: adaptive notch filter, time frequency representation based method, phase locked loop based method, eigensubspace tracking estimation, extended Kalman filter frequency estimation, internal model based method (for an extensive list of references see [15]). The requirements on the frequency estimator, and so the choice of the solution, vary with the application, but typical issues are: accuracy, processing speed or complexity, and ability to handle multiple signals. This paper presents a method for estimating the frequencies of a multi-sinusoidal signal, based on a new property of the elementary symmetric functions on the frequencies. Let $y(t)$ be the sum of n_p sinusoids with unknown amplitudes, frequencies and phases

$$y(t) = \sum_{k=1}^{n_p} A_k \sin(\omega_k t + \Phi_k), \quad t \geq 0. \tag{1}$$

The frequency estimation problem can be stated as the approximation of the function $y(t)$ on a time-segment of the observed signal $[0, T_{obs}]$, where T_{obs} is the observation time, and the unknown parameters, amplitudes, frequencies, and phases $\{A_i, \omega_i, \Phi_i\}_{i=1}^{n_p}$, have to be found from a given discrete sequence of noisy data obtained from some experiment. The most important parameters to be estimated are the frequencies of the sinusoidal components, which once estimated can in turn be used for the computation of the remaining unknown parameters (amplitudes and phases) [16]. For this reason, the attention will be focused on the estimation of the parameters $\{\omega_i\}_{i=1}^{n_p}$, from the available n samples of $y(t)$. The approach used herein, is based on the algebraic derivative method in the frequency domain [17], which yields exact formula in terms of multiple integrals of the signal, when placed in the time domain. The same problem was considered for the first time with analogous techniques in [18,19]. For a good account of the algebraic derivative method, the reader should refer to [20]. It is important to remark that an advantage of the proposed approach is that it can be used with nonuniform sampling. The relation, between the elementary symmetric functions on the frequencies of multi-sine wave signal and multiple integrals of the signal makes the resulting parameters estimation accurate even in the presence of large measurement noise. A theoretical explanation of the robustness of the algebraic derivative method with respect to noise is given in [21,22] by using a non standard analysis; however in the following an analysis of the noise contribution through a deterministic approach, based on the model of the signal plus noise involved, is reported. The paper is organized as follows. In the next section the problem is stated, the main results are derived providing analytical expressions of the estimator, and some computational aspects are also discussed. The third section contains some considerations about the robustness of the proposed method. The fourth section illustrates the performance obtained by laboratory and simulated experiments and it is followed by final conclusions. The proofs of the results are reported in Appendix.

2 Main results

In this section, the major outcomes of the proposed method will be provided. Let us consider a set of m distinct numbers: $X_m = \{x_1, x_2, ..., x_m\}$.

Definition 1 *Let $\sigma(m,q)$ be the function recursively defined as follows:*

$$\begin{aligned}
&\sigma(m,q) = \sigma(m-1,q) + x_m\sigma(m-1,q-1), && m, q \textit{ integer},\\
&\sigma(m,0) = 1, && m = 0, 1, ...,\\
&(q<0) \textit{ or } (m<0) \textit{ or } (q>m) \rightarrow \sigma(m,q) = 0.
\end{aligned} \tag{2}$$

The generating function of $\sigma(m,q)$ [23], [24], is:

$$B_m(x) = \prod_{i=1}^{m}(x - x_i) = \sum_{r=0}^{m}(-1)^{r+m}\sigma(m, m-r)x^r. \tag{3}$$

Therefore $\sigma(m,q)$ is the qth order elementary symmetric function associated to the set X_m which is the sum of all products of q distinct elements chosen from X_m:

$$\begin{cases}
\sigma(m,q) = \sum\limits_{1\leq\pi_1<...<\pi_q\leq m} x_{\pi_1}x_{\pi_2}...x_{\pi_q},\ q = 1, ..., m,\\
\sigma(m,0) = 1.
\end{cases} \tag{4}$$

By eq. (3) the following equality holds:

$$\sum_{r=0}^{m}(-1)^r x_i^r \sigma(m, m-r) = 0, \quad i = 1, 2, ..., m. \tag{5}$$

Eq. (3) is a polynomial identity which means that, when functions σ are known, it is possible to find the elements $x_i,\ i = 1, 2, ..., m$ as roots of the polynomial $B_m(x)$. Let

$$Y(s) = \sum_{k=1}^{n_p} A_k \frac{\cos(\Phi_k)\omega_k + \sin(\Phi_k)s}{s^2 + \omega_k^2} \tag{6}$$

be the unilateral Laplace transform of $y(t)$. An explicit relation between the elementary symmetric functions, defined on the square of the unknown angular frequencies $X_{n_p} = \left\{\omega_1^2, \omega_2^2, ..., \omega_{n_p}^2\right\}$, and multiple integrals of the signal $y(t)$ can be obtained by following an approach similar to that proposed in [25].

Theorem 1

$$\sum_{i=0}^{n_p}\sum_{j=2i}^{2n_p}\binom{2n_p}{j}\binom{2n_p-2i}{2n_p-j}(2n_p-j)!\ s^{j-2i}\frac{d^jY(s)}{ds^j}\sigma(n_p, i) = 0. \tag{7}$$

A sketch of the proof can be found in Appendix. The inverse Laplace transform of eq. (7) gives:

$$\sum_{i=0}^{n_p}\sum_{j=2i}^{2n_p}(-1)^j\binom{2n_p}{j}\binom{2n_p-2i}{2n_p-j}(2n_p-j)!\ \frac{d^{j-2i}}{dt^{j-2i}}\left[t^j y(t)\right]\sigma(n_p, i) = 0. \tag{8}$$

To eliminate the time derivations, which can amplify the effects of noise on the signal $y(t)$, eq. (7) is divided by s^{2n_p+1} thus introducing at least an integral effect on each term which contains the signal $y(t)$:

$$\sum_{i=0}^{n_p}\sum_{j=2i}^{2n_p}\binom{2n_p}{j}\binom{2n_p-2i}{2n_p-j}(2n_p-j)!\frac{1}{s^{2n_p+1+2i-j}}\frac{d^jY(s)}{ds^j}\sigma(n_p, i) = 0. \tag{9}$$

The explicit relation in eq. (9) allows the estimation of the elementary symmetric function on X_{n_p}. Two different approaches could be used. First, one can directly consider the inverse Laplace transform of eq. (9) and work in the time domain to find an estimate $\hat{\sigma}$ of $\sigma = \{\sigma(n_p, i)\}_{i=1}^{n_p}$ in the least-squares sense. Another way is to extend the approach used in [8] to multiple sinusoidal signals, and to solve a linear system of n_p equations. A set of n_p equations can be constructed by considering eq. (9) and the equations obtained by differentiating it with respect to the variable s, $1, 2..., n_p - 1$ times; in this case eq.

(9) can be generalized as:

$$\sum_{i=0}^{n_p}\sum_{j=2i}^{2n_p}\sum_{k=0}^{q}\binom{2n_p}{j}\binom{2n_p-2i}{2n_p-j}\binom{j-2i}{q-k}\binom{q}{k}(2n_p-j)!$$
$$\times(q-k)!\frac{1}{s^{2n_p+1+2i+q-j-k}}\frac{d^{j+k}Y(s)}{ds^{j+k}}\sigma(n_p,i)=0, \quad q=0,1,...,n_p-1. \tag{10}$$

By taking the inverse Laplace transform, eq. (10) can be expressed in time domain as:

$$\sum_{i=0}^{n_p}\beta(q,i,t)\sigma(n_p,i)=0, \tag{11}$$

with

$$\beta(q,i,t)=\sum_{j=2i}^{2n_p}\sum_{k=0}^{q}\binom{2n_p}{j}\binom{2n_p-2i}{2n_p-j}\binom{j-2i}{q-k}\binom{q}{k}$$
$$\times(2n_p-j)!(q-k)!\int^{(2n_p+1+2i+q-j-k)}(-1)^{j+k}t^{j+k}y(t), \tag{12}$$

where we denote by

$$\int^{(j)}\phi(t)$$

the integral expression

$$\int_0^t\int_0^{x_1}...\int_0^{x_{j-1}}\phi(x_j)dx_j...dx_1,$$

with the definition

$$\int^{(1)}\phi(t)=\int_0^t\phi(x_1)dx_1.$$

Note that β also depends on the parameter n_p. From a computational point of view, the multiple integrals on the product between the signal and the time variable involved in $\beta(q,i,t)$, should be expressed in terms of multiple integrals on the measured signal. The following proposition is useful to this aim.

Proposition 1 *Let $y(t)$ be a function which can be expanded in Mc-Laurin series; then:*

$$\int^{(q)}t^k y(t)=\sum_{i=0}^{k}(-1)^i\binom{i+q-1}{q-1}\binom{k}{i}i!\,t^{k-i}\int^{(i+q)}y(t),$$
$$q\geq 1, \quad k\geq 0. \tag{13}$$

Using the result stated in Proposition 1, (see Appendix for the proof), $\beta(q,i,t)$ can be rewritten as:

$$\beta(q,i,t)=\sum_{z=0}^{2n_p+q}\bar{\rho}(z,q,i,t)\int^{(2n_p+1+2i+q-z)}y(t), \tag{14}$$

where

$$\bar{\rho}(z,q,i,t) = \sum_{j=0}^{2n_p}\sum_{k=0}^{q} \frac{(-1)^z}{z!}\binom{2n_p}{j}\binom{2n_p-2i}{2n_p-j}\binom{j-2i}{q-k}$$
$$\times \binom{2n_p+2i+q-z}{k+j-z}\binom{q}{k}(2n_p-j)!(q-k)!(k+j)!t^z. \quad (15)$$

From eq. (14) it is evident that functions $\beta(q,i,t)$ are expressed as a linear combination of multiple integrals of the signal $y(t)$ with coefficients $\bar{\rho}(z,q,i,t)$. The following result proves that the coefficients $\bar{\rho}(z,q,i,t)$ are independent of the index i, thus drastically simplify eq. (15).

Theorem 2

$$\bar{\rho}(z,q,i,t) = \rho(z,q,t) = \frac{(-1)^z}{z!}\binom{4n_p+q-z}{2n_p}(2n_p+q)!t^z. \quad (16)$$

According to the result stated by Theorem 2, (see Appendix for the proof), it follows that:

$$\beta(q,i,t) = (2n_p+q)!\sum_{z=0}^{2n_p+q}\frac{(-1)^z}{z!}\binom{4n_p+q-z}{2n_p} t^z$$
$$\times \int^{(2n_p+1+2i+q-z)} y(t). \quad (17)$$

Remark 1 *Multiple integrals act as a low-pass filter so high frequency zero mean disturbances on the process output and their contribution could be considered negligible. From a theoretical point of view [21], further integrations could be added to more decrease high frequency noise effect in eq. (9).*

2.1 Computational aspects

In this sub-section some computational aspects will be analyzed. First of all, in order to get an estimation in the least-squares sense one can observe that eq. (11) holds for all t. Let $\hat{t} = \{t_1, t_2, ..., t_n\}$ be a set of time instants where eq. (11) will be evaluated, then such equation, for $q = 0$, can be expressed in matrix form as:

$$V\sigma = u \quad (18)$$

where $V \in \mathbb{R}^{n\times n_p}$ and $u \in \mathbb{R}^n$ with

$$V(i,j) = \beta(0,j,t_i), \quad i = 1,2,...,n, \; j = 1,2,...,n_p, \quad (19)$$

$$u(i) = -\beta(0,0,t_i), \quad i = 1,2,...,n. \quad (20)$$

An estimate $\hat{\sigma}$ in the least-squares sense is:

$$\hat{\sigma} = (V^T V)^{-1} V^T u. \tag{21}$$

As far as the second approach is concerned, that is the solution of a linear system, eq. (11) is rewritten in terms of time-varying matrix form as:

$$M(t)\sigma(t) = b(t) \tag{22}$$

where

$$M_{i,j}(t) = \beta(i-1, j, t), \quad i, j = 1, 2, ..., n_p, \tag{23}$$

and

$$b_i(t) = -\beta(i-1, 0, t), \quad i = 1, 2, ..., n_p. \tag{24}$$

Under the hypothesis of non-singularity of the matrix $M(t)$, an estimate $\hat{\sigma}(t)$ can be found as:

$$\hat{\sigma}(t) = M(t)^{-1} b(t). \tag{25}$$

Note that in both the proposed approach, the fundamental step is computation of the coefficients $\beta(i, j, t)$. This can be efficiently accomplished by considering the linear system, driven by the signal $y(t)$:

$$\begin{cases} \dot{x}(t) = Fx(t) + gy(t) \\ \beta(t) = Hx(t) \end{cases} \tag{26}$$

where the state vector

$$x(t) = \left[\int^{(4n_p+q+1)} y(t), \int^{(4n_p+q)} y(t), ..., \int^{(1)} y(t) \right]^T$$

contains the multiple integrals of the signal $y(t)$, and the output $\beta(t)$ is

$$\beta(t) = [\beta(0,0,t) ... \beta(0, n_p, t) \;\; ... \;\; \beta(q-1, 0, t) ... \beta(q-1, n_p, t)]^T . \tag{27}$$

The matrices F, G and H have the following expressions:

$$F = \begin{bmatrix} \mathbf{0_{4n_p+q}} & \mathbf{I_{4n_p+q}} \\ 0 & \mathbf{0^T_{4n_p+q}} \end{bmatrix}, \tag{28}$$

$$g = \begin{bmatrix} \mathbf{0_{4n_p+q}} \\ 1 \end{bmatrix} \tag{29}$$

with $\mathbf{0_n}$ a $n \times 1$ vector of zeros, and $\mathbf{H}$ a matrix of dimensions $(q+1)(n_p+1) \times (4n_p+q+1)$:

$$H = \begin{bmatrix} \mathbf{H_0} \\ \vdots \\ \mathbf{H_q} \end{bmatrix} \tag{30}$$

where the kth block of H is:

$$\mathbf{H_k(i,j)} = \begin{cases} \rho(2i+j+k-2n_p-q-3, k, t), \\ i = 1, ..., n_p+1, \;\; j = 2n_p+q+3-2i-k, ..., 4n_p+q+3-2i, \\ 0, \quad \text{otherwise.} \end{cases} \tag{31}$$

Remark 2 *In the numerical examples we will use both eq. (21) and (25). Eq. (21) is based on a batch process of measurements. Eq. (25) implements a fast, non-asymptotic method for on-line estimation in continuous time. Since the estimation takes place in a fraction of the period of the periodic signal, the unstable filter (see Eqs. (26)-(31)) can be switched off and re-initialized if parameters are expected to change to new constant values.*

In our simulations the integrals are solved, by using the Matlab function *LSIM*, for regularly spaced time samples, as the output of the dynamical system defined in eq. (26). *LSIM* selects the interpolation method automatically based on the smoothness of the signal $y(t)$. For non-uniform samples the integrals are computed by using the function *CUMTRAPZ*.

Remark 3 *Time instants $t_1, t_2, ..., t_n$ in which eq. (11) has to be evaluated, are chosen coincident with the time instants in which the samples of $y(t)$ are collected. Clearly the choice of $t_1, t_2, ..., t_n$, and therefore the instants of sampling, is important for an accurate approximation of the integrals, and then for such reason, some sampling might be better than other [26]. In many cases integrals are numerically solved by using a polynomial which interpolates the function $y(t)$. For all functions not affected by Runge phenomenon, as in the case of sinusoid functions, all choices of interpolation points are theoretically comparable for high value of n but not comparable in terms of numerical accuracy.*

Remark 4 *Considering that at this stage the coefficients σ have been computed, the next step is to calculate the roots of the polynomial:*

$$B_{n_p}(x) = \prod_{k=1}^{n_p}(x-\omega_k^2) = \sum_{r=0}^{n_p}(-1)^{n_p+r}\sigma(n_p, n_p-r)x^r, \tag{32}$$

from which the frequencies of the signal components can be derived. Several *methods exist to find the roots of a polynomial, (for a list of methods see [27]). The proposed approach relies on the fact that the number n_p of the unknown frequencies is known. If n_p is not known, the procedure can be repeated for increasing n_p, however a number of extraneous roots can be introduced. The problem of distinguishing these roots from the true signal-related roots has been discussed by many authors. In [28] it is shown how to choose signal-related roots by investigating both forward and backward linear prediction. In [29] a procedure which uses a test of significance for each new term as it is introduced, is discussed. Such test is based on the maximum likelihood ratio, i.e. the ratio of the maximised likelihood for n_p terms to the maximised likelihood for $(n_p - 1)$ terms. The need of n_pth terms is related to how bigger this quantity is. A classical method relies on the singular value decomposition of the forward linear prediction matrix, [25]. The number of sinusoids in the signal is estimated by comparing the relative magnitudes of the singular values. The signal-related singular values tends to be larger than the noise-related ones. The effect of using a truncated SVD is to increase the SNR in the data prior to obtain the solution. In [30] is presented a modified Kumaresan-Tufts algorithm, where the Hankel structure of the backward prediction matrix is preserved while performing the low-rank approximation. This allows to significantly reduce the noise threshold. More recently [31], the SVD approach is used to develop better tools for interharmonic estimation in frequency power converters in order to avoid possible damage due to their influence. In [32] the same problem is solved by using the Prony's model and the min-norm method. All the above methods are effectively only in case of uniform sampling. Since the proposed approach works also with uniform sampling, then it is possible to apply such methods to estimate the number of sinusoids in the signal. It is interesting, however, to investigate different methods of model order detection which can be used in case of non-uniform sampling, hopefully this work would stimulate further research in this direction.*

3 Analysis of noise effects

In this section the behavior of the proposed method in case of noisy signal is discussed. The presented analysis is divided into two parts. In first one a comparison with an existing method, designed in discrete-time, and the Cramér-Rao lower bound is investigated through numerical simulations. In such case a discrete sequence of noisy data is used as input to the algorithms varying the signal-to-noise ratio. In the second part the behavior of the proposed method, with respect to a continuous-time noise model (see [33]-[35]), is discussed by using an analytic expression of the functions β related to the noise contribution. In the sequel the acronym (**CEF**) is used to refer to the proposed method.

In order to investigate the statistical properties of the estimator, noisy samples generated by computer simulation are used. Three experiments are conducted on the signal $y(t) = A\sin(\omega t + \phi)$, with $A = 1$, $\omega = 2\pi$ and $\phi = \pi/2$ by collecting, during the observation time $T_{obs} = 1$, $n = 256,\ 512,\ 1000$ samples respectively. Noisy samples are obtained by adding white noise samples to the samples of $y(t)$. The SNR is measured in decibels as the logarithm of the average power of the signal's samples and the noise's samples, $r(kT_s)$, over the time of the experiment:

$$\mathrm{SNR} = 10 \log \frac{\sum\limits_{k=1}^{n} y(kT_s)^2}{\sum\limits_{k=1}^{n} r(kT_s)^2}. \tag{33}$$

In the sequel, comparisons between CEF, the iterative modified Prony algorithm for the frequencies estimation proposed in [38] (**MOS**) and the Cramér-Rao lower bound (**CRLB**)

$$CRLB \approx \frac{12}{\mathrm{SNR}\ n(n^2-1)} \frac{1}{(2\pi T_s)^2} \tag{34}$$

are performed. Figs. 1-3 show the mean squared frequency error for each method versus SNR. For each value of the SNR, 1000 iterations were performed by giving to both algorithms the same noisy sampled data sequence.

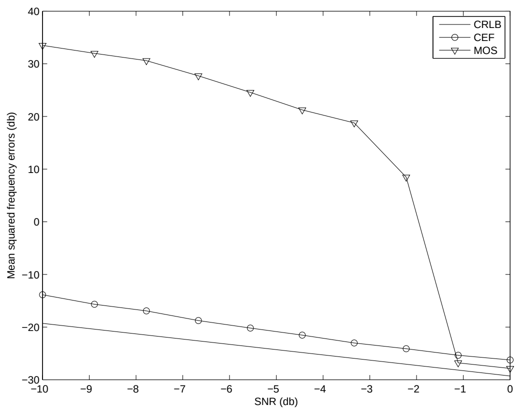

Fig. 1. Mean squared frequency errors versus SNR at $\omega = 2\pi$, $n = 256$, $T_{obs} = 1$.

From the reported results it seems evident that CEF method performs better than MOS in hard situations corresponding to low signal-to-noise ratio. It is important to remark that MOS method is intrinsically iterative, because its formulation is in terms of nonlinear eigenproblem solved by inverse iteration

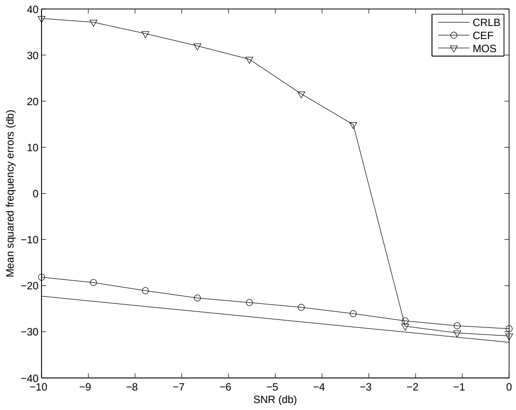

Fig. 2. Mean squared frequency errors versus SNR at $\omega = 2\pi$, $n = 512$, $T_{obs} = 1$.

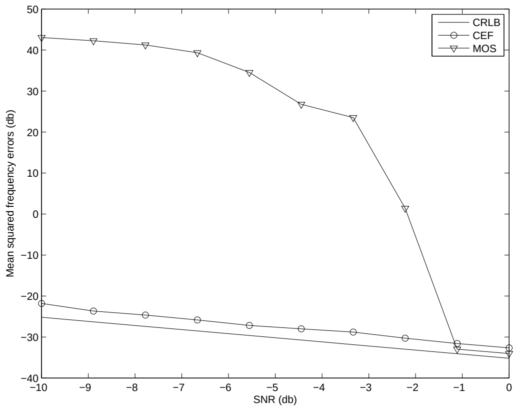

Fig. 3. Mean squared frequency errors versus SNR at $\omega = 2\pi$, $n = 1000$, $T_{obs} = 1$.

method [38]. As n increases the absence of a clearcut boundary to the domain of attraction of the maximum likelihood solution is to be expected for such a non-linear algorithm [38]. For such reasons the values of n have been chosen according to [38].

3.2 A continuous-time noise model

To avoid the mathematical difficulties due to the manipulation of a continuous-time noise, a particular model for the noise contribution has been chosen (see [33]-[35]). Let T_s be the sampling period and assume that $\{r_k\}$ is a zero-mean normally distributed discrete-time noise sequence with covariance:

$$E\left\{r_i r_j^T\right\} = \sigma_r^2 \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker-delta. In addition, $r(t)$ is assumed constant during the sampling interval:

$$r(t) = r(kT_s), \quad \text{for } kT_s \le t < (k+1)T_s. \tag{35}$$

According to standard time series analysis, it follows that the spectral density of $\{r_k\}$ is constant in the frequency range $[-\pi/T_s, \pi/Ts]$. Since the Fourier transform of the sampled signal is periodic with a period equal to the sampling frequency $2\pi/T_s$, then the noise sequence $\{r_k\}$ has constant spectral density for all frequencies. In this way it is possible to obtain a continuous-time white noise representation without adopting the theory of Brownian motion [33]. In the proposed method, the noise attenuation is principally due to the multiple integrals on the signal.

Taking into account the previous assumptions, the noise $r(t)$ can be formally expressed as

$$r(t) = \sum_{k=0}^{n-2} r_k \left\{u\left[t - kT_s\right] - u\left[t-(k+1)T_s\right]\right\} + r_{n-1}u\left[t-(n-1)T_s\right], \quad (36)$$

where

$$u[t] = \begin{cases} 1, \; t \geq 0, \\ 0, \; t < 0. \end{cases}$$

Eq. (36) can be also rewritten as:

$$r(t) = r_0 + \sum_{k=0}^{n-2} \left(r_{k+1} - r_k\right) u\left[t-(k+1)T_s\right], \quad (37)$$

whose multiple integrals of $r(t)$ are explicitly computed as:

$$\int^{(j)} r(t) = \frac{1}{j!}\left\{t^j r_0 + \sum_{k=0}^{n-2} \left[t-(k+1)T_s\right]^j \left(r_{k+1} - r_k\right) u\left[t-(k+1)T_s\right]\right\}. \quad (38)$$

Therefore it is possible to obtain the noise effect on the functions $\beta(q,i,t)$, namely $\beta_r(q,i,t)$, as:

$$\beta_r(q,i,t) = \sum_{z=0}^{2n_p+q} \frac{\rho(z,q,t)}{(2n_p+1+2i+q-z)!} \times \left\{t^{2n_p+1+2i+q-z} r_0 + \sum_{k=0}^{n-2} \left[t-(k+1)T_s\right]^{2n_p+1+2i+q-z} (r_{k+1} - r_k) u\left[t-(k+1)T_s\right]\right\}. \quad (39)$$

For the purpose of our analysis suppose that $|r_k| < \bar{r}_\epsilon$, i.e. $Prob(|r_k| \geq \bar{r}_\epsilon) = \epsilon$, then

$$|\beta_r(q,i,t)| \leq \hat{\beta}_r(q,i,t), \quad (40)$$

where

$$\hat{\beta}_r(q,i,t) = (2n_p+q)! \sum_{z=0}^{2n_p+q} \frac{\binom{4n_p+q-z}{2n_p}}{z!(2n_p+1+2i+q-z)!}$$
$$\times \left\{ t^{2n_p+1+2i+q}\bar{r}_\epsilon + 2\bar{r}_\epsilon t^z \sum_{k=0}^{n-2} [t-(k+1)T_s]^{2n_p+1+2i+q-z} u[t-(k+1)T_s] \right\}. \tag{41}$$

From eq. (41) it is evident that the function $\hat{\beta}_r(q,i,t)$ is a monotonic increasing function in the variable t, then it is straightforward that it assumes the maximum value for $t = T_{obs} = (n-1)T_s$ corresponding to the last time instant of the observation window:

$$\begin{aligned}\hat{\beta}_{r_{max}}(q,i) &= \max_t \left\{ \hat{\beta}_r(q,i,t) \right\} \\ &= \bar{r}_\epsilon (2n_p+q)! \left[(n-1)T_s\right]^{2n_p+1+2i+q} \sum_{z=0}^{2n_p+q} \frac{\binom{4n_p+q-z}{2n_p}}{z!(2n_p+1+2i+q-z)!}.\end{aligned} \tag{42}$$

From eq. (42) it follows that

$$\text{fixed } n, \quad \forall \gamma > 0, \quad \exists \bar{T}_s \mid T_s < \bar{T}_s \Rightarrow \hat{\beta}_r(q,i,t) < \gamma. \tag{43}$$

A different way to interpret eq. (42) could be

$$\forall \gamma > 0, \quad \exists\ \bar{n}, \bar{T}_s \mid (n-1)T_s < (\bar{n}-1)\bar{T}_s \Rightarrow \hat{\beta}_r(q,i,t) < \gamma. \tag{44}$$

Therefore, although the benefits due to the multiple integrals vanish for long observation time T_{obs}, for a given sequence of noise $\{r_k\}$, it is always possible to find a sampling time, or an observation time, which allows to make the $\beta_r(q,i,t)$ contribution negligible.

Remark 5 *Although, in a noise-free context, the observation time may be small, in a noisy environment a strike balance has to be found because, in this case T_{obs} can not be either too small, otherwise the signal contribution on the estimation is almost zero, or too big since, as it is evident from eq. (42) the noise effect grows with T_{obs}. In [21]-[22] a similar discussion has been presented by using the framework of nonstandard analysis.*

An estimation of $\sigma(n_p,i),\ i = 1,...,n_p$ can be obtained by solving the linear system (22) with (23) and (24). If the signal $y(t)$ is corrupted by the noise $r(t)$, then the system (22) becomes

$$(M(t) + \Delta M(t))\tilde{\sigma}(t) = b(t) + \Delta b(t), \tag{45}$$

where

$$\Delta M_{i,j}(t) = \beta_r(i-1, j, t), \quad i, j = 1, 2, ..., n_p, \tag{46}$$

and

$$\Delta b_i(t) = -\beta_r(i-1, 0, t), \quad i = 1, 2, ..., n_p. \tag{47}$$

Following the same approach in [36], to study how sensitive is the solution to perturbation in the data the classical first order bound [37] could be used:

$$\frac{\|\sigma(t) - \tilde{\sigma}(t)\|}{\|\sigma(t)\|} \leq \frac{2\rho(t)\kappa(M(t))}{1 - \rho(t)\kappa(M(t))}, \tag{48}$$

where $\kappa(M(t)) = \|M(t)\| \, \|M(t)^{-1}\|$ is the matrix condition number and $\rho(t)$ measures the level of the noise:

$$\rho(t) = \max\left\{\frac{\|\Delta M(t)\|}{\|M(t)\|}, \frac{\|\Delta b(t)\|}{\|b(t)\|}\right\}, \tag{49}$$

with L_2-vector norm and the induced spectral norm for matrices. This noise measure is thus related to the noise-to-signal ratio.

Such considerations should be better explained by the following experiment in which simulations on one hundred noisy signal $y(t) = \cos(2\pi t) + 2\sin(3\pi t + \pi/3)$ sampled with $T_s = 1 \cdot 10^{-3}$ are carried out. Fig. (4) represents, in logarithmic scale, the average of the functions $\rho(t)$.

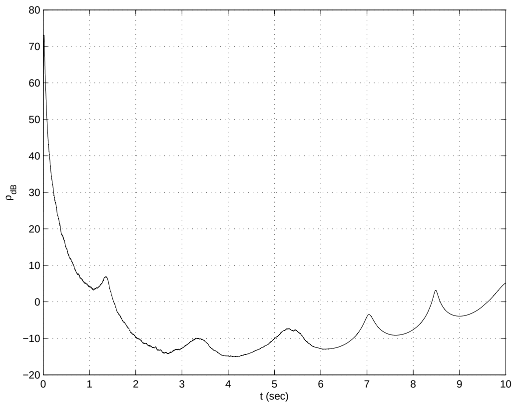

Fig. 4. Mean value of $\rho(t)$ over 100 trials.

These results confirm the necessity of achieving a compromise in the selection of the observation time. From Fig. (4) one can observe that, increasing the observation time, the value of ρ is unacceptable. As the observation time decreases, the value of ρ diminishes until $t^* \approx 4$, which can be considered optimal since it corresponds to the minimum value of ρ. Moreover for $t < t^*$ the value of ρ start to grow up since for small observation time signal contribution becomes insufficient for the estimation. Determining the optimum value for the observation time (in the sense of achieving the best compromise), is still an open question [36].

4 Experimental results

This section includes some numerical results that highlight and point out the advantages and the strengths of the proposed method, in particular it will be devoted to laboratory and simulated experiments. The laboratory experiments were carried out by interfacing Labview and Matlab programs. The experimental data records were acquired by using a National Instruments measurement device, in particular a **NI-DAQPad-6015** acquisition data device with 16 Inputs, 16-bit, 200kS/s Multifunction I/O for USB. The Labview software used is **NI-DAQ 7** which allows to create a DAQ-assistant for the signal acquisition. The acquired signals were generated by a 15MHz waveform generator **Agilent 33120A**. The goodness of the proposed method will be measured on m experiments in terms of

- the mean value of the frequency relative errors:

$$\hat{e}_1 = \frac{1}{m} \sum_{i=1}^{m} e_{r_i}; \tag{50}$$

- the bias:

$$\hat{e}_2 = \frac{1}{m} \sum_{i=1}^{m} \left(f - \hat{f}_i \right); \tag{51}$$

- the normalized root mean squared error:

$$\hat{e}_3 = \frac{1}{f} \sqrt{\frac{1}{m} \sum_{i=1}^{m} \left(f - \hat{f}_i \right)^2}. \tag{52}$$

Experiment 1

The first experiment consists in the acquisition in one period of $n = 1000$ samples for each of the following signals (in this case $n_p = 1$):

$$y_i(t) = 5 \sin(2\pi 50t + \Phi_i), \quad i = 1, 2, ..., 500, \tag{53}$$

where the parameters Φ_i, $i = 1, 2, ..., 500$ are constants and uniformly distributed in $[0, \pi]$.

Naturally, the collected data are corrupted by noise present in the acquisition process. In order to investigate the robustness against harmonic components in the signals, in each experiment the 3rd and the 121st harmonics were added via software:

$$y_{n_i}(t) = y_i(t) + r(t), \quad i = 1, 2, ..., 500, \tag{54}$$

with

$$r(t) = A_n \left[\sin(2\pi 150t) + \sin(2\pi 6050t) \right]. \tag{55}$$

In eq. (55) A_n is chosen according to the desired value of the signal-to-interference ratio (SIR):

$$\mathrm{SIR} = 10\log_{10}\left(\frac{P_{y_i}}{P_r}\right), \tag{56}$$

where

$$P_f = \frac{1}{T_{obs}}\int_0^{T_{obs}} f(t)^2 dt. \tag{57}$$

Figures 5, 6 and 7 show the comparisons between CEF and MOS for indexes $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ respectively, versus SIR in $[5dB, 31dB]$ for $T_{obs} = 0.02s$. As it is evident from the experiments, the proposed method seems to be robust with respect to the harmonic noise and comparable with MOS.

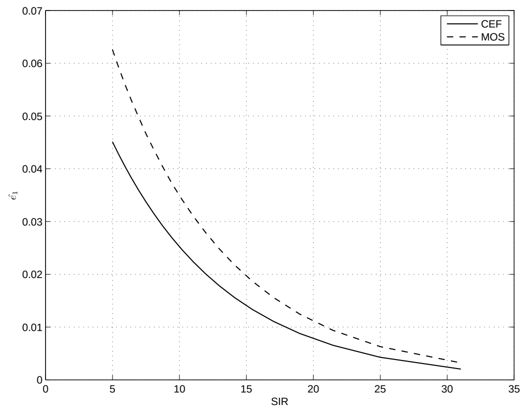

Fig. 5. Example 1. Comparisons between CEF and MOS respect to $\hat{e}_1$.

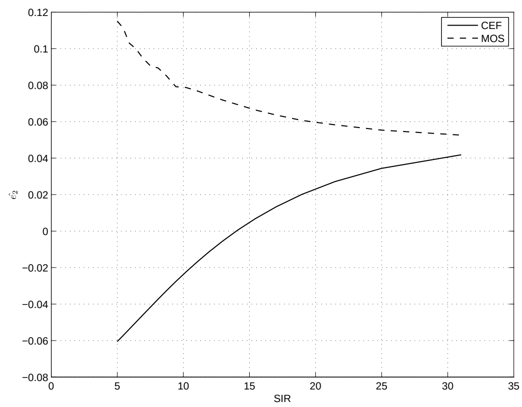

Fig. 6. Example 1. Comparisons between CEF and MOS respect to $\hat{e}_2$.

Experiment 2

In the second experiment two signals, which are not explicitly expressed in the form (1), but that can be led back to it by using a Fourier series expansion, were considered.

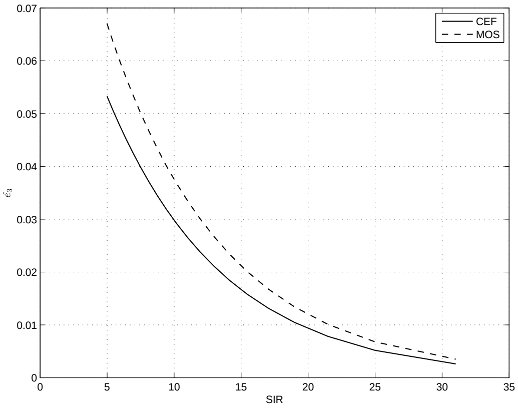

Fig. 7. Example 1. Comparisons between CEF and MOS respect to $\hat{e}_3$.

⋆ **Case 1:**

First, the acquisition of $n = 1000$ samples in one period of a 50Hz periodic square waveform, with $T_s = 2.002 \cdot 10^{-5}$ and $T_{obs} = 0.02$ was done. In Fig. 8 the periodic square waveform acquired is shown.

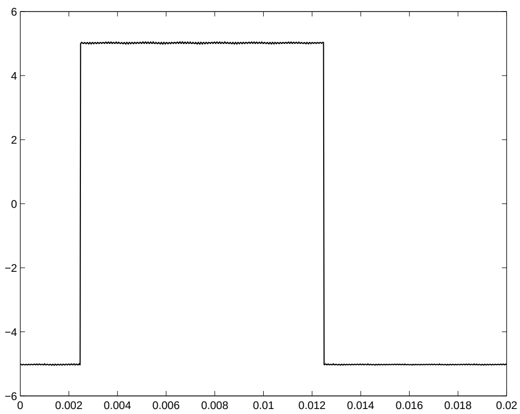

Fig. 8. Square waveform with an observation time given by $T_{obs} = 0.02$.

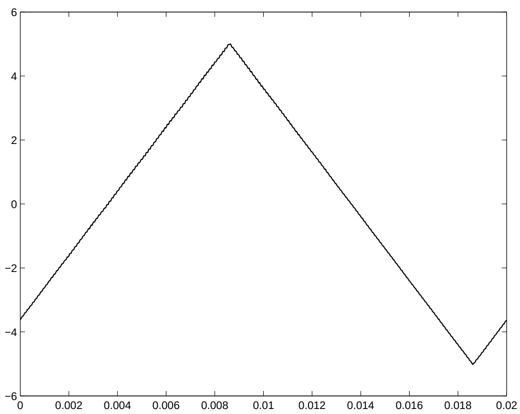

Fig. 9. Triangular waveform with an observation time given by $T_{obs} = 0.02$.

The relative error between the fundamental frequency and the estimated one is $\hat{e}_1 = 2.58 \cdot 10^{-2}$. The second signal considered is represented by a

50Hz periodic triangular waveform as depicted in Fig. 9, also in this case $n = 1000$, $T_s = 2.002 \cdot 10^{-5}$ and $T_{obs} = 0.02$. For this signal the relative error between the fundamental frequency and the estimated one is $\hat{e}_1 = 1.00 \cdot 10^{-2}$.

$\star$ **Case 2:**

In this experiment, the estimation of the first two harmonics of the Fourier series expansion of the previous acquired signals is performed, ($n_p = 2$). On the estimation given for the square waveform the relative error on the fundamental frequency is $\hat{e}_1 = 1.60 \cdot 10^{-3}$ and $\hat{e}_1 = 6.26 \cdot 10^{-2}$ for the second harmonic. For the triangular waveform the relative error on the first frequency is $\hat{e}_1 = 1.37 \cdot 10^{-3}$ and $\hat{e}_1 = 6.48 \cdot 10^{-2}$ for the second one.

Remark 6 *We would like to note that in the Fourier series expansion of the considered signals the even harmonics are absent since both waveforms possess certain symmetrical properties.*

Experiment 3

This experiment, ($n_p = 1$), is aimed to stress that our method can also be used in case of non-uniform samples. The considered signals have the following expression:

$$y(t) = 5\sin(2\pi 50t + \pi/3). \tag{58}$$

Three simulations were conducted by corrupting the signal with a zero mean white noise with different values of SNR. In each simulation $n = 10000$ samples of the signal were generated by using a uniform distribution on the observation window $[0, 0.02]$. For each simulation, with a fixed SNR, 10000 iterations were performed. Table 1 reports the values of $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ for different SNR.

SNR	$\hat{e}_1$	$\hat{e}_2$	$\hat{e}_3$
-10dB	$3.01 \cdot 10^{-2}$	$1.51 \cdot 10^{0}$	$3.81 \cdot 10^{-2}$
0dB	$9.58 \cdot 10^{-3}$	$4.79 \cdot 10^{-1}$	$1.20 \cdot 10^{-2}$
10dB	$2.99 \cdot 10^{-3}$	$1.49 \cdot 10^{-1}$	$3.75 \cdot 10^{-3}$

Table 1
Example 3. Indexes $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ over 10000 tests.

Experiment 4

In order to present the quality, precision and velocity in the computation of frequencies estimation of multi-sinusoidal signal, an experiment is carried out by comparing the proposed method with that one proposed in [39], namely (**XIA**).

It is assumed that the following signal with two frequencies is available for measurement:

$$y(t) = \sin(t) + 1.35\sin(5t). \tag{59}$$

The estimation was done by solving, in each sampling time, the linear system represented by eq. (22). In [39] the following sixth-order estimator is proposed:

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= \xi_4 \\
\dot{\xi}_4 &= -3\xi_1 - 10\xi_2 - 5\xi_3 - 2.5\xi_4 + y(t) \\
\dot{\theta}_1 &= 7500\,(y_i(t) - y(t))\,\xi_3 \\
\dot{\theta}_2 &= 7500\,(y_i(t) - y(t))\,\xi_1,
\end{aligned} \tag{60}$$

with

$$y_i(t) = 2.5\xi_4 + 10\xi_2 + (5 - \theta_1)\xi_3 + (3 - \theta_2)\xi_1. \tag{61}$$

In simulation the sampling time $T_s = 4 \cdot 10^{-3}$ is chosen and all initial conditions are set to be zero. Fig. 10 shows the comparison between CEF method (solid line) and XIA one (dashed line). From the reported results it can be seen that the new method produces a very quick estimate of the system frequency since the estimation takes place in a fraction of time respect to the period of the signal.

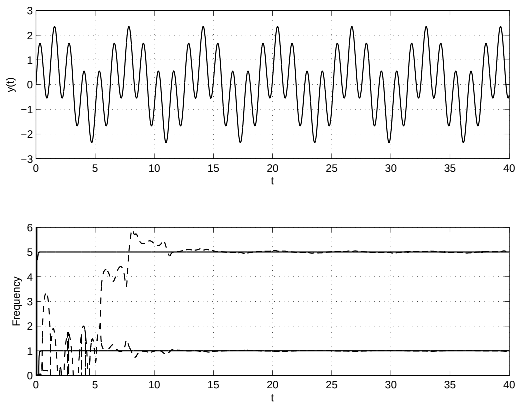

Fig. 10. Observed signal in Experiment 4 and the obtained estimations (dashed line - XIA, solid line - CEF).

Experiment 5

This experiment is carried out by comparing the proposed method with a new approach which guarantees a globally convergence presented by Bobtsov in [40]. The considered signal is represented by a biased sinusoid signal and is equal to the considered one by Bobtsov:

$$y(t) = -2 + 2\sin(0.5t). \tag{62}$$

The simulation was carried out by choosing the same designed simulation parameters required by Bobtsov's method (see examples section in [40]) and a sampling time $T_s = 4 \cdot 10^{-5}$. Note that, the proposed method does not consider explicitly the case of a biased multi-sinusoidal signal, but anyway a bias can be viewed as a particular sine wave with zero frequency and phase multiple of $\frac{\pi}{2}$. Therefore, an estimation of a biased sinusoid signal can be obtained by adapting opportunely the proposed method to such case by considering $n_p = 2$. In this case eq. (11) becomes

$$\beta(0,0,t) + \beta(0,1,t)\sigma(2,1) + \beta(0,2,t)\sigma(2,2) = 0. \tag{63}$$

Note that, since the bias term was considered to be a zero frequency sinusoid, one component of the elementary symmetric functions is equal to zero, in particular $\sigma(2,2) = 0$, then eq. (63) can be rewritten as:

$$\beta(0,0,t) + \beta(0,1,t)\sigma(2,1) = 0.$$

Moreover, to perform the comparison between the two approaches an "on-line" version for our method was used

$$\sigma(2,1) = -\frac{\beta(0,0,t)}{\beta(0,1,t)}, \quad \forall \beta(0,1,t) \neq 0.$$

The results of the computer simulation for the proposed method and the Bobtsov's method are depicted in Fig. 11.

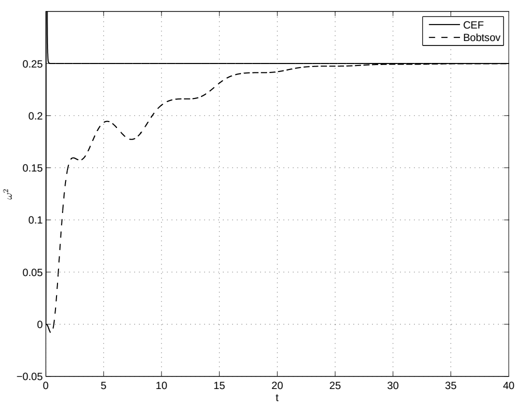

Fig. 11. Comparison between the estimation of the proposed method and Bobtsov's method.

As it is evident, the estimation takes place in a fraction of time while Bobtsov's method, which is essentially an asymptotic globally convergent observer, clearly requires a significant convergence period as well as a priori design parameters.

Experiment 6

In several cases it is difficult to estimates a signal with frequencies which are close to each other, even when there is no noise [29]. For example, in the following signal

$$y(t) = \cos(\omega_1 t) + \cos(\omega_2 t), \tag{64}$$

with $\omega_1 = 0.1841$ and $\omega_2 = 0.2$, the beat phenomenon arises. Although the data seem to represent a periodic signal, it seems that a frequency close to the two considered one appears with a much lower frequency associated with the evolution of the envelope [29]:

$$y(t) = 2\cos\left(\frac{\omega_1 + \omega_2}{2} t\right)\cos\left(\frac{\omega_1 - \omega_2}{2} t\right). \tag{65}$$

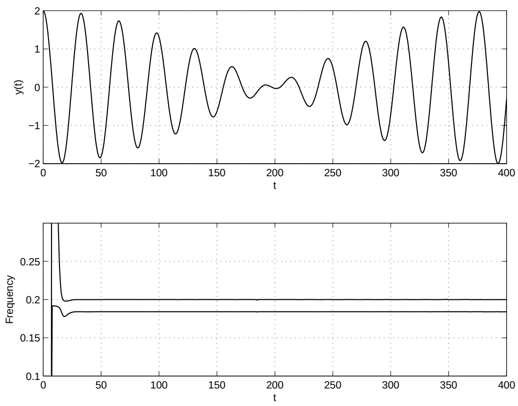

Fig. 12. Observed signal in Experiment 6 and the obtained estimations.

The estimation was done by solving, in each sampling time, the linear system represented by eq. (22). In the performed simulation the sampling time $T_s = 7.828 \cdot 10^{-1}$ is chosen. Also in this difficult case, the new method produces a quick estimation of the frequencies of the considered signal. Note that MOS algorithm fails to accurately estimate the two frequencies, in fact it gives $\hat{\omega}_1 = 0.1444$ and $\hat{\omega}_2 = 0.1564$.

5 Conclusion

A relation between the elementary symmetric functions on the frequencies of multi-sine wave signal and its multiple integrals has been investigated. Such relation has been used to derive two different approaches for estimating the frequencies of such signal from a finite number of noisy discrete-time measurements. From a computational point of view the new algorithm is simple to implement. The experiments have shown very good performance of the estimator and high accuracy at the remarkably high noise level in the signal, i.e. the estimated frequencies are close to the real ones. The proposed method has given good estimation results in the presence of noise thanks to the use of multiple integrals of the measured signal acting as an intrinsic low-pass filter. The proposed estimator needs further investigation because, for example, it does not appear ready for an immediate extension to the case of time-varying frequency signals, hopefully this work would stimulate further research in this direction.

Appendix

The proof of the statements of Proposition 1, Theorem 1 and 2 are given in the following.

Theorem 1:

$$\sum_{i=0}^{n_p} \sum_{j=2i}^{2n_p} \binom{2n_p}{j} \binom{2n_p - 2i}{2n_p - j} (2n_p - j)! \; s^{j-2i} \frac{d^j Y(s)}{ds^j} \; \sigma(n_p, i) = 0. \tag{A-1}$$

Proof. We first introduce the following lemma whose proof is omitted since it is easily obtained by induction and by standard algebraic manipulations.

Lemma 1

$$\frac{d^q \left[s^k Y(s)\right]}{ds^q} = s^{k-q} \sum_{j=0}^{q} \binom{k}{q-j} \binom{q}{j} (q-j)! \frac{d^j Y(s)}{ds^j} s^j. \tag{A-2}$$

By eq. (6) one has

$$Y(s) \prod_{i=1}^{n_p} (s^2 + \omega_i^2) = \sum_{k=1}^{n_p} A_k (cos(\Phi_k)\omega_k + sin(\Phi_k)s) \times \prod_{i=1, i\neq k}^{n_p} (s^2 + \omega_i^2). \tag{A-3}$$

Taking into account Lemma 1 and differentiating eq. (A-3) $2n_p$ times with respect to s, eq. (A-1) follows. ■

Proposition 1: Let $y(t)$ a function which can be expanded in Mc-Laurin series then:

$$\int^{(q)} t^k y(t) = \sum_{i=0}^{k} (-1)^i \binom{i+q-1}{q-1} \binom{k}{i} i! \; t^{k-i} \int^{(i+q)} y(t),$$
$$q \geq 1, \quad k \geq 0. \tag{A-4}$$

Proof. Let us suppose that the right-hand side of eq. (A-4) holds, then, by taking into account the Mc-Laurin series of $y(t)$:

$$y(t) = \sum_{l=0}^{\infty} \frac{y^{(l)}(0)}{l!} t^l \tag{A-5}$$

it follows that:

$$\begin{aligned}&\sum_{i=0}^{k} (-1)^i \binom{i+q-1}{q-1} \binom{k}{i} i!\ t^{k-i} \int^{(i+q)} y(t)\\ &= \sum_{l=0}^{\infty} \frac{y^{(l)}(0)}{l!} \sum_{i=0}^{k} (-1)^i \binom{i+q-1}{q-1} \binom{k}{i} i!\ t^{k-i} \int^{(i+q)} t^l\end{aligned} \tag{A-6}$$

Since

$$\int^{(i+q)} t^l = \frac{l!}{(i+q+l)!} t^{i+q+l},$$

then eq. (A-6) becomes

$$\begin{aligned}&\sum_{l=0}^{\infty} \frac{y^{(l)}(0)}{l!} \sum_{i=0}^{k} (-1)^i \binom{i+q-1}{q-1} \binom{k}{i} i!\ t^{k-i} \int^{(i+q)} t^l\\ &= \sum_{l=0}^{\infty} \frac{y^{(l)}(0)}{l!} \sum_{i=0}^{k} (-1)^i \binom{i+q-1}{q-1} \binom{k}{i} i! \frac{l!}{(i+q+l)!} t^{k+q+l}.\end{aligned} \tag{A-7}$$

By standard arguments, [41], and taking into account the following equality:

$$\sum_{i=0}^{k} (-1)^i \binom{i+q-1}{q-1} \binom{k}{i} i! \frac{1}{(i+q+l)!} = \frac{(l+k)!}{l!(l+q+k)!}, \tag{A-8}$$

eq. (A-4) follows. ■

Theorem 2:

$$\bar{\rho}(z,q,i,t) = \rho(z,q,t) = \frac{(-1)^z}{z!} \binom{4n_p+q-z}{2n_p} (2n_p+q)! t^z. \tag{A-9}$$

Proof. In the expression (15) a variable change $j+k=c$ gives:

$$\begin{aligned}\bar{\rho}(z,q,i,t) = \sum_{j=0}^{2n_p} \sum_{c=0}^{2n_p+q} \frac{(-1)^z}{z!} \binom{2n_p}{j} \binom{2n_p-2i}{2n_p-j} \binom{2n_p+2i+q-z}{c-z}\\ \times \binom{j-2i}{q+j-c} \binom{q}{c-j} (2n_p-j)!(q+j-c)!\ c!\ t^z.\end{aligned} \tag{A-10}$$

Since the summation on the index j has an explicit closure [41]:

$$\sum_{j=0}^{2n_p} \binom{2n_p}{j} \binom{2n_p - 2i}{2n_p - j} \binom{j - 2i}{q + j - c} \binom{q}{c - j} (2n_p - j)!$$
$$\times (q + j - c)! = \frac{\Gamma(2n_p - 2i + 1)\Gamma(2n_p + q + 1)}{\Gamma(c + 1)\Gamma(c - 2i - q + 1)\Gamma(2n_p + q - c + 1)}, \tag{A-11}$$

then eq. (A-10) can be expressed as:

$$\bar{\rho}(z, q, i, t) = \frac{(-1)^z}{z!} \sum_{c=0}^{2n_p+q} \frac{\Gamma(2n_p - 2i + 1)\Gamma(2n_p + q + 1)}{\Gamma(c - z + 1)\Gamma(2n_p + 2i + q - c + 1)}$$
$$\times \frac{\Gamma(2n_p + 2i + q - z + 1)}{\Gamma(c - 2i - q + 1)\Gamma(2n_p + q - c + 1)} t^z. \tag{A-12}$$

Eq. (A-12) is rearranged in terms of hypergeometric function ${}_3F_2$ [42]:

$$\bar{\rho}(z, q, i, t) = (-1)^z \frac{\Gamma(2n_p - 2i + 1)\Gamma(2n_p + 2i + q - z + 1)}{\Gamma(1 - 2i - q)\Gamma(2n_p + 2i + q + 1)\Gamma(1 - z)\Gamma(z + 1)} t^z$$
$$\times {}_3F_2 \left(\left. \begin{matrix} 1, -2n_p - q, -2i - 2n_p - q \\ 1 - 2i - q, 1 - z \end{matrix} \right| 1 \right). \tag{A-13}$$

Let us indicate with $\Lambda_{3,2}$ the hypergeometric function involved in (A-13). By using the following property of the hypergeometric functions [43]:

$${}_3F_2 \left(\left. \begin{matrix} a, b, c \\ e, f \end{matrix} \right| 1 \right) = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s + b)\Gamma(s + c)}$$
$$\times {}_3F_2 \left(\left. \begin{matrix} e - a, f - a, s \\ s + b, s + c \end{matrix} \right| 1 \right),$$
$$s = e + f - a - b - c, \; s \neq 0, \tag{A-14}$$

$\Lambda_{3,2}$ can be rewritten as:

$$\Lambda_{3,2} = \frac{\Gamma(1 - 2i - q)\Gamma(1 - z)\Gamma(4n_p + q - z + 1)}{\Gamma(2n_p - z + 1)\Gamma(2n_p - 2i - z + 1)}$$
$$\times {}_3F_2 \left(\left. \begin{matrix} -2i - q, -z, 4n_p + q - z + 1 \\ 2n_p - z + 1, 2n_p - 2i - z + 1 \end{matrix} \right| 1 \right). \tag{A-15}$$

The hypergeometric function in eq. (A-15) satisfies the Saalschütz's conditions

[43]:

$$
{}_3F_2\left(\begin{matrix} -n, a, b \\ c, 1+a+b-c-n \end{matrix}\middle| 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n},
$$

$$
n \text{ positive integer or zero,} \tag{A-16}
$$

with $(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$, then the following identity holds:

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} -2i-q, -z, 4n_p+q-z+1 \\ 2n_p-z+1, 2n_p-2i-z+1 \end{matrix}\middle| 1\right) \\
&= \frac{\Gamma(2i-2n_p)\Gamma(2n_p+2i+q+1)\Gamma(2n_p+1-z)\Gamma(z-2n_p-q)}{\Gamma(2n_p+1)\Gamma(-2n_p-q)\Gamma(2n_p+2i+q+1-z)\Gamma(2i+z-2n_p)}.
\end{aligned} \tag{A-17}
$$

Taking into account eqs. (A-13), (A-15) and (A-17), and standard properties of gamma function, eq. (16) follows. ■

Acknowledgments

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