# Can Compact Currents be Uniquely Determined ?

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## Introduction

The EEG/MEG inverse problem is ill-posed and its solutions have two independent sources of non-uniqueness. The problem is ill-posed by the nature of the physics because infinitely many different current configurations can give rise to the same electro-magnetic fields. It is also mathematically underdetermined because the number of available mathematically independent data points is less than the dimension of the solution space.

Since electro-magnetic inverse solutions are non-unique, some criteria must be chosen by which to select a particular solution. Here we assume that neuronal generators of EEG/MEG data, and hence our desired solutions, are compact. Such an assumption, in one form or another, ranging from dipolar sources to sources being activations of small areas of arbitrary shape and location in the cortex, has been used in most neuro-magnetic inverse solutions. Here we take the latter case to be our target solutions. Intercortical recordings have shown a much greater range of activity in the cortex. We maintain, however, that our assumption is a likely one, since EEG/MEG data reflect only the synchronous activations on the order of at least 10,000-100,000 neurons. The spatial extent of the individual synchronous activations, at least in normal functional cases, is fairly small, as suggested by physiological data, by functional images from PET and fMRI, and by the EEG/MEG field waveforms. The degree of compactness of the currents, however, is not entirely known.

Although numerous algorithms have been developed to find localized solutions, the solutions from these algorithms do not necessarily agree. This suggests that the assumption of compactness by itself does not resolve the non-uniqueness of the electro-magnetic inverse problem. Since neural currents are not known to satisfy any particular optimization criterion, such as the minimum $l_1$ norm, we do not know which of the competing compact solutions is correct, as long such non-uniqueness prevails. This paper establishes the conditions under which the electro-magnetic inverse problem becomes unique.

Due to the space limitations here we were only able to provide a sparse summary of the results of our analysis. The full details of the analysis and the treatment of the equally important issue of methodologies for unambiguous identification of neural currents as suggested by the analysis, are presented in [1].

Uniqueness of electric field generators has also been studied in [2], where interpolation of finite measurements is suggested as a way to deal with the mathematical non-uniqueness part of the problem. Our results show that, in fact, interpolation cannot resolve this non-uniqueness.

## Background

**Forward model:** The contributions to electric and magnetic fields from electric currents at different head locations add up linearly. The forward model describing this relationship is simply the equation

$$A\mathbf{x} = \mathbf{b}, \tag{1}$$

where $\mathbf{x}$ is a vector of $n$ values of the current at $n$ discrete points in the head, $\mathbf{b}$ is a vector containing $m$ values of the electric/magnetic field measured at the $m$ sensors and $A$ is the $m \times n$ matrix that connects the two. To obtain a discrete representation of the current we subdivide the head into nodes spaced at some desired interval and represent the continuous current value at each node by a static dipole. The forward model $A$ is found by discretizing a quasi-static approximation of Maxwell's equations. Expressions for forward models of virtually any complexity, as a function of the primary currents only, are now available, and we assume here that one such model has been selected. The analysis is independent of the model $A$ that is chosen and applies to EEG as well as MEG data. By a generator we mean the set of all neurons that are firing simultaneously. By a source we mean a set of simultaneously firing neurons located in one continuous anatomical neighborhood.

We use the lead field formulation (1) is used in the analysis here, but another, namely a non-linear formulation of the inverse problem, is also possible. The results derived from the linear formulation, however, pertain to properties of the electric currents themselves, meaning that these results are true for any compact electro-magnetic inverse solution, irrespective of the forward model it is derived from.

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**Compact solutions:** The compactness of a current distribution relates to the number of non-zero entries in $\mathbf{x}$ . In solutions representing currents restricted to a few small regions in the brain, we expect most of the entries of $\mathbf{x}$ to be zero, indicating no current at those locations, and a few non-zero entries, indicating the presence of current at those few locations.

We define compact solutions (and currents) as *solutions for which the number of nonzero entries in* $\mathbf{x}$ *is less than the number of data* $m$*.*

The number $m$ is the natural boundary for defining spatially compact solutions for a number of reasons. First, it provides mathematical tractability for the uniqueness results. Second, the mathematical properties of compact and non-compact solutions so defined are very distinct. For purely algebraic reasons, systems such as (1) always have an infinite number of solutions with $m$ or more non-zero entries, regardless of the actual current that generated the given data. Alternatively, no compact solutions may exist for data generated by a non-compact source. The number of compact solutions, when they do exist, is very limited and can be one, as shown by the theory presented below. Finally, the assumption that sources don't extend over more than $m$ grid nodes is already implicit in most popular temporal methods, such as adaptive filtering and MUSIC.

The dependence of the definition of compact currents on the choice of the grid spacing may appear to be a cause of concern, but such dependence cannot be avoided. The widely used temporal methods, for example, implicitly have this dependence. Here we assume that an 'optimal' grid has been chosen, that is, one that optimizes the trade-off between the smoothness of the sampled functions and the achievable resolution as determined by the level of noise. The topic of finding such an optimal grid is outside the scope of this paper. The whole issue should not be a concern however, as the results below show that the sizes of the currents of interest need to be comfortably below the $m/2$ threshold to assure their unique localization. Thus small changes in the grid spacing will not affect the definition of compactness for those sources.

For the analysis, it is also convenient to refer to the size of generators by their dimension, which we define as, *the number of non-zero entries in* $x$. Since each node is typically modeled by three components of $x$, the number of active nodes is $1/3$ the dimension of $x$.

## Results

**Separation of physical and mathematical ill-posedness:** The non-uniqueness of inverse solutions to (1) is embodied in the null space of $A$, denoted by $\mathcal{N}(\mathcal{A})$ . It consists of all vectors $\{x_n\}$ that are mapped into the zero vector by the matrix $A$, i.e. $Ax_n = 0$ . Thus non-uniqueness of solutions occurs because any current $x_r$ can be replaced by a different current $x_r + x_n$ to produce the same data $b$. When $\mathcal{N}(\mathcal{A})$ exists, its set of vectors $\{x_n\}$ is infinite.

The null space exists under two conditions, when either $n > m$ for the matrix $A$ in (1) or when the rank of $A$ is less than $m$. The first condition, $A$ having more columns than rows, relates to the mathematical non-uniqueness arising from the limited number of data, and the second, linear dependence within $m$ columns of $A$, relates to the physical ill-posedness. This distinction between the two kinds of non-uniqueness should be obvious, but a more extensive discussion of this issue can be found in [1]. The distinction is also a key element that enables us to carry out the uniqueness analysis of the electro-magnetic inverse problem.

**Non-uniqueness due to the physics of the electromagnetic problem:** In a spherically symmetric, spherically homogeneous conductor, the radial component of the current produces no external magnetic field. In the general case of an arbitrarily shaped conductor, which includes the spherically symmetric case, a primary current $\mathbf{J}_0$ that is normal to a closed surface and has constant dipole moment per unit area is silent both electrically and magnetically. Any two primary currents differing by such $\mathbf{J}_0$ are indistinguishable magnetically and electrically. For two sources to differ by the same current $J_0$, they must be adjacent or overlapping. This means that non-unique generators of external electric/magnetic fields must be adjacent or overlapping as well. The same result for the electric field only was shown in [2].

We draw an important insight on the nature of physical non-uniqueness from the above discussion. For compact generators which consist of small spatially separated sources, the physics contributes only to local non-uniqueness within individual sources and does not affect the general locations of the sources. That is, the ambiguity is limited to the amplitude variations within each source and possibly to the shape of the boundaries of the sources only.

The problem of non-uniqueness is compounded significantly by the presence of noise and the finite precision of the instruments. We list some examples here. It is well known that the fields from two closely spaced dipoles with opposite orientations practically cancel out and cannot be detected. This effect can be observed on a relatively large scale, for example, in the lateral segment of the superior olivary nucleus of the cat, which consists of many neurons with variable orientations [3]. When oriented in the same direction, two closely spaced dipoles can appear, electrically and magnetically, as a quadrapole. Fields from two small, closely spaced, and not necessary point sources, can be fit within comparable error using a deeper and stronger single dipole [4]. We add that in fact the single dipole and the two (or one extended) sources, in the last example, are the lower and upper boundaries of a cone, within which an infinite number of current distributions can provide a comparable fit the same fields.

The columns of $A$ describing non-unique solutions in the above examples are not linearly dependent, but they are nearly linearly dependent. These columns become effectively linearly dependent in the presence of noise, since the currents they describe are indistinguishable electrically and magnetically. Thus the ambiguity due to the physics extends to a greater number of currents in the presence of noise, but what is important is that it remains local.

## Algebraic non-uniqueness of inverse solutions:

To study the algebraic non-uniqueness we transform our problem into one that excludes the physical non-uniqueness, by assuming the following property of $A$.

**Linear independence property:** *Given a linear system (1), any $m$ columns of $A$ are linearly independent.*

The analysis in this section will refer to the transformed system only, and to simplify the presentation we will use the term (non-)uniqueness here to mean the algebraic (non-)uniqueness only. The key results are stated as theorems here. Their proofs are given in [1].

### Theory in the noise free case

The following theorem provides the necessary and sufficient conditions for non-uniqueness of compact inverse solutions given EEG/MEG sampled fields.

**Theorem 1** *Two generators $\mathbf{x}_1$ and $\mathbf{x}_2$ of EEG/MEG data are non-unique iff the vector $(\mathbf{x}_1 - \mathbf{x}_2) \in \mathcal{N}(\mathcal{A})$, where $\mathcal{N}(\mathcal{A})$ denotes the null space of $A$.*

Three main results come from Theorem 1. One is that compact non-unique inverse solutions can exist, because we can easily construct any two such compact sources satisfying the condition of Theorem 1. The second is that mathematical non-uniqueness is not dependent on location, that is non-unique generators of a certain size (see below) can occur anywhere in the cortex. What determines whether two or more generators become non-unique are the relative amplitudes of their currents, which must match exactly a vector in $\mathcal{N}(\mathcal{A})$ . The likelihood of this happening depends on how many vectors are contained in $\mathcal{N}(\mathcal{A})$ , i.e. the size of the null space of $A$. For EEG/MEG the size of $\mathcal{N}(\mathcal{A})$ is very large, on the order of $5,000 - 10,000$ dimensions. Thus the occurrence of non-unique generators is not that unlikely. The final result is that the combination of two non-unique generators must always extend over $m$ voxels or more. There is however no restriction on the size of an individual non-unique generator, which can vary from a single dipole to a distribution of several sources extending jointly over $m - 1$ voxels.

The following example gives an illustration of the last result. Suppose the measurement system consists of 150 gradiometers and the activity is well approximated by a single dipole described by 3 components. The third result of Theorem 1 guarantees that no other distribution with less than 148 components can produce the same measurements. The gap in the size of the two solutions is quite large, so the smaller source is easily distinguishable. This is largely responsible for the success of many algorithms in finding sources of very small dimensions.

We now present the two conditions, besides the obvious utilization of priors from other modalities, under which non-uniqueness can be resolved.

**Maximally sparse generators:** The second result from Theorem 1 implies that there can never be more than one generator of dimension less than or equal to $m/2$. Thus, if we know that the true activation does not extend over $m/6$ nodes, we can always find it as the unique maximally sparse solution. The maximally sparse solution is the solution with the smallest number of nonzero terms. The following theorem formally states this result.

**Theorem 2** *If the number of data samples is at least twice the dimension of the true generator, this generator can always be found as the unique maximally sparse solution.*

Note that the uniqueness of the generators of continuous fields shown in [2] follows directly from Theorem 2.

**Temporal information:** Because the vector made up of currents of two non-unique generators must match some particular null vector of $A$, the non-uniqueness of these two generators cannot be maintained if the amplitudes of the currents change. Since the intensity of neural activation tends to change rapidly, and provided that activations of the individual sources are at least partially uncorrelated, the data over several time points can consistently reflect only one compact generator. This also assumes that neural activations do not move rapidly in the cortex. Given that the assumptions are satisfied, consistency of a solution across several time points provides unambiguous means of resolving non-uniqueness.

**Theory in the presence of Noise:** The presence of noise makes non-uniqueness more prevalent, because candidate solutions need to provide only an approximate fit to the data. In the exact case, the two generators have to match a null vector of $A$ exactly to be non-unique. With the addition of noise, this match doesn't have to be exact. Since the null space is orthogonal to the range space, even moderate perturbations in the current values of non-unique generators have a negligible effect on the data and thus their non-uniqueness properties. Hence the set of non-unique currents is increased.

The uniqueness result of Theorem 2 is also affected adversely by noise. If a true maximally sparse generator makes up a significant component of a vector that lies entirely or largely in $\mathcal{N}(\mathcal{A})$ , there exist other sparse solutions with a comparable fit to the data. Some of these solutions are likely to be sparser than the true one. They also all share a common component in the range space of $A$. For these reasons, the algorithms for maximally sparse solutions are likely to find one of the 'phantom' solutions rather than the true one. For a true generator to make up a significant component of a vector in $\mathcal{N}(\mathcal{A})$ , however, it has to be relatively large. Thus the described problem does not affect relatively small generators such as ones consisting of a few small sources, but becomes noticeable as the size of a generator approaches the $m/2$ threshold.

Note that the non-uniqueness described here should not be confused with the problem addressed by regularization, where the nature of incorrect solutions depends on the condition number of $A$.

## Summary and Conclusions

The mathematical non-uniqueness manifests itself in the ambiguity in the general locations of the sources in the cortex. The physical ill-posedness manifests itself in ambiguity over current values within neighborhoods of individual sources. Thus estimation of a compact current is a two step process. We first need to find the general locations of the sources that make up the generator. We then estimate the optimal solution constrained to lie within these locations.

We can resolve the mathematical non-uniqueness in three ways: 1) use priors, such as fMRI or PET images; 2) when currents are known to be very small, solve for the maximally sparse solution with the understanding that this estimate is not always reliable; 3) use temporal information, provided the assumptions listed above are satisfied. Given the potential perils of the last two methods, they may be best combined into one algorithm. The optimal way of resolving physical non-uniqueness is to find the minimum norm estimate constrained by the approximate source locations from step 1 and by any other available information. Further discussion of these issues and the methodologies for finding compact sources motivated by the analysis are presented in [1].

## References


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