ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.10(2010) No.3,pp.313-319

Relative Multifractal Analysis in a Probability Space

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(Received 28 May 2010, accepted 23 July 2010)

Abstract: In this paper, we mainly research a multifractal analysis of one probability measure with respect to another. We explore the properties of the multifractal Hausdorff measure and the multifractal packing measure with one to another in a probability space, and calculate the relative results of multifractal spectra function in a probability space.

Keywords: packing measure; Hausdorff measure; relative; probability space;

1 Introduction

The fractal properties have been a basic question in fractal geometry. Many authors [1–10] have studied them. In recent years multifractal theory has been discussed by numerous authors and it is developing rapidly. In 1986, physicist Halsey drew to the conception of multifractal spectrum. Olsen was motivated by the heuristics of Halsey. In 1995 Olsen established a multifractal formalism (see [1]). In 2000, Cole researched relative multifractal analysis which was based on [1](see [2]). Billingsley defined the Haussdorff measure in a probability space (see [3, 4]). Then, Dai and Taylor developed the packing measure and packing dimension in a probability space (see [5]). In fact, multifractal Hausdorff measure is a generalization of the centered Hausdorrff measure and multifractal packing measure is a generalization of the packing measure. Applying the above idea, Li and Dai generalized the Hausdorff measure and packing measure in a probability space (see [6]). So we can establish a general formalism for the multifractal analysis of one probability measure with respect to another in a probability space.

2 Preliminaries

In this paper, we want to yield a generalization multifractal formalism in a probability space.

Let we start by defining Hausdorff ϕ-measure and packing ϕ-measure. We start with a fixed stochastic process $\{X_n, n \in N\}$ on a probability space $\{\Omega, \mathscr{F}, \mu\}$ taking values in a finite or countable state space E. A cylinder set C of rank n is of the form

$$C = \{\omega : X_i(\omega) = k_i,\ i = 1, 2, \cdots n\}$$

with $k_i \in E$. For each $\omega_0 \in \Omega$ there is a unique cylinder set of rank n, denoted by $u_n(\omega_0)$, which contains ω_0. Thus

$$u_n(\omega_0) = \{\omega : X_i(\omega) = X_i(\omega_0),\ i = 1, 2, \cdots n\}.$$

We assume the process is $\mathscr{F}$-measure, that is that $\mathscr{C} \subset \mathscr{F}$, where $\mathscr{C}$ is the class of all cylinder set. We use sets in $\mathscr{C}$ for both covering and packing. In this paper, we will assume that μ is $\mathscr{H} = \sigma(\mathscr{C})$ continuous, that is

$$\lim_{n\to\infty} \mu(u_n(\omega)) = 0\ \ \forall\, \omega \in \Omega.$$

Any function $\phi : [0, \delta] \to [0, 1]$ which is continuous, monotone increasing, with $\phi(0) = 0$, is called a measure function. The Billingsley [3] definition of Hausdorff ϕ-measure follows, for $\delta > 0$, $A \subset \Omega$, define

$$\mathscr{L}^{\phi}_{\mu,\delta}(A) := \inf\Big\{ \sum_i \phi(\mu(C_i)) : A \subset \cup_i C_i, \{C_i\} \subset \mathscr{C}, \mu(C_i) < \delta \Big\},$$

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IJNS.2010.12.15/421

$$\mathscr{L}_{\mu}^{\phi}(A) := \lim_{\delta\to 0} \mathscr{L}_{\mu,\delta}^{\phi}(A).$$

Dai and Taylor [5] defined the packing ϕ-premeasure and ϕ-measure by

$$\widetilde{\mathscr{P}}_{\mu,\delta}^{\phi}(A) := \sup\big\{ \sum_i \phi(\mu(C_i)) : \{C_i\} \text{ are disjoint, } \mu(C_i) < \delta \text{ and } C_i = u_n(\omega) \text{ with } \omega \in A\big\}.$$

$$\widetilde{\mathscr{P}}_{\mu}^{\phi}(A) := \lim_{\delta\to 0} \widetilde{\mathscr{P}}_{\mu,\delta}^{\phi}(A),$$

$$\mathscr{P}_{\mu}^{\phi}(A) := \inf\big\{ \sum_i \widetilde{\mathscr{P}}_{\mu}^{\phi}(A_i) : A \subset \cup_i A_i\big\}.$$

In particular, if $\phi = s^{\alpha}$, we simply write $\mathscr{L}_{\mu}^{\alpha}(\cdot)$, $\mathscr{P}_{\mu}^{\alpha}(\cdot)$ and $\widetilde{\mathscr{P}}_{\mu}^{\alpha}(\cdot)$ for $\mathscr{L}_{\mu}^{\phi}(\cdot)$, $\mathscr{P}_{\mu}^{\phi}(\cdot)$ and $\widetilde{\mathscr{P}}_{\mu}^{\phi}(\cdot)$ respectively.

We now define multifractal generalizations of the Hausdorff ϕ-measure and packing ϕ-measure. For $q \in \mathbb{R}$, define $\phi_q : [0,\infty) \to \overline{\mathbb{R}}_+ = [0,\infty]$ by

$$\phi_q(x) = \begin{cases} \infty & \text{for } x = 0, \\ x^q & \text{for } x > 0, \end{cases} \quad \text{for } \ q < 0,$$

$$\phi_q(x) = 1 \text{ for } q = 0,$$

$$\phi_q(x) = \begin{cases} 0 & \text{for } x = 0, \\ x^q & \text{for } x > 0, \end{cases} \quad \text{for } \ q > 0.$$

Let $\emptyset \neq A \subset \Omega$ and $\delta > 0$, suppose ν is a probability measure on $(\Omega, \mathscr{F})$. For $q \in \mathbb{R}$, write

$$\widetilde{\mathscr{H}}_{\mu,\nu,\delta}^{q,t}(A) = \inf\left\{ \sum_i \phi_q(\nu(C_i))\phi_t(\mu(C_i)) : A \subset \cup_i C_i,\ \mu(C_i) < \delta \ \text{ and } C_i = u_n(\omega) \ \text{ with } \ \omega \in A\right\}.$$

$$\widetilde{\mathscr{H}}_{\mu,\nu}^{q,t}(A) = \lim_{\delta\to 0} \widetilde{\mathscr{H}}_{\mu,\nu,\delta}^{q,t}(A) = \sup_{\delta>0} \widetilde{\mathscr{H}}_{\mu,\nu,\delta}^{q,t}(A).$$

It is easy to check that if $A \subset B$, then a centered δ-covering of B is not necessarily a centered δ-covering of A, thus $\widetilde{\mathscr{H}}_{\mu,\nu}^{q,t}$ is not necessarily monotone. So, we put

$$\mathscr{H}_{\mu,\nu}^{q,t}(A) := \sup_{A_i \subset A} \widetilde{\mathscr{H}}_{\mu,\nu}^{q,t}(A_i)$$

Clearly, $\mathscr{H}_{\mu,\nu}^{q,t}$ is of monotony, and we can prove that $\mathscr{H}_{\mu,\nu}^{q,t}$ is of subadditivity, thus $\mathscr{H}_{\mu,\nu}^{q,t}$ is an outer measure, we call the set function $\mathscr{H}_{\mu,\nu}^{q,t}(A)$ multifractal Hausdorff measure in a probability space. We also make the dual definitions. Write

$$\begin{aligned} \widetilde{\mathscr{P}}_{\mu,\nu,\delta}^{q,t}(A) \quad &:= \sup\left\{ \sum_i \phi_q(\nu(C_i))\phi_t(\mu(C_i)) : C_i \cap C_j = \emptyset, i \neq j, \right. \\ &\left. \qquad \mu(C_i) < \delta \text{ and } C_i = u_n(\omega) \text{ with } \omega \in A\right\}, \end{aligned}$$

$$\widetilde{\mathscr{P}}_{\mu,\nu,}^{q,t}(A) := \lim_{\delta\to 0} \widetilde{\mathscr{P}}_{\mu,\nu,\delta}^{q,t}(A) = \inf_{\delta\to 0} \widetilde{\mathscr{P}}_{\mu,\nu}^{q,t}(A)$$

It is imperative to point out that the subadditivity is the false for the pre-measure $\widetilde{\mathscr{P}}_{\mu,\nu,}^{q,t}$, and we can apply the method of Munroe(see [7]Theorem 11.3) to the pre-measure $\widetilde{\mathscr{P}}_{\mu,\nu}^{q,t}(\cdot)$ to obtain an outer measure

$$\mathscr{P}_{\mu,\nu}^{q,t}(A) := \inf\big\{ \sum_i \widetilde{\mathscr{P}}_{\mu,\nu}^{q,t}(A_i) : A \subset \cup_i A_i\big\},$$

we call the set function $\widetilde{\mathscr{P}}_{\mu,\nu}^{q,t}(\cdot)$ multifractal packing measure in a probability space.

Definition 1 *Let*

$$\dim_{\mu,\nu}^{q}(A) := \sup\{t : \mathscr{H}_{\mu,\nu}^{q,t}(A) = +\infty\} = \inf\{t : \mathscr{H}_{\mu,\nu}^{q,t}(A) = 0\},$$

$$\textit{Dim}_{\mu,\nu}^{q}(A) := \sup\{t : \mathscr{P}_{\mu,\nu}^{q,t}(A) = +\infty\} = \inf\{t : \mathscr{P}_{\mu,\nu}^{q,t}(A) = 0\},$$

$$\Delta_{\mu,\nu}^{q}(A) := \sup\{t : \widetilde{\mathscr{P}}_{\mu,\nu}^{q,t}(A) = +\infty\} = \inf\{t : \widetilde{\mathscr{P}}_{\mu,\nu}^{q,t}(A) = 0\},$$

which are, respectively, called multifractal Hausdorff dimension, packing dimension, pre-packing dimension with respect to μ in a probability space.

3 The relative multifractal spectrum in a probability space

In order to investigate the relative multifractal spectrum, we will define the local pointwise dimension.

Let $\omega \in \Omega$, define the upper, lower local dimension of θ with respect to ν at a point ω, respectively, by

$$\overline{\gamma}_{\theta,\nu}(\omega) = \limsup_{n\to\infty} \frac{\log \theta(u_n(\omega))}{\log \nu(u_n(\omega))},$$

$$\underline{\gamma}_{\theta,\nu}(\omega) = \liminf_{n\to\infty} \frac{\log \theta(u_n(\omega))}{\log \nu(u_n(\omega))}.$$

If $\overline{\gamma}_{\theta,\nu}(\omega) = \underline{\gamma}_{\theta,\nu}(\omega)$, then the common value, known as the local dimension of θ with respect to ν at ω, is denoted by $\gamma_{\theta,\nu}(\omega)$.

Then, if $\alpha \geq 0$, let us introduce the fractal sets

$$\overline{\Pi}^{\alpha} := \{\omega \in \mathrm{supp}\mu \cap \mathrm{supp}\nu : \overline{\gamma}_{\mu,\nu}(\omega) \leq \alpha\},$$

$$\underline{\Pi}_{\alpha} := \{\omega \in \mathrm{supp}\mu \cap \mathrm{supp}\nu : \underline{\gamma}_{\mu,\nu}(\omega) \geq \alpha\},$$

$$\Pi(\alpha) = \underline{\Pi}_{\alpha} \cap \overline{\Pi}^{\alpha} = \{x \in \mathrm{supp}\mu \cap \mathrm{supp}\nu \mid \gamma_{\mu,\nu}(x) = \alpha\}.$$

Also, let

$$\overline{\gamma}_{\nu,\mu}(\omega) = \limsup_{n\to\infty} \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))},$$

$$\underline{\gamma}_{\nu,\mu}(\omega) = \liminf_{n\to\infty} \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))}.$$

Given θ, μ, ν are probability measures, for α, $\beta \geq 0$, set

$$\Pi(\overline{\alpha}, \overline{\beta}) := \{\omega \in \mathrm{supp}\mu \cap \mathrm{supp}\nu | \ \overline{\gamma}_{\theta,\nu}(\omega) \leq \alpha \text{ and } \overline{\gamma}_{\nu,\mu}(\omega) \leq \beta\},$$

$$\Pi(\underline{\alpha}, \overline{\beta}) := \{\omega \in \mathrm{supp}\mu \cap \mathrm{supp}\nu | \ \alpha \leq \underline{\gamma}_{\theta,\nu}(\omega) \text{ and } \overline{\gamma}_{\nu,\mu}(\omega) \leq \beta\},$$

$$\Pi(\overline{\alpha}, \underline{\beta}) := \{\omega \in \mathrm{supp}\mu \cap \mathrm{supp}\nu | \ \overline{\gamma}_{\theta,\nu}(\omega) \leq \alpha \text{ and } \beta \leq \underline{\gamma}_{\nu,\mu}(\omega)\},$$

$$\Pi(\underline{\alpha}, \underline{\beta}) := \{\omega \in \mathrm{supp}\mu \cap \mathrm{supp}\nu | \ \alpha \leq \underline{\gamma}_{\theta,\nu}(\omega) \text{ and } \beta \leq \underline{\gamma}_{\nu,\mu}(\omega)\}.$$

Also, let

$$\begin{aligned}\Pi(\alpha, \beta) &:= \Pi(\overline{\alpha}, \overline{\beta}) \cap \Pi(\underline{\alpha}, \overline{\beta}) \cap \Pi(\overline{\alpha}, \underline{\beta}) \cap \Pi(\underline{\alpha}, \underline{\beta}) \\ &= \{\omega \in \mathrm{supp}\mu \cap \mathrm{supp}\nu | \ \gamma_{\theta,\nu}(\omega) = \alpha, \gamma_{\nu,\mu}(\omega) = \beta\}.\end{aligned}$$

Fix α, $\beta \geq 0$, $q, t \in \mathbb{R}$ and δ_1, $\delta_2 > 0$ such that $\delta_1 \leq \alpha q + t$ and $\delta_2 \leq \beta(\alpha q + t - \delta_1)$. Then the following inequalities hold.

Theorem 3.1 *Let $\{\Omega, \mathscr{F}\}$ be a probability space and θ, ν, μ are probability measures on $\{\Omega, \mathscr{F}\}$. Then we have the following:*

(i) $\mathscr{L}_{\mu}^{\beta(\alpha q+t+\delta_1)+\delta_2}(\Pi(\overline{\alpha}, \overline{\beta})) \leq \mathscr{H}_{\theta,\nu}^{q,t}(\Pi(\overline{\alpha}, \overline{\beta}))$ for $q \geq 0$.

(ii) $\mathscr{L}_{\mu}^{\beta(\alpha q+t+\delta_1)+\delta_2}(\Pi(\underline{\alpha}, \overline{\beta})) \leq \mathscr{H}_{\theta,\nu}^{q,t}(\Pi(\underline{\alpha}, \overline{\beta}))$ for $q \leq 0$.

Proof. The statements are true for $q = 0$.

(i) For $m \in \mathbb{N}$, write

$$\begin{aligned}T_m := \Big\{\omega \in \Pi(\overline{\alpha}, \overline{\beta}) | \ & \frac{\log \theta(u_n(\omega))}{\log \nu(u_n(\omega))} \leq \alpha + \frac{\delta_1}{q} \\ & \text{and } \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))} \leq \beta + \frac{\delta_2}{\alpha q + t + \delta_1}, \text{ for } 0 < \mu(u_n(\omega)) < \frac{1}{m}\Big\}.\end{aligned}$$

Now given $m \in \mathbb{N}$, and $0 < \rho < \frac{1}{m}$ let $\{C_i = u_n(\omega), \omega \in T_m\}$ be a centered ρ-covering of T_m. Then clearly

$$\frac{\log \theta(C_i)}{\log \nu(C_i)} \leq \alpha + \frac{\delta_1}{q},$$

$$\theta(C_i) \geq \nu(C_i)^{\alpha + \frac{\delta_1}{q}},$$
$$\theta(C_i)^q \geq \nu(C_i)^{\alpha q + \delta_1},$$
$$\theta(C_i)^q \nu(C_i)^t \geq \nu(C_i)^{\alpha q + t + \delta_1}.$$

Also, we have,

$$\frac{\log \nu(C_i)}{\log \mu(C_i)} \leq \beta + \frac{\delta_2}{\alpha q + t + \delta_1}$$

$$\nu(C_i) \geq \mu(C_i)^{\beta + \frac{\delta_2}{\alpha q + t + \delta_1}},$$
$$\nu(C_i)^{\alpha q + t + \delta_1} \geq \mu(C_i)^{\beta(\alpha q + t + \delta_1) + \delta_2}.$$

Putting these together we see that

$$\theta(C_i)^q \nu(C_i)^t \geq \mu(C_i)^{\beta(\alpha q + t + \delta_1) + \delta_2}.$$

Hence

$$\sum_i \theta(C_i)^q \nu(C_i)^t \geq \sum_i \mu(C_i)^{\beta(\alpha q + t + \delta_1) + \delta_2} \geq \mathscr{L}_{\mu,\rho}^{\beta(\alpha q + t + \delta_1) + \delta_2}(T_m).$$

From this we can deduce that for $0 < \rho < \frac{1}{m}$

$$\mathscr{L}_{\mu,\rho}^{\beta(\alpha q + t + \delta_1) + \delta_2}(T_m) \leq \widetilde{\mathscr{H}}_{\theta,\nu,\rho}^{q,t}(T_m).$$

Thus letting $\rho \to 0$ now yields

$$\mathscr{L}_{\mu}^{\beta(\alpha q + t + \delta_1) + \delta_2}(T_m) \leq \widetilde{\mathscr{H}}_{\theta,\nu}^{q,t}(T_m) \leq \mathscr{H}_{\theta,\nu}^{q,t}(T_m) \text{ for } m \in \mathbb{N}.$$

Clearly $T_m \uparrow \Pi(\overline{\alpha}, \overline{\beta})$ from the definition of T_m, whence

$$\mathscr{L}_{\mu}^{\beta(\alpha q + t + \delta_1) + \delta_2}(\Pi(\overline{\alpha}, \overline{\beta})) = \sup_m \mathscr{L}_{\mu}^{\beta(\alpha q + t + \delta_1) + \delta_2}(T_m) \leq \sup_m \mathscr{H}_{\theta,\nu}^{q,t}(T_m) \leq \mathscr{H}_{\theta,\nu}^{q,t}(\Pi(\overline{\alpha}, \overline{\beta})).$$

(ii) For $q < 0$, for any $m \in \mathbb{N}$, write

$$T_m := \Big\{\omega \in \Pi(\underline{\alpha}, \overline{\beta}) | \ \frac{\log \theta(u_n(\omega))}{\log \nu(u_n(\omega))} \geq \alpha + \frac{\delta_1}{q} \\ \text{and } \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))} \leq \beta + \frac{\delta_2}{\alpha q + t + \delta_1}, \text{ for } 0 < \mu(u_n(\omega)) < \frac{1}{m}\Big\}.$$

and proceed as in case (i). ■

Theorem 3.2 *Let $\{\Omega, \mathscr{F}\}$ be a probability space and θ, ν, μ are probability measures on $\{\Omega, \mathscr{F}\}$. Then we have the following:*

(i) $\mathscr{P}_{\mu}^{\beta(\alpha q + t + \delta_1) + \delta_2}(\Pi(\overline{\alpha}, \overline{\beta})) \leq \mathscr{P}_{\theta,\nu}^{q,t}(\Pi(\overline{\alpha}, \overline{\beta}))$ *for* $q \geq 0$.

(ii) $\mathscr{P}_{\mu}^{\beta(\alpha q + t + \delta_1) + \delta_2}(\Pi(\underline{\alpha}, \overline{\beta})) \leq \mathscr{P}_{\theta,\nu}^{q,t}(\Pi(\underline{\alpha}, \overline{\beta}))$ *for* $q \leq 0$.

Proof. The statements are true for $q = 0$.

(i) For $m \in \mathbb{N}$, write

$$T_m := \Big\{\omega \in \Pi(\overline{\alpha}, \overline{\beta}) | \ \frac{\log \theta(u_n(\omega))}{\log \nu(u_n(\omega))} \leq \alpha + \frac{\delta_1}{q} \\ \text{and } \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))} \leq \beta + \frac{\delta_2}{\alpha q + t + \delta_1}, \text{ for } 0 < \mu(u_n(\omega)) < \frac{1}{m}\Big\}.$$

Now given $m \in \mathbb{N}$, $B \subset T_m$ and $0 < \eta < \frac{1}{m}$ let $\{C_i = u_n(\omega), \omega \in B\}$ be a centered η-packing of B. Then clearly

$$\frac{\log \theta(C_i)}{\log \nu(C_i)} \leq \alpha + \frac{\delta_1}{q},$$

hence

$$\theta(C_i)^q \nu(C_i)^t \geq \nu(C_i)^{\alpha q + t + \delta_1}.$$

Also, we have,

$$\frac{\log \nu(C_i)}{\log \mu(C_i)} \leq \beta + \frac{\delta_2}{\alpha q + t + \delta_1}$$

$$\nu(C_i)^{\alpha q + t + \delta_1} \geq \mu(C_i)^{\beta(\alpha q + t + \delta_1) + \delta_2}.$$

Putting these together we see that

$$\theta(C_i)^q \nu(C_i)^t \geq \mu(C_i)^{\beta(\alpha q + t + \delta_1) + \delta_2}.$$

Hence

$$\sum_i \theta(C_i)^q \nu(C_i)^t \geq \sum_i \mu(C_i)^{\beta(\alpha q + t + \delta_1) + \delta_2}.$$

From this we can deduce that for $0 < \eta < \frac{1}{m}$

$$\widetilde{\mathscr{P}}^{\beta(\alpha q + t + \delta_1) + \delta_2}_{\mu,\eta}(B) \leq \widetilde{\mathscr{P}}^{q,t}_{\theta,\nu,\eta}(B).$$

Thus letting $\eta \to 0$ gives that for all $B \subset T_m$

$$\widetilde{\mathscr{P}}^{\beta(\alpha q + t + \delta_1) + \delta_2}_{\mu}(B) \leq \widetilde{\mathscr{P}}^{q,t}_{\theta,\nu}(B).$$

Now let $T_m \subset \cup_i B_i$, then the above inequality implies that

$$\begin{aligned}
\mathscr{P}^{\beta(\alpha q + t + \delta_1) + \delta_2}_{\mu}(T_m) &= \mathscr{P}^{\beta(\alpha q + t + \delta_1) + \delta_2}_{\mu}(\cup_i (T_m \cap B_i)) \\
&\leq \sum_i \mathscr{P}^{\beta(\alpha q + t + \delta_1) + \delta_2}_{\mu}(T_m \cap B_i) \\
&\leq \sum_i \widetilde{\mathscr{P}}^{q,t}_{\theta,\nu}(T_m \cap B_i) \\
&\leq \sum_i \widetilde{\mathscr{P}}^{q,t}_{\theta,\nu}(B_i),
\end{aligned}$$

Hence

$$\mathscr{P}^{\beta(\alpha q + t + \delta_1) + \delta_2}_{\mu}(T_m) \leq \mathscr{P}^{q,t}_{\theta,\nu}(T_m) \;\; \forall m \in \mathbb{N}.$$

Since $T_m \uparrow \Pi(\overline{\alpha}, \overline{\beta})$, $\Pi(\overline{\alpha}, \overline{\beta}) = \cup_m T_m$. From the monotonicity of the measure, we have

$$\mathscr{P}^{\beta(\alpha q + t + \delta_1) + \delta_2}_{\mu}(\Pi(\overline{\alpha}, \overline{\beta})) = \sup_m \mathscr{P}^{\beta(\alpha q + t + \delta_1) + \delta_2}_{\mu}(T_m) \leq \sup_m \mathscr{P}^{q,t}_{\theta,\nu}(T_m) \leq \mathscr{P}^{q,t}_{\theta,\nu}(\Pi(\overline{\alpha}, \overline{\beta})).$$

(ii) For $q < 0$, for any $m \in \mathbb{N}$, write

$$\begin{aligned}
T_m := \Big\{\omega \in \Pi(\underline{\alpha}, \overline{\beta}) |\; &\frac{\log \theta(u_n(\omega))}{\log \nu(u_n(\omega))} \geq \alpha + \frac{\delta_1}{q} \\
&\text{and } \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))} \leq \beta + \frac{\delta_2}{\alpha q + t + \delta_1}, \text{ for } 0 < \mu(u_n(\omega)) < \frac{1}{m}\Big\}.
\end{aligned}$$

and proceed as in case (i). ■

Theorem 3.3 *(i) If $A \subseteq K(\overline{\alpha}, \underline{\beta})$ is Borel then, for $q \leq 0$,*

$$\mathscr{H}^{q,t}_{\theta,\nu}(A) \leq \mathscr{L}^{\beta(\alpha q + t - \delta_1) - \delta_2}_{\mu}(A).$$

(ii) If $A \subseteq K(\underline{\alpha}, \underline{\beta})$ is Borel then, for $0 \leq q$,

$$\mathscr{H}^{q,t}_{\theta,\nu}(A) \leq \mathscr{L}^{\beta(\alpha q + t - \delta_1) - \delta_2}_{\mu}(A).$$

Proof. An exhaustive proof of this theorem would require considerable repetition. Thus we only prove (ii). The statement is well-known for $q = 0$.

(1) For $m \in \mathbb{N}$, let us set

$$T_m := \{x \in A | \frac{\log \theta(u_n(\omega))}{\log \nu(u_n(\omega))} \geq \alpha - \frac{\delta_1}{q}$$
$$\text{and } \frac{\log \nu(u_n(\omega))}{\log \mu(u_n(\omega))} \geq \beta - \frac{\delta_2}{\alpha q + t - \delta_2}, \text{ for } 0 < \mu(u_n(\omega)) < \frac{1}{m}\}$$

Fix $m \in \mathbb{N}$, $E \subseteq T_m$ and $0 < \eta < \frac{1}{m}$ let $\{C_i : C_i = u_n(\omega), \omega \in B\}$ be a centered η-covering of E. Then we have

$$\frac{\log \theta(C_i)}{\log \nu(C_i)} \geq \alpha - \frac{\delta_1}{q},$$
$$\theta(C_i) \leq \nu(C_i)^{\alpha - \frac{\delta_1}{q}},$$
$$\theta(C_i)^q \nu(C_i)^t \leq \nu(C_i)^{\alpha q + t - \delta_1}.$$

Also, we have,

$$\frac{\log \nu(C_i)}{\log \mu(C_i)} \geq \beta - \frac{\delta_2}{\alpha q + t - \delta_1}$$
$$\nu(C_i)^{\alpha q + t - \delta_1} \leq \mu(C_i)^{\beta(\alpha q + t - \delta_1) - \delta_2},$$

Putting these together we see that

$$\theta(C_i)^q \nu(C_i)^t \leq \mu(C_i)^{\beta(\alpha q + t - \delta_1) - \delta_2}.$$

Hence

$$\sum_i \theta(C_i)^q \nu(C_i)^t \leq \sum_i \mu(C_i)^{\beta(\alpha q + t - \delta_1) - \delta_2}$$
$$\leq \widetilde{\mathscr{P}}_{\mu,\eta}^{\beta(\alpha q + t - \delta_1) - \delta_2}(E).$$

From this we can deduce that for $\eta < \frac{1}{m}$

$$\widetilde{\mathscr{P}}_{\theta,\nu,\eta}^{q,t}(E) \leq \widetilde{\mathscr{P}}_{\mu.\eta}^{\beta(\alpha q + t - \delta_1) - \delta_2}(E).$$

Thus letting $\eta \to 0$ gives that for all $E \subseteq T_m$

$$\widetilde{\mathscr{P}}_{\theta,\nu}^{q,t}(E) \leq \widetilde{\mathscr{P}}_{\mu}^{\beta(\alpha q + t - \delta_1) - \delta_2}(E).$$

Finally, let $(E_i)_{i \in N}$ be a covering of T_m.Then we have

$$\mathscr{P}_{\theta,\nu}^{q,t}(T_m) \leq \mathscr{P}_{\mu,\nu}^{q,t}(\cup_i (T_m \cap (E_i)))$$
$$\leq \sum_i \mathscr{P}_{\mu,\nu}^{q,t}(T_m \cap E_i)$$
$$\leq \sum_i \widetilde{\mathscr{P}}_{\mu,\nu}^{q,t}(T_m \cap E_i)$$
$$\leq \sum_i \widetilde{\mathscr{P}}_{\mu}^{\beta(\alpha q + t - \delta_1) - \delta_2}(T_m \cap E_i)$$
$$\leq \sum_i \widetilde{\mathscr{P}}_{\mu}^{\beta(\alpha q + t - \delta_1) - \delta_2}(E_i).$$

Hence

$$\mathscr{P}_{\theta,\nu}^{q,t}(T_m) \leq \mathscr{P}_{\mu}^{\beta(\alpha q + t - \delta_1) - \delta_2}(T_m)$$

and the result follows since $A = \cup_m T_m$. ■

Theorem 3.4 *(i) If $A \subseteq K(\overline{\alpha}, \underline{\beta})$ is Borel then, for $q \leq 0$,*

$$\mathscr{P}_{\theta,\nu}^{q,t}(A) \leq \mathscr{P}_{\mu}^{\beta(\alpha q + t - \delta_1) - \delta_2}(A).$$

(ii) If $A \subseteq K(\underline{\alpha}, \underline{\beta})$ is Borel then, for $0 \leq q$,

$$\mathscr{P}_{\theta,\nu}^{q,t}(A) \leq \mathscr{P}_{\mu}^{\beta(\alpha q + t - \delta_1) - \delta_2}(A).$$

The proof is the same as Theorem 3.3.

4 Acknowledgments

Research is supported by the Education Foundation of Jiangsu Province (08KJB110003).

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