An Analytical Approximation for the Excess Noise Factor of Avalanche Photodiodes with Dead Space

Majeed M. Hayat, *Member, IEEE*, Zikuan Chen, and Mohammad A. Karim, *Senior Member, IEEE*

***Abstract*—Approximate analytical expressions are derived for the mean gain and the excess noise factor of avalanche photodiodes including the effect of dead space. The analysis is based on undertaking a characteristic-equation approach to obtain an approximate analytical solution to the existing system of recurrence equations which characterize the statistics of the random multiplication gain. The analytical expressions for the excess noise factor and the mean gain are shown to be in good agreement with the exact results obtained from numerical solutions of the recurrence equations for values of the dead space reaching up to 20% of the width of the multiplication region.**

I. Introduction

THERE has been an increased recent interest in avalanche photodiodes (APD's) with a thin multiplication region (MR) for their low avalanche multiplication noise [1]–[6]. Experiments have shown that the excess noise factor, which is a measure of the avalanche multiplication noise, is reduced as the width of the MR of the device is decreased [1], [2]. This observation indicates that the excess noise factor is not only a function of the mean gain and the hole-to-electron ionization coefficient ratio but also dependent on the MR width. This dependence cannot be explained within the context of the conventional (McIntyre) avalanche multiplication theory [7], [8] which asserts that the excess noise factor is a function only of the mean gain and the ionization coefficient ratio.

The dependence of the excess noise factor on the width of the MR has been attributed, in part, to the nonlocalized nature of the impact ionization coefficients [2], [4]–[6] which is based on the physical assumption that a newly generated carrier must travel a certain distance, called the dead space, in order to gain sufficient energy before it is capable of impact ionizing [9]. The dead space is primarily a function of the electric field and the impact ionization energy associated with the material. For an APD with a thin MR, the value of dead space becomes relatively high and the effect of dead space on the multiplication noise therefore becomes more significant. Both theoretical [9], [10]–[12] and Monte-Carlo [5], [6] studies confirm that dead space results in a reduction in the excess noise factor. Spinelli *et al.* [13] has recently developed an approximate expression for the mean gain by employing a small-perturbation approach to obtain an approximate solution to the recurrence equations reported in the dead-space multiplication theory [10]. However, the approximation reported in [13] does not address the excess noise factor.

In this letter, we develop an approximate analytical solution to the recurrence equations given in [10] using a technique which involves the characteristic equations corresponding to the recurrence equations. Approximate analytical expressions for both the mean gain and the excess noise factor are determined under conditions of constant ionization coefficients. To our knowledge, no prior closed-form analytical expression for the excess noise factor has been reported for the dead-space model. Our results for the mean gain and the excess noise factor are in good agreement with the exact numerical solutions. Furthermore, the accuracy of the reported approximation of the mean gain is generally comparable to the perturbation approximation reported in [13]. The analysis reported in this letter has the potential to provide a simple alternative tool to simulation and extensive numerical methods traditionally used to characterize avalanche multiplication noise in thin APD's.

II. Results and Discussion

Consider the multiplication region of a pure-electron injection APD extending from $x = 0$ to $x = w$, where w is the width of the avalanche multiplication region. A parent electron is injected at $x = 0$ and travels under the effect of the electric field in the x-direction. After traveling a fixed dead space d_e, the electron becomes capable of impact ionizing with an ionization coefficient α. Upon ionizing, the parent and secondary electrons are required to travel a minimum distance d_e from the point of generation before they are capable of further impact ionizations. The hole travels in the $-x$-direction and becomes capable of impact ionizing with an ionization coefficient β only after traveling a dead space d_h. This process continues until all carriers exit the multiplication region resulting in a net random gain G.

To characterize the statistics of G, the random counts $Z(x)$ and $Y(x)$ were introduced in [10] as the total number of carriers generated as a result of an initial electron or hole, respectively, located at position x in the multiplication region. The random gain is then $G = 0.5(Z(0) + Y(0))$, which can be further reduced to $G = 0.5(Z(0) + 1)$, since $Y(0) = 1$. Clearly, if the mean, $z(x) = \langle Z(x)\rangle$, and the second moment, $z_2(x) = \langle Z^2(x)\rangle$, are determined then the mean gain and excess noise factor can be computed as $\langle G\rangle = 0.5(z(0) + 1)$

Manuscript received December 7, 1998; revised February 24, 1999.

M. M. Hayat is with the Electro-Optics Program and the Department of Electrical and Computer Engineering, University of Dayton, Dayton, OH, 45469-0245 USA.

Z. Chen and M. A. Karim are with the Department of Electrical Engineering, University of Tennessee, Knoxville, TN 37996-2100 USA.

Publisher Item Identifier S 0741-3106(99)05670-0.

and

$$F = \frac{\langle G^2 \rangle}{\langle G \rangle^2} = \frac{1}{4\langle G \rangle^2}(z_2(0) + 4\langle G \rangle - 1)$$

where $z(x)$, $y(x)$, $z_2(x)$, $y_2(x)$ obey the following differential equations (differential forms of the recurrence equations (14), (15), (18), and (19) in [10]). For $0 \le x \le w - d_e$:

$$z'(x) - \alpha[z(x) - 2z(x+d_e) - y(x+d_e)] = 0 \quad (1)$$

$$z_2'(x) - \alpha[z_2(x) - 2z_2(x+d_e) - y_2(x+d_e)] \\ = -2\alpha z(x+d_e)(2y(x+d_e) + z(x+d_e)) \quad (2)$$

and for $d_h \le x \le w$:

$$y'(x) + \beta[y(x) - 2y(x-d_h) - z(x-d_h)] = 0 \quad (3)$$

$$y_2'(x) + \beta[y_2(x) - 2y_2(x-d_h) - z_2(x-d_h)] \\ = 2\beta y(x-d_h)(2z(x-d_h) + y(x-d_h)) \quad (4)$$

with the boundary conditions $z(x) = z_2(x) = 1$, if $w - d_e \le x \le w$, and $y(x) = y_2(x) = 1$, if $0 \le x \le d_h$. The approach we undertake to approximately solve for $z(x)$ and $y(x)$ is based on proposing exponential solutions. The desired exponents are then found by substituting these assumed exponential forms in (1) and (3), and obtaining an algebraic characteristic equation characterizing the exponent that results in self consistency in (1) and (3). (This approach is similar to the standard method used to derive the solution of linear differential equations with constant coefficients where the self consistency of the proposed exponential solution is represented by a polynomial characteristic equation whose roots are the desired exponents.) Specifically, assume a solution of the form $z(x) = c_1 e^{rx}$ and $y(x) = c_2 e^{rx}$, and substitute these forms into (1) and cancel out all the terms that involve x to yield a linear equation in c_1 and c_2 involving the unknown parameter r. We now follow the same procedure using (3) in place of (1), and obtain another equation involving c_1 and c_2. These two equations can be written as

$$\begin{bmatrix} r - \alpha + 2\alpha e^{rd_e} & \alpha e^{rd_e} \\ -\beta e^{-rd_h} & r + \beta - 2\beta e^{-rd_h} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (5)$$

For a nontrivial solution to c_1 and c_2 in (5), we require that the matrix above is singular (its determinant must be zero) which results in the nonlinear characteristic equation characterizing r

$$(r - \alpha + 2\alpha e^{rd_e})(r + \beta - 2\beta e^{-rd_h}) + \alpha\beta e^{r(d_e - d_h)} = 0. \quad (6)$$

For brevity, we consider the case when the electron and hole ionizations are unequal in which case (6) has two roots: 1) $r_1 = 0$ and 2) $r_2 = r \ne 0$. (The exponent r is computed using MATLAB and its built-in function *fzero*.) Now by setting $z(x)$ and $y(x)$ to a linear combination of the two exponentials (corresponding to the two roots) and applying the boundary conditions $z(w - d_e) = y(d_h) = 1$, we can solve for $z(x)$ and $y(x)$ and obtain

$$\langle G \rangle = \frac{\rho + e^{rd_h}}{\rho e^{r(w-d_e)} + e^{rd_h}},$$

$$\text{where } \rho = -\alpha e^{rd_e}/(r - \alpha + 2\alpha e^{rd_e}). \quad (7)$$

To find an approximate solution to (2) and (4), first note that the right-hand side of (2) and (4) is explicitly determined by substituting the previously calculated approximate expressions for $z(x)$ and $y(x)$, and it consists of a constant plus a weighted sum of the terms e^{rx} and e^{2rx}. We can therefore assume a solution (a combination of the homogeneous and particular solutions) to the unknown functions $z_2(x)$ and $y_2(x)$ as the superposition of terms of the form $z_2(x) = p_1 e^{rx} + p_2 e^{2rx} + p_3 x e^{rx} + p_4 x + p_5$ and $y_2(x) = q_1 e^{rx} + q_2 e^{2rx} + q_3 x e^{rx} + q_4 x + q_5$. The exponent r turns out to satisfy the same characteristic equation as in (6). Upon substituting the proposed forms (with known r but unknown coefficients) into (2) and (4), and applying the conditions $z_2(w - d_e) = y_2(d_h) = 1$, we obtain a system of ten linear equations involving the ten unknown coefficients $p_1, \ldots, p_5$, and $q_1, \ldots, q_5$. After some algebra, the unknown coefficients are determined and the final expression for the excess noise factor is found to be

$$F = \frac{1}{4\langle G \rangle^2}(w_1 + w_3 + w_5 + 4\langle G \rangle - 1) \quad (8)$$

where the parameters w_1, w_3, and w_5, are obtained by solving the linear system of equations $\mathbf{q} = \mathbf{P}\mathbf{v}$, where $\mathbf{q} = [1, 1, b_1, b_2, 0, b_3, a_1, a_2, a_3]^T$, $\mathbf{v} = [w_1, \ldots, w_9]^T$. The nonzero entries of $\mathbf{P}$ and $\mathbf{q}$ are (p_{ij} refers to the ith row and the jth column):

$$\begin{aligned}
&p_{11} = p_{26} = 1, \quad p_{12} = w - d_e, \quad p_{13} = e^{r(w-d_e)}, \\
&p_{14} = (w - d_e)e^{r(w-d_e)}, \quad p_{15} = e^{2r(w-d_e)}, \\
&p_{22} = -d_h, \quad p_{27} = e^{rd_h}, \quad p_{28} = d_h e^{rd_h}, \\
&p_{29} = e^{2rd_h}, \quad p_{31} = p_{36} = \alpha, \quad p_{32} = 1 + \alpha d_e, \\
&p_{43} = p_{54} = r - \alpha + 2\alpha e^{rd_e}, \quad p_{44} = 1 + 2\alpha d_e e^{rd_e}, \\
&p_{47} = p_{58} = \alpha e^{rd_e}, \quad p_{48} = \alpha d_e e^{rd_e}, \\
&p_{65} = 2r - \alpha + 2\alpha e^{2rd_e}, \quad p_{69} = \alpha e^{2rd_e}, \\
&p_{71} = p_{76} = -\beta, \quad p_{72} = -(1 + \beta d_h), \\
&p_{83} = -\beta e^{-rd_h}, \quad p_{88} = 1 + 2\beta d_h e^{-rd_h}, \\
&p_{87} = r + \beta - 2\beta e^{-rd_h}, \quad p_{84} = \beta d_h e^{-rd_h}, \\
&p_{95} = -\beta e^{-2rd_h}, \quad p_{99} = 2r + \beta - 2\beta e^{-2rd_h}, \\
&\quad b_1 = 2\alpha\gamma_2^2, \quad b_2 = 4\alpha\rho\gamma_1\gamma_2 e^{rd_e}, \\
&\quad b_3 = -2\alpha\rho(\rho + 2)\gamma_1^2 e^{2rd_e}, \quad a_1 = -2\beta\gamma_2^2, \\
&\quad a_2 = 4\beta\rho\gamma_1\gamma_2 e^{-rd_h}, \quad a_3 = 2\beta(2\rho + 1)\gamma_1^2 e^{-2rd_h}
\end{aligned}$$

where $\gamma_1 = 2/(\rho e^{r(w-d_e)} + e^{rd_h})$, and $\gamma_2 = 1 - \gamma_1 e^{rd_h}$. Equation (8) is the main contribution of this letter. To our knowledge, this is the first time that an analytical expression for the excess noise factor for the dead-space multiplication model is reported.

We now compare our characteristic-equation method (CM) approximations for the mean and the excess noise factor [(7) and (8)] to the exact numerical method (ENM) reported in [10]. We also compare our expression for the mean gain to the perturbation-method (PM) approximation reported in [13]. For simplicity, we assume that the electron and hole dead spaces are equal with a common value d. It is seen from Fig. 1 that both the CM and PM approximations are accurate for the "small" dead space case of $d/w = 0.05$. In the case of $k = 0.1$ and $d/w = 0.1$, the CM approximation outperforms

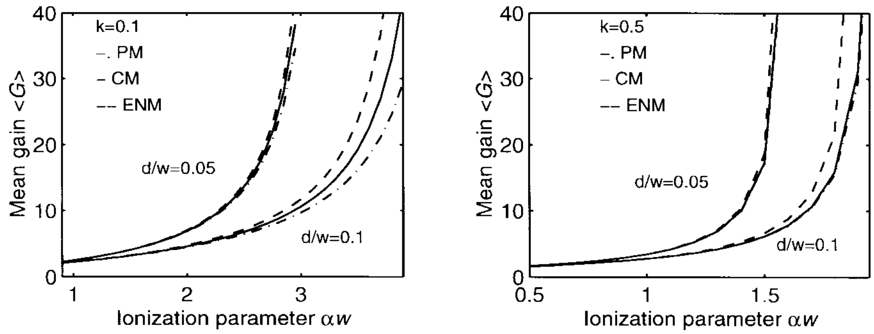

Fig. 1. Dependence of the mean gain $\langle G\rangle$ on the electron ionization parameter αw. Solid lines represent the characteristic-equation method (CM) approximation, dashed-dotted lines represent the perturbation-method (PM) approximation and the dashed lines correspond to the exact numerical method (ENM). Two cases for the hole-electronionization coefficient ratio k and two cases for the relative dead space parameter d/w are considered: $k = 0.1$ and $k = 0.5$; $d/w = 0.05$ and $d/w = 0.1$.

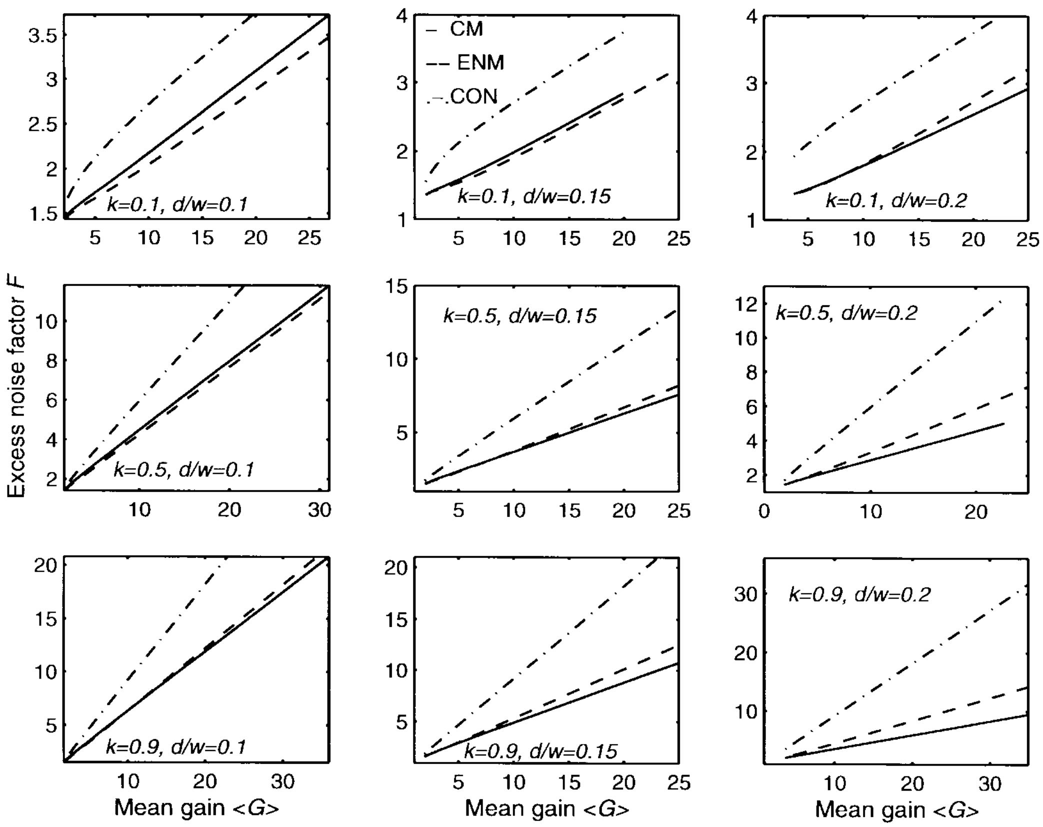

Fig. 2. Excess-noise factor F as a function of the mean gain $\langle G\rangle$. Solid lines represent the characteristic-equation method (CM) approximation and the dashed lines correspond to the exact numerical method (ENM). Three cases for the hole-electron ionization coefficient ratio k and three cases for the relative dead space d/w are considered: $k = 0.1$, $k = 0.5$, and $k = 0.9$; $d/w = 0.1$, $d/w = 0.15$, and $d/w = 0.2$. For comparison, the F versus $\langle G\rangle$ characteristics for the case $d/w = 0$ are plotted according to the conventional theory (CON) and represented by a dashed-dotted line.

the PM approximation. The main contribution of this letter is summarized in Fig. 2. It is seen that the CM approximation of the excess noise factor, as a function of the mean gain, is in good agreement with the exact results obtained using the ENM method. Even in the case when $d/w = 0.2$, the CM approximation performs well in capturing the dead space effect which can be seen by comparing the CM graph to the graph from the conventional theory (CON) reported in [7] for which $d/w = 0$.

In summary, we have developed approximate expressions for the mean gain and the excess noise factor of APD's in a dead-space model which are in good agreement with the exact results obtained using extensive numerical solutions. These approximations provide a valuable tool for designing APD's with thin multiplication regions which are known to be sensitive to the dead space effect.

References

[1] C. Hu, K. A. Anselm, B. G. Streetman, and J. C. Campbell, "Noise characteristics of thin multiplication region avalanche photodiodes," *Appl. Phys. Lett.*, vol. 69, no. 24, pp. 3734–3736, 1996.

[2] K. A. Anselm, P. Yuan, C. Hu, C. Lenox, H. Nie, G. Kinsey, J. C. Campbell, and B. G. Streetman, "Characteristics of GaAs and AlGaAs

homojunction avalanche photodiodes with thin multiplication regions," *Appl. Phys. Lett.*, vol. 71, pp. 3883–3885, 1997.

[3] V. Chandramouli, C. M. Maziar, and J. C. Campbell, "Design considerations for high-performance avalanche photodiode multiplication layers," *IEEE Trans. Electron Devices*, vol. 41, pp. 648–654, May 1994.

[4] K. F. Li, D. S. Ong, J. P. R. David, P. N. Robson, R. C. Tozer, G. J. Reez, and R. Grey, "Low excess noise characteristics in thin avalanche region GaAs diodes," *Electron Lett.*, vol. 34, pp. 125–126, 1998.

[5] D. S. Ong, K. F. Li, G. J. Rees, G. M. Dunn, J. P. R. David, and P. N. Robson, "A Monte Carlo investigation of multiplication noise in thin p^+-i-n^+ GaAs avalanche photodiodes," *IEEE Trans. Electron Devices*, vol. 45, pp. 1804–1809, Aug. 1998.

[6] D. S. Ong, K. F. Li, G. J. Rees, J. P. R. David, and P. N. Robson, "Monte Carlo estimation of multiplication noise in thin p^+-i-n^+ GaAs diodes," *Appl. Phys. Lett.*, vol. 72, pp. 232–234, 1998.

[7] R. J. McIntyre "Multiplication noise in avalanche photodiodes," *IEEE Trans. Electron Devices*, vol. ED-13, pp. 164–168, Jan. 1966.

[8] J. N. Hollenhorst, "A theory of multiplication noise," *IEEE Trans. Electron Devices*, vol. 37, pp. 781–788, Mar. 1990.

[9] Y. Okuto and C. R. Crowell, "Ionization coefficients in semiconductors: A nonlocalized property," *Phys. Rev.*, vol. B10, pp. 4284–4296, 1973.

[10] M. M. Hayat, B. E. A. Saleh, and M. C. Teich, "Effect of dead space on gain and noise of double-carrier multiplication avalanche photodiodes," *IEEE Trans. Electron Devices*, vol. 39, pp. 546–552, 1992.

[11] M. M. Hayat, W. L. Sargeant, and B. E. A. Saleh, "Effect of dead space on gain and noise in Si and GaAs avalanche photodiodes," *IEEE J. Quantum Electron.*, vol. 5, pp. 1360–1365, 1992.

[12] J. S. Marsland, R. C. Woods, and C. A. Brownhill, "Lucky drift estimation of excess noise factor for conventional avalanche photodiodes including dead space effect," *IEEE Trans. Electron Devices*, vol. 39, pp. 1129–1135, 1992.

[13] A. Spinelli and A. L. Lacaita, "Mean gain of avalanche photodiodes in a dead space model," *IEEE Trans. Electron Devices*, vol. 43, pp. 23–31, Jan. 1996.