

# The cosmological principle and honeycombs

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## Abstract

We present the possibility that the gravitational growth of primordial density fluctuations leads to what can be considered a weak version of the cosmological principle. The large-scale mass distribution associated with this principle must have the geometrical structure known as a regular honeycomb. We give the most important parameters that characterize the honeycombs associated with the closed, open, and flat Friedmann–Lemaître–Robertson–Walker models. These parameters can be used to determine by means of observations which is the appropriate honeycomb. For each of these honeycombs, and for a nearly flat universe, we have calculated the probability that a randomly placed observer could detect the honeycomb as a function of the density parameters  $\Omega_0$  and  $\Omega_{\Lambda 0}$ .

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## 1. Introduction

In recent works (see [1–5]), it has been speculated that the cosmological large-scale matter distribution may form repetitive structures analogous to the crystalline ones formed with polyhedra. The geometrical structures of largest symmetry into which a homogeneous space can be decomposed are known as regular honeycombs. A regular honeycomb is a decomposition of the space into congruent regular polyhedra (see [6, 7]). In section 2 we give a new version of the cosmological principle that we call the weak cosmological principle. This version is the most natural way to extend the cosmological principle to a universe with inhomogeneities. The geometrical structures that fit it are precisely the regular honeycombs. Because closed, open, and flat Friedmann–Lemaître–Robertson–Walker (FLRW) models correspond to the three possible homogeneous spaces: elliptic, hyperbolic, and Euclidean respectively, we have given the distances, angles and other characteristic parameters for the regular honeycombs of these spaces (sections 3 and 4). The

calculation of the distances and angles appears in the appendix. Observe that among these three spaces only the Euclidean one does not have a proper scale. The other two have a characteristic scale, namely, the curvature radius (see [8]). Thus, in these spaces there are no homothetic polyhedra and the honeycombs are rigid in the sense that the size of the basic cell cannot be arbitrary as is the case for the Euclidean honeycombs. In particular, in these spaces there cannot be fractal structures because fractals do not have any characteristic scale. In section 5 we present a possible interpretation of this geometric scenario in terms of the cosmological dynamics, by interpreting the spherical and hyperbolic honeycombs as the pattern of large-scale mass distribution. Such crystalline structures would be the result of the evolution of a homogeneous one. If the basic cell of the honeycomb lies in the interior of the particle horizon, then the appropriate honeycomb could be determined by observations, providing a check on the parameter values that we have calculated in sections 3 and 4. In section 6 we have detailed some observational prospects for detection of the honeycomb structure. The case in which the size of the basic cell is much smaller than the particle horizon is commented on in subsection 6.1. In subsection 6.2 we have studied the case where the particle horizon is of the order of the basic cell's size, and we have calculated, for each regular honeycomb, the probability that a randomly placed observer could detect the honeycomb as a function of the density parameters  $\Omega_0$  and  $\Omega_{\Lambda 0}$ .

## 2. A weak version of the cosmological principle

The assumption of large-scale homogeneity, together with that of large-scale isotropy, is called the cosmological principle (CP) (see, for example, [11]). This principle applies for continuous mass distribution. For the discrete case, the symmetry of the regular honeycombs is the most natural definition of discrete homogeneity and isotropy. A regular (or homogeneous) honeycomb is a decomposition of the space into congruent regular polyhedra, which are called the cells of the honeycomb. Any motion that takes a cell into another also takes the whole honeycomb into itself, i.e., belongs to the group of symmetries of the honeycomb. The homogeneity corresponds to the decomposition of the space into regular polyhedra (see [7]), and the isotropy corresponds to the symmetry of regular polyhedra. When all the matter is distributed homogeneously at the vertices of a honeycomb, we say that it obeys the discrete cosmological principle (DCP). We can consider another version of the CP, in which the matter is distributed in a continuous way, in the pattern of a honeycomb, with a hierarchical distribution of matter densities, increasing through the sequence: interior, faces, edges, and vertices of the basic cells. In general, if a distribution of matter has the symmetry of a honeycomb, we say that it obeys the weak cosmological principle (WCP). It includes the CP when the matter is distributed homogeneously and isotropically in the basic cell, and also includes the DCP considered above as a limit case. Another weak version of the cosmological principle has been considered in [9, 10]. In that version the universe is locally homogeneous and isotropic but not necessarily globally homogeneous and isotropic.

In a space of dimension 3 with constant curvature, the honeycombs are classified by means of three integer numbers  $\{p, q, r\}$  called the Schläfli symbols, which completely characterize the honeycomb (see [7]). Specifically,  $\{p, q\}$  characterizes the polyhedron which is the basic cell of the honeycomb,  $p$  is the number of vertices (or edges) of each of the regular polygons that constitute the faces of a cell,  $q$  is the number of faces (or edges) having a common vertex in each cell, and  $r$  is the number of cells having a common edge. Therefore the dihedral angle of each cell,  $\alpha$ , equals  $2\pi/r$ . Note that because the dihedral angle must be a divisor of  $2\pi$ , not all the regular polyhedra can be the cells of a regular honeycomb. It is easy to see that the

**Table 1.** Three-dimensional spherical honeycombs.

Name	Schläfli symbol	$N_0$	$N_1$	$N_2$	$N_3$	Basic cell
5-cell	{3, 3, 3}	5	10	10	5	Tetrahedron
8-cell	{4, 3, 3}	16	32	24	8	Cube
16-cell	{3, 3, 4}	8	24	32	16	Tetrahedron
24-cell	{3, 4, 3}	24	96	96	24	Octahedron
120-cell	{5, 3, 3}	600	1200	720	120	Dodecahedron
600-cell	{3, 3, 5}	120	720	1200	600	Tetrahedron

numbers of vertices,  $V$ , edges,  $E$ , and faces,  $F$ , in a polyhedron with Schläfli symbols  $\{p, q\}$  can be given in terms of  $p, q$  by

$$V = \frac{2}{1 - q(\frac{1}{2} - \frac{1}{p})}, \quad E = \frac{qV}{2}, \quad \text{and} \quad F = \frac{qV}{p}. \quad (1)$$

### 3. Honeycombs in the closed FLRW models

In this case the space-like sections of the universe are three-dimensional spheres,  $S^3$ . Honeycombs in  $S^3$  are in one-to-one correspondence with regular polyhedra in  $R^4$ . The correspondence can be described as follows. The convex hull of a set  $M$  is the minimal convex set containing this set; it is the intersection of all convex sets containing  $M$ . Then the convex hull in  $R^4$  of the set of vertices of a honeycomb in  $S^3$  is a regular polyhedron inscribed in  $S^3$ , and conversely if  $P$  is a regular polyhedron inscribed in  $S^3$ , then the central projection of its faces onto  $S^3$  forms a honeycomb in  $S^3$ . From the six regular polyhedra of  $R^4$  (see [6, 7]) we get the following six regular honeycombs in  $S^3$ :

The regular simplex of  $R^4$  with Schläfli symbols {3, 3, 3} gives the honeycomb 5-cell of  $S^3$ , which is composed of five spherical tetrahedra.

The regular cube of  $R^4$  ({4, 3, 3}) gives the 8-cell of  $S^3$ , which is composed of eight spherical cubes.

The regular cocube of  $R^4$  ({3, 3, 4}) gives the 16-cell of  $S^3$ , which is composed of eight spherical tetrahedra.

The regular 24-hedron ({3, 4, 3}) gives the 24-cell of  $S^3$ , which is composed of 24 spherical octahedra.

The 120-cell honeycomb ({5, 3, 3}) is composed of 120 spherical dodecahedra.

Finally, the 600-cell honeycomb ({3, 3, 5}) is composed of 600 spherical tetrahedra.

Reversing the order of the Schläfli symbols yields the so-called dual honeycombs. The vertices of the honeycomb  $\mathcal{P}^*$ , dual to the honeycomb  $\mathcal{P}$ , should be taken as the centres of the cells of  $\mathcal{P}$ . The symmetry groups of  $\mathcal{P}$  and  $\mathcal{P}^*$  coincide. The 8-cell and 16-cell honeycombs are dual to one another, and the same holds for the 120-cell and 600-cell honeycombs. For symmetric Schläfli symbols, dual honeycombs are congruent. This is the case of the 5-cell and the 24-cell. In our interpretation of the honeycombs as the patterns of the large-scale matter distribution, if the higher density is at the vertices of a honeycomb then the lower density is at the vertices of the corresponding dual honeycomb.

Table 1 gives the following characteristics of these honeycombs: the Schläfli symbols  $\{p, q, r\}$ ; the numbers of vertices,  $N_0$ , edges,  $N_1$ , faces,  $N_2$ , and polyhedra,  $N_3$ . Note that  $N_0 - N_1 + N_2 - N_3$  is the Euler characteristic of  $S^3$ , so it is zero. In table 2 we also give the

**Table 2.** Characteristic parameters associated with the three-dimensional spherical honeycombs ( $R(t) = 1$ ).

Name	$d$	$r_c$	$d_E$	$r_i$	$d_F$	Vol	$\rho$
5-cell	1.8235	1.3181	1.1503	0.9117	1.1503	3.9478	0.2533
8-cell	1.0472	1.0472	0.9553	0.7854	0.7854	2.4674	0.8105
16-cell	1.5708	1.0472	0.7854	0.5236	0.9553	1.2337	0.4023
24-cell	1.0472	0.7854	0.6155	0.5236	0.6155	0.8224	1.2158
120-cell	0.2709	0.3881	0.3648	0.3141	0.2318	0.1644	30.3964
600-cell	0.6283	0.3881	0.2318	0.1354	0.3649	0.0329	6.0793

**Table 3.** Other characteristic parameters associated with the three-dimensional spherical honeycombs.

Name	Dihedral angle	Interior angle	$C_V$	$E_V$
5-cell	120°	109.47°	4	4
8-cell	120°	109.47°	4	4
16-cell	90°	90°	8	6
24-cell	120°	70.53°	6	8
120-cell	120°	109.47°	4	4
600-cell	72°	63.43°	20	12

distance between adjacent vertices (or edge length),  $d$ ; the distance from the centre  $C$  of a cell to a vertex  $V$  (or circum-radius),  $r_c$ ; that to an edge  $E$ ,  $d_E$ ; that to a face  $F$  (or in-radius),  $r_i$ ; and the distance from the centre of a face to a vertex of that face,  $d_F$ . The calculation of these parameters is shown in the appendix.

The distances in the RW spherical space for any cosmic time  $t$  are the above multiplied by the expansion function of the universe,  $R(t)$ . To obtain the corresponding recession velocities we have to multiply the above distances by the Hubble parameter,  $H(t)$ .

We can also obtain the volume of a cell, Vol, as the quotient of the volume of  $S^3$ ,  $2\pi^2 R(t)^3$ , and the number  $N_3$  of cells of the honeycomb. The density of vertices  $\rho$  is the quotient of  $N_0$  and the volume of  $S^3$ . In table 2 we list the values of the volume Vol and  $\rho$  for the six honeycombs considered. Other interesting parameters of the honeycombs are: the number of edges that share a vertex,  $E_V$ , which is given by  $E_V = 2N_1/N_0$ , and the number,  $C_V$ , of cells that share a vertex, which is given by  $C_V = rE_V/q$ . This number corresponds also to the number of vertices of the basic cell of the dual honeycomb, and thus is given by  $C_V = 1/(1 - q(\frac{1}{2} - \frac{1}{r}))$ . In table 3 we list the values of these parameters, as well as the dihedral angles, and the interior angles  $\phi$  of the polygons constituting the faces of each cell.

#### 4. Honeycombs in the open and flat FLRW models

The space-like sections of the universe, in the open FLRW model, are three-dimensional spaces of constant negative curvature, and these spaces are isomorphic to  $H^3$ , the hyperbolic space (or Lobachevskij space) of dimension 3. If we restrict ourselves to honeycombs with bounded cells it follows that there are only four regular honeycombs in  $H^3$  [7] (see the appendix). Their Schläfli symbols and dihedral and interior angles are listed in table 4. We have also calculated the characteristic distances and angles of the basic cells of these honeycombs, as well as the volume of the basic cell, the number,  $E_V$ , of edges that share a vertex, the number,  $C_V$ , of

**Table 4.** Characteristic parameters associated with the three-dimensional bounded honeycombs of the open RW space.

Name	Schläfli symbol	Basic cell	Dihedral angle	Interior angle	$C_V$	$E_V$
$i_{120}$	{3, 5, 3}	Icosahedron	120°	41.81°	12	20
$d_{90}$	{5, 3, 4}	Dodecahedron	90°	90°	8	6
$c_{72}$	{4, 3, 5}	Cube	72°	63.43°	20	12
$d_{72}$	{5, 3, 5}	Dodecahedron	72°	63.43°	20	12

**Table 5.** Other characteristic parameters associated with the three-dimensional bounded honeycombs of the open RW space ( $R(t) = 1$ ).

Name	$d$	$r_c$	$d_E$	$r_i$	$d_F$	Vol	$\rho$
$i_{120}$	1.7366	1.3826	0.9726	0.8683	0.9727	4.6860	0.2134
$d_{90}$	1.0613	1.2265	1.0613	0.8085	0.8425	4.3062	0.5806
$c_{72}$	1.6169	1.2265	0.8425	0.5306	1.0613	1.7225	0.2322
$d_{72}$	1.9927	1.9028	1.4391	0.9964	1.4321	11.1991	0.0893

cells that share a vertex, the volume of the basic cell, Vol, and the density,  $\rho$ . Tables 4 and 5 give the values of all these parameters. Note that the honeycombs  $d_{90}$  and  $c_{72}$  are dual to one another, and that  $d_{120}$  and  $d_{72}$  are self-dual.

The flat universe corresponds to the Euclidean tridimensional space. The regular polyhedra of this space are the five platonic polyhedra. Among these polyhedra only the cube has a dihedral angle divisor of  $2\pi$ . Thus the only possible regular honeycomb is formed by cubes. Its Schläfli symbols are {4, 3, 4}, the dihedral and the interior angles are both of 90°,  $C_V = 8$ , and  $E_V = 6$ . But because in a flat space there is not a proper length, the basic cube can be of any size and there is no characteristic distance.

We can also consider whether there are honeycomb structures for models of the universe with local constant curvature but with topology different to the usual one. Models of universes of this type have been considered; see [12, 13, 20]. See also [14] for a recent result that constrains the possible topologies of these spaces. Special attention has been paid to the locally flat and the locally hyperbolic ones. The reason is that one can then have universes that are compact and flat, and universes that are compact and have negative constant curvature. One which is very popular is known as the Seifert–Weber dodecahedral space (see [8, 21]). This space is obtained from the above  $d_{72}$  hyperbolic honeycomb. To construct this space we have to glue together the opposite faces of the basic dodecahedron using a clockwise twist of 3/10 of a revolution. Another example, this one with positive constant curvature, is the Poincaré dodecahedral space. This space is associated with the 120-cell spherical honeycomb. To obtain it, opposite faces of the basic honeycomb are glued together using this time a twist of 1/10 of a revolution. For the flat space, identification of the opposite faces of a cube gives the 3-torus, which is a compact flat model of the universe. Only this last space, among all spaces with non-trivial topology, admits a regular honeycomb structure. This is because for constant non-zero curvature there are not two basic polyhedra whose distances between vertices are such that one is a divisor of the other.

*Comment.* The relation of the above regular honeycombs and the multiply connected spherical orientable spaces (see [15–17]) is as follows: to obtain a spherical orientable 3-manifold by identifying the faces of a platonic polyhedron  $\Sigma$ , the polyhedron must obey two conditions

(see [18]): (1) the dihedral angle must be a submultiple of  $2\pi$ , say  $2\pi/r$ , and (2) the number of edges of  $\Sigma$  must be divisible by  $r$ . By definition, the basic polyhedron cell of any honeycomb obeys (1), but there are two spherical honeycombs, the 16-cell and the 600-cell, that do not satisfy (2). With the remaining four spherical honeycombs we can associate globally homogeneous spherical 3-manifolds. These manifolds are single-action spherical manifolds. The single-action spherical manifolds are those for which the members of a subgroup  $R$  of  $S^3$  act as pure right-handed Clifford translations (see [19]). With the honeycomb 5-cell, whose basic cell is a tetrahedron, we can associate the lens space  $L(5, 1)$ , which is the single-action manifold associated with the cyclic group  $Z_5$ . With the honeycomb 8-cell, whose basic cell is a cube, we can associate the Montesinos quaternionic space, which is a prism manifold associated with the binary dihedral group  $D_2^*$ . With the honeycomb 24-cell, whose basic cell is an octahedron, we can associate the Montesinos octahedral space, which is the single-action manifold associated with the binary tetrahedral group  $T^*$ . Finally, with the honeycomb 120-cell, whose basic cell is a dodecahedron, we can associate the above-mentioned Poincaré dodecahedral space, which is the single-action manifold associated with the binary icosahedral group  $I^*$ . These associations provide a way to show that  $S^3$  is a covering space of the above manifolds.

Among the four hyperbolic honeycombs, only the  $i_{120}$  and the  $d_{72}$  obey the above condition (2). By identification of opposite faces of the basic icosahedron cell of  $i_{120}$  we get a hyperbolic compact manifold, the 3-torus  $T^3$ . The basic dodecahedron cell of the honeycomb  $d_{72}$  gives rise, by identification of opposite faces, to the above-mentioned Seifert–Weber dodecahedral space. Finally, [18] gives other possible manifolds associated with these honeycombs.

## 5. Interpretation in terms of standard FLRW cosmology

Analysis of the power spectrum of density perturbations and the correlation function have shown that galaxies appear to be gathered into immense sheets and filaments surrounding very large voids (see [1–3]). The most symmetric distributions of matter, after the homogeneous and isotropic one, are those associated with the honeycomb structures. These structures give the most natural generalization of the CP. We have named this generalization the WCP. Then we propose that the large-scale structure of the universe could have the structure of a honeycomb.

We have seen that there are eleven suitable honeycombs, six corresponding to a closed universe, four to an open one, and one to a flat one. We have calculated the different parameters that characterize these honeycombs.

The model that we propose is very speculative, but we think that it could be useful in looking for new ways to interpret the inhomogeneities that have been discovered on large cosmological scales. To make this scenario feasible we have to assume that, initially, there was a homogeneous and isotropic distribution of dark matter or of some other non-observable kind of matter. We accept also that inhomogeneities with higher energy density than the mean, formed during the cosmic evolution, are distributed in the most homogeneous and isotropic manner possible, which we assume to have the honeycomb structure. From this, we can speculate that the visible matter is concentrated in these inhomogeneities of higher density, with a hierarchical distribution of densities, increasing through the sequence: interior, faces, edges, and vertices of the basic cells.

The above symmetric distribution may be considered as the limit attractor of the less symmetrical present distribution consisting of a huge net of filaments made up of clusters of galaxies. This net would evolve, seeking the stability associated with the symmetry of any of the above-described honeycombs. At the present time we could be just in the phase transition

that goes from a more or less homogeneous distribution to a crystalline one. We do not know what the precise dynamics governing the above process might be. Presumably, it would be a very complex one, with the extragalactic magnetic field as a principal actor. It is possible that the seeds of these structures were generated in the first moments after the big bang, perhaps before the inflation due to the strong magnetic fields generated by the turbulence of the charged plasma (see [5, 22–24]). If this were the case, there would be a suppression of cosmological density fluctuations on scales beyond the size of the basic cell, similar to what happens in small universe models (see [19, 25]). Then the honeycomb models could also explain the existence of a cut-off in the cosmic microwave background (CMB) angular power spectrum on large angular scales (see [26]).

## 6. Observational prospects

### 6.1. The case in which the size of the basic cell of the honeycomb is much smaller than the particle horizon

If the size of the basic cell of the honeycomb is much smaller than the particle horizon we could verify the correctness of the above model. A possible way would be to study the distribution of high red-shifts,  $z$ , in any direction. They should exhibit peaks with periodic separations given in  $\log(1+z)$ . The period should depend on the periodic structure of the honeycomb and, therefore, on the observational direction. Sufficient observations of this kind would enable the determination of the appropriate honeycomb. Regularities of this type have been reported by Broadhurst *et al* (see [3]). They found that in regions of small area around the northern and southern galactic polar caps, the high and low density alternate with a rather constant step of  $128 \text{ h}^{-1} \text{ Mpc}$ . In other directions the regularity is much less pronounced.

Another possible observational parameter is the number of filaments that converge on a supercluster. The open space honeycombs only admit 6, 12, or 20 filaments; the possibilities for the closed space are 4, 6, 8, and 12; the flat one only admits six filaments (see tables 3 and 4).

Once we know the honeycomb, we can use its characteristic distances to determine the present curvature of the universe  $R_0$ . The value of  $R_0$  may then be used to sharpen the value of the density parameter,  $\Omega_{\text{tot}}$  (it can be calculated from  $\Omega_{\text{tot}} = 1 - kc^2/(R_0 H_0)^2$ ;  $k = 1, -1$  for the closed and open cases respectively), as well as other cosmological parameters.

We can also look for observable effects of these structures on gravitational waves, analogous to the x-ray diffraction of crystals. Another possible observational feature is the lens effect of these periodic structures on electromagnetic waves.

### 6.2. The case where the particle horizon is of the order of the basic cell's size

If the particle horizon is of the order of the basic cell's size, we may observe only a part of that basic cell, but the data ratio between the characteristic distances of the cell as well as the values of  $E_V$  and  $C_V$  can be enough to determine which is the appropriate honeycomb. To this end, it would be important that we can observe at least one vertex of the honeycomb, because in that case we can observe  $E_V$ ,  $C_V$ , and the dihedral and interior angles, and if these values are ones given in the above tables, then we will have evidence that we are in a honeycomb. Also these values will be enough to determine the appropriate honeycomb in all the cases except for the pair of spherical honeycombs the 5-cell and 8-cell, and hyperbolic  $c_{72}$  and  $d_{72}$ , for which the values of these four parameters coincide. To discriminate between these cases, we must use the observable distances to the vertices, edges, and faces to reconstruct the basic cell. Now

we will calculate, for each of the honeycombs, the probability that a randomly placed observer can detect a vertex of the honeycomb. This probability,  $p_r$ , will depend on the horizon radius  $r_h$  considered.

In the following we assume that the universe can be described by the RW metric, and that matter is made up from dust of density  $\rho_m$  and a cosmological constant  $\Lambda$ . The Friedmann equation is then given by

$$H^2 = \frac{8\pi G\rho_m}{3} - \frac{kc^2}{R^2} + \frac{\Lambda}{3}, \quad (2)$$

where  $H = \dot{R}/R$  is the Hubble parameter,  $G$  is Newton's constant, and  $k = 1, 0, -1$  for an open, flat, and closed universe respectively.

Moreover, we have that  $\rho_m = (R/R_0)^3 \rho_{m0}$ , and the red-shift  $z$  is given by  $z = R_0/R - 1$ , where the subscript 0 denotes evaluation at the present time.

The RW metric gives  $dr = (1/R)c dt$  for the photon equation. Integrating this equation, and taking into account the above relations, we can find (see [27]) that the comoving distance  $r(z)$ , run over by a photon as function of the red-shift  $z$ , is given by

$$r(z) = \sqrt{|1 - \Omega_{\text{tot}}|} \int_0^z (\Omega_{\Lambda 0} + (1 - \Omega_{\text{tot}})(x + 1)^2 + \Omega_0(x + 1)^3)^{-1/2} dx, \quad (3)$$

where  $\Omega_0$ ,  $\Omega_{\Lambda 0}$ , and  $\Omega_{\text{tot}}$  are the density parameters given by  $\Omega_0 = \frac{8\pi G\rho_{m0}}{3H_0^2}$ ,  $\Omega_{\Lambda 0} = \frac{\Lambda c^2}{3H_0^2}$ , and  $\Omega_{\text{tot}} = \Omega_0 + \Omega_{\Lambda 0}$ . The above distance,  $r(z)$ , is given in units of the curvature radius  $R_0$ . The horizon radius,  $r_h$ , corresponds to  $z = \infty$ ; the last scattering surface radius,  $r_{\text{LSS}}$ , associated with the cosmic microwave background (CMB), corresponds to  $z \approx 1100$ ; and for the quasars and the clusters of galaxies, we could take red-shift cut-offs of  $z \approx 6$  and  $z \approx 1$  respectively. The probability  $p_r$  associated with any radius  $r(z)$  is given by the fraction of the basic cell volume in which the distance to a vertex is smaller than  $r(z)$ , that is

$$p_r = \frac{V_r}{V}, \quad (4)$$

where  $V_r$  is the volume of the region of the basic cell such that the distances from its points to a vertex are less than  $r(z)$ , and  $V$  is the volume of the basic cell of the honeycomb. If  $r(z) > r_c$ , then  $V_r = V$ , and  $p_r = 1$ . If  $2r(z)$  is less than the edge length  $d$ , the spheres with centres at the vertices and radius  $r(z)$  do not intersect, and  $V_r$  is given by

$$V_r = V \frac{V_s(r(z))}{C_V}, \quad (5)$$

where  $V_s(r(z))$  is the volume of the sphere of radius  $r(z)$ ,  $C_V = \frac{2}{1-q(\frac{1}{2}-\frac{1}{r})}$  is the number of cells around a vertex, and  $V = \frac{2}{1-q(\frac{1}{2}-\frac{1}{r})}$  is the number of vertices of a cell. The volumes of a sphere of radius  $r$  in  $S^3$  and in  $\Pi^3$  are given respectively by

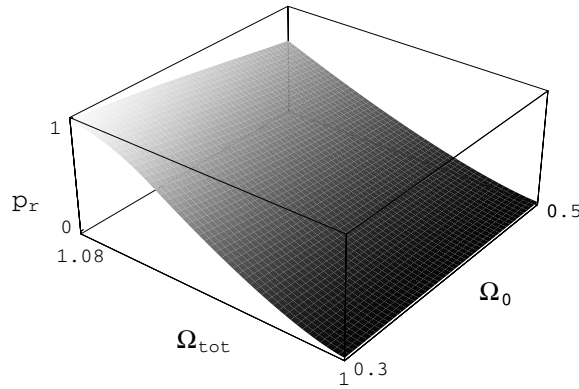
$$V_s(r) = \pi(2r - \sin 2r), \quad V_s(r) = \pi(2r - \sinh 2r), \quad (6)$$

where we have taken the curvature radius  $R = 1$ .

If  $d/2 < r(z) < \text{distance from the centre of a face to a vertex of that face } d_F$ , there are no points common to more than two spheres. The volume of the region of the intersection of two spheres is double the volume of the spherical cup,  $V_c$ , corresponding to the height  $r(z) - d/2$ , multiplied by the number of edges of a cell,  $E$ , and divided by the number of cells with a common edge,  $r$ . Thus we have

$$V_r = \frac{V}{C_V} V_s(r) - \frac{E}{r} 2V_c(r - d/2). \quad (7)$$





**Figure 1.** The probability,  $p_r$ , that a randomly located observer can detect the spherical honeycomb 16-cell, for  $0.3 < \Omega_0 < 0.5$ ,  $1 < \Omega_{\text{tot}} < 1.08$ , and  $z = \infty$ .

**Table 6.** The probability that a randomly located observer detects a given honeycomb, for red-shift  $z = 1, 6, 1100, \infty$ . For the honeycombs of the closed universe we have taken  $(\Omega_0, \Omega_{\text{tot}}) = (0.3, 1.03)$ , and for the open universe  $(\Omega_0, \Omega_{\text{tot}}) = (0.3, 0.95)$ .

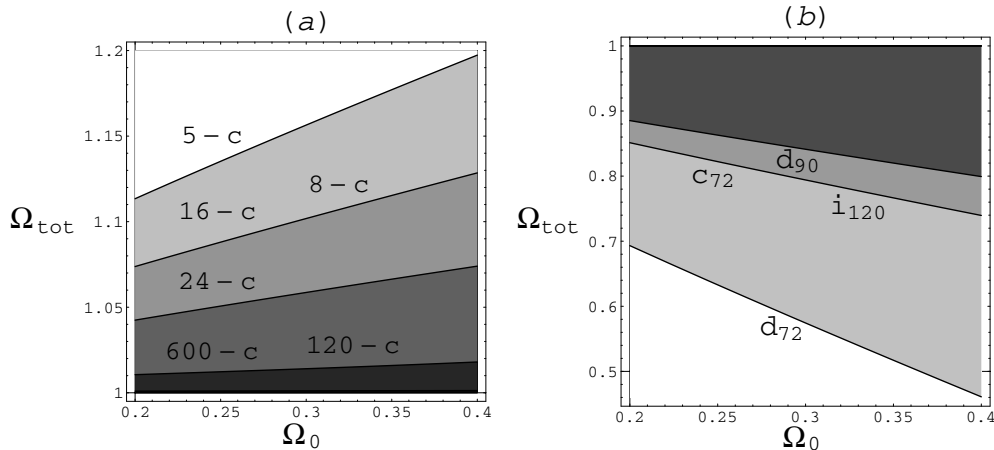
Honeycomb	$z = 1$	$z = 6$	$z = 1100$	$z = \infty$
$r(z)(0.3, 1.03)$	0.135	0.337	0.558	0.576
5-cell	0.003	0.040	0.173	0.190
8-cell	0.008	0.127	0.546	0.590
16-cell	0.004	0.064	0.277	0.304
24-cell	0.012	0.191	0.809	0.860
120-cell	0.311	–	1	1
600-cell	0.062	0.913	1	1
$r(z)(0.3, 0.95)$	0.170	0.423	0.706	0.729
$i_{120}$	0.004	0.070	0.347	0.386
$d_{90}$	0.012	0.191	0.740	0.779
$c_{72}$	0.005	0.076	0.374	0.420
$d_{72}$	0.002	0.029	0.145	0.161

The volume  $V_c(r - d/2)$  equals  $V_s(r)/2$  minus the volume of the spherical segment of height  $d/2$ ,  $V_{\text{seg}}(d/2)$ . But  $V_{\text{seg}}(h) = \int_0^h A(y) dy$ , where  $A(y)$  is the area of the circle of radius  $y$ . Taking into account that for the spherical and hyperbolic cases we have respectively:  $A(y) = \pi \sin^2 y$ ,  $\cos r = \cos y \cos x$ , and  $A(y) = \pi \sinh^2 y$ ,  $\cosh r = \cosh y \cosh x$ , we obtain, for the spherical and hyperbolic segment volumes respectively,

$$V_{\text{seg}}(h) = (h - \cos^2 r \tan h), \quad V_{\text{seg}}(h) = (h - \cosh^2 r \tanh h). \quad (8)$$

With the above expressions we can calculate the probability  $p_r$  in all the cases except when  $d_F < r(z) < r_c$ . In this case we can approximate the value of  $p_r$  by interpolation. Using equation 3 we can express  $p_r$  as a function of the density parameters  $\Omega_0$  and  $\Omega_{\text{tot}}$ . As an example, we have shown in figure 1 the probability for a randomly located observer of detecting the spherical honeycomb 16-cell, for  $0.3 < \Omega_0 < 0.5$ ,  $1 < \Omega_{\text{tot}} < 1.08$ , and  $z = \infty$ . Note that if  $\Omega_{\text{tot}} \rightarrow 0$ , then  $r(z) \rightarrow 0$ , and the probability  $p_r$  also goes to 0.

Table 6 gives the probability of detecting any of the regular honeycombs considered above for  $z = 1, 6, 1100, \infty$ . For the honeycombs of the closed universe we have taken



**Figure 2.** The region of the  $(\Omega_0, \Omega_{\text{tot}})$  plane where  $r_{\text{LSS}}$  is greater than the circum-radius,  $r_c$ ; that is, the region in which the probability of detecting the honeycomb is 1. (a) Shows these regions for the spherical honeycombs. The points at which  $r_{\text{LSS}} < r_c$  correspond to the region above the marked line ( $r_{\text{LSS}} = r_c$ ). In the same way, (b) gives these regions for the hyperbolic honeycombs. In this case the points with  $r_{\text{LSS}} < r_c$  are in the regions below the marked lines.

$(\Omega_0, \Omega_{\text{tot}}) = (0.3, 1.03)$ , and for the open universe  $(\Omega_0, \Omega_{\text{tot}}) = (0.3, 0.95)$ . These values of the density parameters are in the range  $0.9 < \Omega_{\text{tot}} < 1.1$  for the nearly flat universes that have been found from recent observations [28]. The value omitted in the table corresponds to a value of  $r(z)$  such that  $d_F < r(z) < r_c$ , and, as has been pointed out previously, it cannot be calculated by the above procedure. Notice that in the closed universe the higher probabilities correspond to the 120-cell and 600-cell honeycombs, and to  $d_{90}$  in the open case.

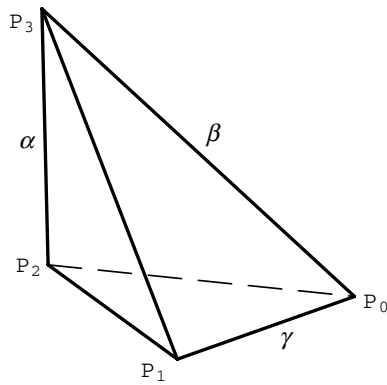
We have also calculated for each regular honeycomb the region of the  $(\Omega_0, \Omega_{\text{tot}})$  plane where  $r_{\text{LSS}}$  is greater than the circum-radius,  $r_c$ , that is, the region in which the probability of detecting the honeycomb is 1. Figure 2(a) shows these regions for the spherical honeycombs, and figure 2(b) gives these regions for the hyperbolic honeycombs. Observe that the honeycombs that are easiest to detect in a nearly flat universe are the spherical 120-cell and 600-cell.

## 7. Summary

In this article, we have considered the possibility that the gravitational growth of primordial density fluctuations leads to what can be considered a weak version of the cosmological principle, for which the large-scale matter distribution has the pattern of a regular honeycomb. In a recently published paper (see [29]) we studied honeycombs in the space of relativistic velocities and in the Milne cosmological model. In both cases the honeycombs were the hyperbolic ones. In that paper we advanced some of the ideas of this one.

There are six regular honeycombs associated with the closed FLRW universe, and four, with bounded cells, associated with the open case. We have calculated the most important parameters characterizing these honeycombs.

We have also given some observational prospects for detecting the honeycomb. Moreover, we have calculated, for each honeycomb, and for a nearly flat universe, the probability that a randomly placed observer could detect the honeycomb as a function of the density parameters  $\Omega_0$  and  $\Omega_{\Lambda 0}$ .



**Figure 3.** One of the  $2pF$  identical double-rectangular tetrahedra which any regular polyhedron is decomposed into.

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**Appendix**

In this appendix we will calculate the characteristic distances and angles of the regular honeycombs considered in this paper. To calculate these distances as functions of the Schläfli symbols  $\{p, q, r\}$  of the honeycomb we proceed as follows. First, we decompose each polyhedron into  $F$  identical pyramids with the apex in the centre of the polyhedron. Each of these pyramids is then decomposed into  $2p$  double-rectangular tetrahedra by dropping perpendicular lines from the apex onto the faces and onto the lines bounding the faces. The vertices of this tetrahedron are: the centre of the cell,  $P_3$ , the centre of a face,  $P_2$ , the centre of an edge,  $P_1$ , and a vertex of the cell,  $P_0$  (see figure 3). We recall that a tetrahedron  $P_0P_1P_2P_3$  is said to be double rectangular if its edge  $P_3P_2$  is orthogonal to the face  $P_0P_1P_2$  and its edge  $P_1P_0$  is orthogonal to the face  $P_1P_2P_3$ . Thus, three out of the six dihedral angles are right angles. Thus, the double-rectangular tetrahedron is determined by its dihedral angles  $\alpha, \beta,$  and  $\gamma$  corresponding to the edges  $a = P_3P_2, b = P_3P_0,$  and  $c = P_1P_0$  respectively. Then, using spherical trigonometry, we have (see [7])

$$\tan a \tan \alpha = \tan b \tan \left(\frac{\pi}{2} - \beta\right) = \tan c \tan \gamma = \frac{\sqrt{\Delta}}{\cos \alpha \cos \gamma}, \tag{9}$$

where  $\Delta = \sin^2 \alpha \sin^2 \gamma - \cos^2 \beta$ .

By definition of  $p, q,$  and  $r$  it follows that  $\alpha = \pi/p, \beta = \pi/q,$  and  $\gamma = \pi/r$ . Therefore, by substituting these values in equation (9) we obtain  $a, b,$  and  $c$ . The above-defined characteristic distances of a honeycomb are then given by:  $d = 2c, r_c = b, d_E = \arg \sin(\sin a/\sin \gamma), r_i = a,$  and  $d_E = \arg \sin(\sin c/\sin \alpha)$  (see table 2).

Moreover, we can get the interior angle  $\phi$  of the polygons constituting the faces of each cell by solving for the hyperbolic triangle  $P_0P_1P_2$ . In fact, we have  $\sin \frac{\phi}{2} = \cos \alpha/\cos c$ . In this way we have obtained the values of  $\phi$  in table 3.

The space-like sections of the universe, in the open FLRW model, are three-dimensional spaces of constant negative curvature, and these spaces are isomorphic to  $H^3$ . We have

followed the Vinberg and Shvartsman [7] classification of hyperbolic honeycombs, which does not include as honeycombs those with cells inscribed in horospheres instead of finite spheres. As in the spherical case, for a honeycomb with Schläfli symbols  $\{p, q, r\}$  the dihedral angle of each cell,  $\alpha$ , equals  $2\pi/r$ , but in the hyperbolic case  $\alpha$  has the restriction  $\alpha_{\min} \leq \alpha < \alpha_{\text{Euc}}$ , where  $\alpha_{\min}$  is the minimal possible dihedral angle in such a regular polyhedron in hyperbolic space, and  $\alpha_{\text{Euc}}$  is the dihedral angle of the corresponding polyhedron in Euclidean space. From this fact, if we restrict ourselves to honeycombs with bounded cells, it follows that there are only four regular honeycombs in  $H^3$  [7]. Their Schläfli symbols and dihedral and interior angles are listed in table 4. In our paper [29], we calculated the characteristic distances and angles of the basic cell of these honeycombs, as well as the volume of the basic cell, the number,  $E_V$ , of edges that share a vertex, the number,  $C_V$ , of cells that share a vertex, the volume of the basic cell, Vol, and the density,  $\rho$ . Tables 4 and 5 give the values of all these parameters.

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