The coherence of complemented ideals in the space of real analytic functions

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Abstract

We characterize when an ideal of the algebra $A(\mathbb{R}^d)$ of real analytic functions on $\mathbb{R}^d$ which is determined by the germ at $\mathbb{R}^d$ of a complex analytic set V is complemented under the assumption that either V is homogeneous or $V \cap \mathbb{R}^d$ is compact. The characterization is given in terms of properties of the real singularities of V. In particular, for an arbitrary complex analytic variety V complementedness of the corresponding ideal in $A(\mathbb{R}^d)$ implies that the real part of V is coherent. We also describe the closed ideals of $A(\mathbb{R}^d)$ as sections of coherent sheaves.

In this paper we study ideals in the algebra $A(\mathbb{R}^d)$ of real analytic functions on $\mathbb{R}^d$ and, in particular, under which conditions they are complemented, i.e. there is a continuous linear projection onto an ideal J. This question has been studied in the papers [21, 22] in the following cases: in [21] it was assumed that

$$J = J_X(\mathbb{R}^d) = \{f \in A(\mathbb{R}^d) \,:\, f|_X = 0\},$$

where X is a compact coherent real-analytic subvariety of $\mathbb{R}^d$ and in [22] it was assumed that $J = (P)$ the principal ideal generated by a polynomial P (which without restriction of generality may be assumed irreducible). In both cases the characterizing condition is that a complex variety V satisfies a certain condition (local Phragmèn-Lindelöf condition) in each of its real singular points. In the first case V is the global complexification of X and in the second case $V = \{z \in \mathbb{C}^d \,:\, P(z) = 0\}$.

In the present paper we consider ideals of the form $J = J_V(\mathbb{R}^d)$, where V is a complex-analytic variety in a neighborhood of $\mathbb{R}^d$ and $J_V(\mathbb{R}^d)$ is the ideal of all functions in

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$A(\mathbb{R}^d)$ such that some/any extension to a holomorphic function on a neighborhood of $\mathbb{R}^d$ vanishes in a neighborhood of X in V. We observe that this covers, in particular, both cases mentioned above. We show that if an ideal $J_V(\mathbb{R}^d)$ is complemented in $A(\mathbb{R}^d)$ then V must be the smallest complex analytic variety among those which have the same real part and, moreover, must satisfy PL_{loc} in every real singular point (Theorem 2.4). The latter condition is also sufficient for compact or homogeneous V (Theorem 3.3). This result unifies and generalizes the results of [21, 22] and removes from [21] the unnecessary assumption of coherence. In particular we get for arbitrary $\mathbb{C}$-analytic X that complementedness of the ideal $J_X(\mathbb{R}^d)$ of real analytic functions vanishing on X implies that X is of type PL (for definitions see below), which implies coherence (Corollary 2.5).

In [22] it was observed that the complementedness of the principal ideal P is equivalent to the existence of a continuous linear division operator $T : A(\mathbb{R}^d) \longrightarrow A(\mathbb{R}^d)$ such that $P \cdot T(f) = f$ for $f \in (P)$. Surprisingly, we will find a partial generalization of this result for finitely generated ideals $(f_1, \ldots, f_m)$, see the remarks after the proof of Theorem 3.3, comp. Lemma 3.2.

We also observe that for any ideal J in $A(\mathbb{R}^d)$ its closure is just the ideal of sections of the coherent sheaf of ideals generated by J (in fact, the sheaf is generated also on some neighbourhood of $\mathbb{R}^d$). On the other hand, for any sheaf of ideals its set of sections over $\mathbb{R}^d$ is a closed ideal in $A(\mathbb{R}^d)$ which leads to a characterization of closed ideals in $A(\mathbb{R}^d)$ and a description of ideals of the form $J_V(\mathbb{R}^d)$ (Theorem 1.1). It might happen that there are more than one sheaf of ideals with the same set of sections on $\mathbb{R}^d$, the smallest among them is always coherent.

Let us denote by $A(X)$, $X \subseteq \mathbb{R}^d$, the space of real analytic functions on X, i.e., those functions which around every point of X develop into a convergent power series. By $H(K)$ we denote spaces of germs of holomorphic functions on K.

1 Ideals in $A(\mathbb{R}^d)$

In the present paper we study the following class of ideals in $A(\mathbb{R}^d)$. Let Ω_0 be a complex holomorphically convex neighborhood of $A(\mathbb{R}^d)$, V a complex analytic subvariety of Ω_0 and $X := V \cap \mathbb{R}^d$ its real part. We set

$$J_V(\mathbb{R}^d) = \{f \in A(\mathbb{R}^d) \,:\, f_a \in J_{V_a} \text{ for all } a \in \mathbb{R}^d\}.$$

Here f_a denotes the germ of f in $\mathscr{O}_a$, V_a the germ of V in a and J_{V_a} the ideal of V_a.

$J_V(\mathbb{R}^d)$ is also the set of all functions in $A(\mathbb{R}^d)$ such that some/any extension to a holomorphic function on a neighborhood of $\mathbb{R}^d$ vanishes in a neighborhood of X in V.

Let $\mathscr{J}_V$ be the sheaf of ideals on Ω_0 for which the stalk over a equals J_{V_a}. Then $\mathscr{J}_V$ is a coherent sheaf of ideals on Ω_0 and $J_V(\mathbb{R}^d) = \Gamma(\mathbb{R}^d, \mathscr{J}_V)$.

Let J be an arbitrary ideal in $A(\mathbb{R}^d)$. We define $J_a := \mathscr{O}_a \cdot J = \{\sum_j g_j f_j \,:\, g_j \in \mathscr{O}_a,\, f_j \in J\}$. This defines a sheaf $\mathscr{O} \cdot J$. By $\mathrm{Rad}(J_a)$ we denote the radical of J_a.

We have the following description of closed ideals in $A(\mathbb{R}^d)$ and ideals $J_V(\mathbb{R}^d)$ among them which is surprisingly parallel to the well-known complex analytic case, see [11, Sec. III.J].

Theorem 1.1 *(a) An ideal J in $A(\mathbb{R}^d)$ is closed if and only if there is a coherent sheaf $\mathscr{G}$ on a holomorphically convex neighborhood Ω_0 of $\mathbb{R}^d$ such that $J = \Gamma(\mathbb{R}^d, \mathscr{G})$. In this case we have $\mathscr{G} = \mathscr{O} \cdot J$ on $\mathbb{R}^d$.*

(b) An ideal J in $A(\mathbb{R}^d)$ is of the form $J_V(\mathbb{R}^d)$ for some complex analytic set V if and only if it is closed and $\mathrm{Rad}(J_a) = J_a$ for all $a \in \mathbb{R}^d$.

Corollary 1.2 *Any finitely generated ideal in $A(\mathbb{R}^d)$ is closed.*

Proof: By [11, Cor. III.I4], $(f_1, \ldots, f_n) = \Gamma(\mathbb{R}^d, \mathscr{G})$, where $\mathscr{G}$ is the sheaf generated by $f_1, \ldots, f_n$. □

Remarks. (1) It is worth observing that closed ideals in $A(\mathbb{R}^d)$ need not be finitely generated (see Example on page 114 Vol. III of [11]) and that the radical of a closed ideal in $A(\mathbb{R}^d)$ need not be of the form $J_V(\mathbb{R}^d)$ (use the same example as above).

(2) If $V \cap \mathbb{R}^d$ is compact (comp. the proof of [11, Vol. III, Th. J8]) or V is homogeneous (observe that the germ of the ideal at zero is generated by homogeneous polynomials and they generate also other germs via homogeneity) then $J_V(\mathbb{R}^d)$ is finitely generated.

(3) If $d = 1$ all closed ideals are determined by the so-called *multiplicity varieties* $M = ((x_i)_{i\in I}, (m_i)_{i\in I})$ for some finite or countable set I, where $(x_i)_{i\in I}$ is a discrete family in $\mathbb{R}$ and $(m_i)_{i\in I}$ is a family of positive natural numbers (so called multiplicities). Then every closed ideal J in $A(\mathbb{R})$ is of the form:

$$J(M) := \{f \in A(\mathbb{R}) : f^{(\alpha)}(x_i) = 0 \quad \text{for all } \alpha \in \mathbb{N}_0,\, i \in I,\, \alpha < m_i\}.$$

$J(M)$ is of the type $J_V(\mathbb{R}^d)$ if and only if $m_i = 1$ for every $i \in I$.

The proof of Theorem 1.1 will be contained in the following lemmata.

We set $D_n := \{x \in \mathbb{R}^d : |x| \le n\}$. First we consider ideals in $H(D_n)$. We will use several times a classical result of Cartan and Grauert that every open set in $\mathbb{R}^d$ has a basis of holomorphically convex complex neighbourhoods [3].

From [9, Théorème (I,9)] (cf. [20]) we derive:

Lemma 1.3 *$H(D_n)$ is noetherian.*

From this we obtain:

Lemma 1.4 *Every ideal J in $H(D_n)$ is closed and, moreover, there is a coherent sheaf $\mathscr{G}$ of ideals on an open holomorphically convex neighborhood of D_n such that $J = \Gamma(D_n, \mathscr{G})$.*

Proof: By Lemma 1.3, there are $f_1, \dots, f_m \in J$ such that $J = (f_1, \dots, f_m)$. Let U be an open holomorphically convex neighborhood of D_n such that $f_1, \dots, f_m \in J \cap H(U)$ and let $\mathscr{G}$ be the $\mathscr{O}$-sheaf on U generated by $f_1, \dots, f_m$. It is coherent. Obviously $J \subset \Gamma(D_n, \mathscr{G})$. To show the converse inclusion let $f \in \Gamma(D_n, \mathscr{G})$ and $W \subset U$ be a holomorphically convex neighborhood of D_n such that $f \in \Gamma(W, \mathscr{G})$. We consider the exact sequence of sheaves on W

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{O}^m \xrightarrow{q} \mathscr{G} \longrightarrow 0$$

where $q(g_1, \dots, g_m) = \sum_j g_j f_j$ and $\mathscr{K}$ denotes the kernel sheaf. Since it is coherent (see [12, Chap. IV, Sec. B, 12. Proposition] and [12, Chap. IV, Sec. C, 1. Theorem] or [11, Th. III.B15]) we obtain that $H^1(W, \mathscr{K}) = 0$ hence $f_1, \dots, f_m$ generate $\Gamma(W, \mathscr{G})$. Therefore $f = \sum_j g_j f_j$ with $g_1, \dots, g_m \in H(W)$ which implies $f \in J$.

Since $H(D_n)$ is an LS-space, by [7, 7.2], it suffices to show that $\Gamma(D_n, \mathscr{G})$ is sequentially closed. Let us take a sequence $(f_j) \subset \Gamma(D_n, \mathscr{G})$ convergent to $f \in H(D_n)$. Thus (f_j) and f are holomorphic on a fixed complex neighbourhood U of D_n and (f_j) is uniformly convergent there to f. By the closure of modules theorem [11, Cor. II.H12], $f \in \Gamma(D_n, \mathscr{G})$. □

Now, we consider ideals in $A(\mathbb{R}^d)$.

Lemma 1.5 *Let J be an ideal in $A(\mathbb{R}^d)$and J_n be the ideal in $H(D_n)$ generated by J.*
1. J_n is generated by finitely many elements in J.
2. J is dense in J_n with respect to the topology of $H(D_n)$.
3. $J = \mathrm{proj}_n J_n$ if and only if J is closed in $A(\mathbb{R}^d)$.

Proof: 1. By Lemma 1.3, J_n is generated by finitely many elements in J_n each of which is generated by finitely many elements in J.

2. Let $f \in J_n$ hence $f = \sum_{j=1}^m g_j f_j$, $g_j \in H(K)$, $f_j \in J$. For every g_j there is a sequence of polynomials $g_j^{(n)}, n \in \mathbb{N}$ which converges to g_j in $H(K)$. Then $f_n := \sum_{j=1}^m g_j^{(n)} f_j \in J$ for all n and f_n converges to f in $H(K)$.

3. One inclusion follows from 2. the other from Lemma 1.4. □

Lemma 1.6 *For every sheaf of ideals $\mathscr{G}$ the set of continuous sections $\Gamma(\mathbb{R}^d, \mathscr{G})$ is a closed ideal in $A(\mathbb{R}^d)$.*

Proof: Since

$$\Gamma(\mathbb{R}^d, \mathscr{G}) = \mathrm{proj}_{n \in \mathbb{N}} \Gamma(D_n, \mathscr{G}).$$

this follows from Lemma 1.4 or Lemma 1.5. $\square$

Lemma 1.7 *Let J be an ideal in $A(\mathbb{R}^d)$. Then the closure of J is equal to $\Gamma(\mathbb{R}^d, \mathscr{O} \cdot J)$ and $\mathscr{O} \cdot J$ is the restriction of a coherent sheaf of ideals $\mathscr{G}$ defined on some holomorphically convex neighbourhood of $\mathbb{R}^d$, in particular, $\mathscr{O} \cdot J$ is coherent on $\mathbb{R}^d$.*

Proof: In the first step we consider $D_n \subset \mathbb{R}^d$. By Lemma 1.5, there are generators $f_1, \ldots, f_m \in J$ of the ideal J_n in $H(D_n)$ generated by J. There is a holomorphically convex neighborhood Ω_n of $\mathbb{R}^d$ such that all f_j extend to holomorphic functions on Ω_n. By $\mathscr{G}_n$ we denote the subsheaf of $\mathscr{O}$ on Ω_n generated by $f_1, \ldots, f_m$. It is coherent and $J_n = \Gamma(D_n, \mathscr{G}_n)$ (see the proof of Lemma 1.4).

For $\mathscr{G}_n$ we choose generators $f_1^n, \ldots, f_m^n$ and for $\mathscr{G}_{n+1}$ generators $f_1^{n+1}, \ldots, f_M^{n+1}$. We may assume that the generators for $\mathscr{G}_n$ are among those for $\mathscr{G}_{n+1}$ and $\Omega_{n+1} \subset \Omega_n$. For every $z \in \Omega_{n+1}$ the stalk of $\mathscr{G}_{n+1}$ is larger than the stalk of $\mathscr{G}_n$ (more generators!). For every $k \in \{1, \ldots, M\}$ there is a representation in $H(D_n)$ of the form

$$f_k^{n+1} = \sum_{j=1}^{m} g_{k,j} f_j^n, \qquad g_{k,j} \in H(D_n).$$

This extends to a representation of the same form in $H(U)$ with $g_{k,j} \in H(U)$, where $U \subset \Omega_{n+1}$ is a neighborhood of D_n. Therefore on U the stalks of $\mathscr{G}_n$ and $\mathscr{G}_{n+1}$ coincide.

Thus, we may find $\varepsilon_n > \varepsilon_{n+1}$ such that $\mathscr{G}_n$ and $\mathscr{G}_{n+1}$ coincide on $U_n := \{z = x + iy : |x| < n, \, |y| < \varepsilon_{n+1}\} \subset \Omega_{n+1}$. We set $U = \bigcup_n U_n$. Then the sequence of sheaves $(\mathscr{G}_n)$ defines a coherent sheaf $\mathscr{G}$ on U which on $\mathbb{R}^d$ equals $\mathscr{O} \cdot J$. We find a holomorphically convex neighborhood $\Omega \subset U$ of $\mathbb{R}^d$. By definition we have $J \subset \Gamma(\mathbb{R}^d, \mathscr{G})$ and, by Lemma 1.5, J is dense in $\Gamma(\mathbb{R}^d, \mathscr{G})$. By Lemma 1.6, the latter ideal is closed. $\square$

This completes the proof of Theorem 1.1 (a).

Proof of Theorem 1.1 (b): Necessity is obvious. To prove sufficiency we choose a coherent sheaf $\mathscr{G}$ on a neighborhood $\Omega \subset \mathbb{R}^d$ such that $J = \Gamma(\mathbb{R}^d, \mathscr{G})$, hence $\mathscr{G} = \mathscr{O} \cdot J$ on $\mathbb{R}^d$. We set $V = \{z \in \Omega : \mathscr{G}_a \neq \mathscr{O}_a\}$. Then V is a complex-analytic set in Ω and $J \subset J_V(\mathbb{R}^d)$. We assume $\mathrm{Rad}(J_a) = J_a$ for all $a \in \mathbb{R}^d$. For $f \in J_V(\mathbb{R}^d)$ we have then, by Rückert's Nullstellensatz [11, Th. II.E2], $f \in J_a$ for all $a \in \mathbb{R}^d$, which implies $f \in \Gamma(\mathbb{R}^d, \mathscr{G}) = J$. $\square$

Let now $X \subset \mathbb{R}^d$ be an arbitrary set. We define the ideal

$$J_X(\mathbb{R}^d) = \{f \in A(\mathbb{R}^d) \,:\, f|_X = 0\}.$$

We now set it in relation to the previously defined ideals of type $J_V(\mathbb{R}^d)$.

A set $X \subset \mathbb{R}^d$ is called $\mathbb{C}$-analytic if it is the common zero set of a family (equivalently, finite family) of functions in $A(\mathbb{R}^d)$ (see [25, p. 154], [3, Proposition 15]). Therefore $J_X(\mathbb{R}^d) = J_{\widehat{X}}(\mathbb{R}^d)$, where $\mathbb{C}$-analytic set $\widehat{X}$ is defined as $\{x : f(x) = 0, \ \forall f \in A(\mathbb{R}^d)\}$. It can happen that $X \neq \mathbb{R}^d$ but $J_X(\mathbb{R}^d) = 0$ (see [3, Proposition 16]) which means that $\widehat{X} = \mathbb{R}^d$. In this case our ideal becomes trivial and we exclude this case. Without restriction of generality we may assume that X is a $\mathbb{C}$-analytic subvariety of $\mathbb{R}^d$.

It is also known that $\mathbb{C}$-analytic sets are exactly those which support a coherent sheaf of ideals [3, Proposition 15]. If X is $\mathbb{C}$-analytic it has a (global) complexification V. That is, there is a complex subvariety V of some holomorphically convex neighborhood of $\mathbb{R}^d$ in $\mathbb{C}^d$, such that for every $f \in A(\mathbb{R}^d)$, which vanishes on X, the germ of f on V also vanishes (see [17, Proposition 16]). Thus $J_X(\mathbb{R}^d) = J_V(\mathbb{R}^d)$ and the problem of complementedness of $J_X(\mathbb{R}^d)$ is a particular case of the problem considered in the present paper. Let us observe that every $\mathbb{C}$-analytic set X has the weak extension property (see [18, Theorem 1]), i. e. every real analytic function on X which extends to a neighborhood of X in $\mathbb{R}^d$, extends to the whole of $\mathbb{R}^d$. Thus the problem of complementedness of $J_X(\mathbb{R}^d)$ is equivalent to the problem of existence of linear continuous operator from the space of germs of real analytic functions over X to $A(\mathbb{R}^d)$.

2 Complementedness of ideals $J_V(\mathbb{R}^d)$, necessary condition

To analyze the consequences of the existence of a continuous linear projection in $A(\mathbb{R}^d)$ onto $J_V(\mathbb{R}^d)$ we proceed like in [22].

Let Ω_0 be a complex holomorphically convex neighborhood of $A(\mathbb{R}^d)$, V a complex analytic subvariety of Ω_0 and $X := V \cap \mathbb{R}^d$ its real part. Set

$$H_V(X) := \{(f, \Omega) \,:\, \Omega \text{ open neighborhood of } X \text{ in } V, f \text{ holomorphic on } \Omega\}$$

with $(f_1, \Omega_1) = (f_2, \Omega_2)$ if there exists an open set $\Omega \subset V$ with $X \subset \Omega \subset \Omega_1 \cap \Omega_2$ and $f_1|_\Omega = f_2|_\Omega$.

We equip $H_V(X)$ with the projective limit topology of the spaces $H_V(X \cap D_n)$, which denotes the space of germs of holomorphic functions defined on a neighborhood of $X \cap D_n$ in V. This topology makes it a (PLS)-space (comp. [6]).

We define the natural restriction map $\rho : A(\mathbb{R}^d) \longrightarrow H_V(X)$ by $\rho(f) = F|_V$, where F is an extension of f to a holomorphic function on an open neighborhood of $\mathbb{R}^d$.

Lemma 2.1 *The sequence*

$$0 \longrightarrow J_V(\mathbb{R}^d) \hookrightarrow A(\mathbb{R}^d) \stackrel{\rho}{\longrightarrow} H_V(X) \longrightarrow 0 \tag{1}$$

is topologically exact. Thus $J_V(\mathbb{R}^d)$ is complemented in $A(\mathbb{R}^d)$ if and only if the exact sequence (1) splits, that is if and only if ρ has a continuous linear right inverse.

Proof: Obviously, we have $\ker \rho = J_V(\mathbb{R}^d)$. We have to show the surjectivity of ρ. For given (f, Ω) we find, by use of the Cartan-Grauert Theorem, an open pseudoconvex set $\omega \subset \mathbb{C}^d$ so that $\mathbb{R}^d \subset \omega$ and $\omega \cap V \subset \Omega$. By the Cartan-Oka theory there exists an $F \in H(\omega)$ so that $F|_{\omega \cap V} = f$.

To prove the openness of ρ we use the same argument as before to show that for every n the natural restriction map $\rho_n : A(D_n) = H(D_n) \longrightarrow H_V(X \cap D_n)$ is surjective. So the family (ρ_n) constitutes a surjective map between the two projective spectra:

$$(\rho_n) : (H(D_n))_{n\in\mathbb{N}} \to (H_V(X \cap D_n))_{n\in\mathbb{N}}.$$

Since, by [6, Proposition 1.5], $\mathrm{Proj}^1_n\, H(D_n) = \mathrm{Proj}^1\, A(\mathbb{R}^d) = 0$ it follows from the long exact cohomology sequence (see [24, 3.1.8])

$$0 \longrightarrow J_V(\mathbb{R}^d) \longrightarrow A(\mathbb{R}^d) \stackrel{\rho}{\longrightarrow} H_V(X) \longrightarrow \mathrm{Proj}^1 \ker \rho \longrightarrow \dots$$
$$\dots \longrightarrow \mathrm{Proj}^1_{n\in\mathbb{N}} H(D_n) \stackrel{(\rho_n)}{\longrightarrow} \mathrm{Proj}^1_{n\in\mathbb{N}} H_V(X \cap D_n) \longrightarrow 0$$

that also $\mathrm{Proj}^1_n\, H_V(X \cap D_n) = 0$ and $H_V(X) = \mathrm{limproj}_n\, H_V(X \cap D_n)$ is ultrabornological, by [24, Cor. 3.3.10] or [6, Proposition 1.4]. Openness of ρ follows from de Wilde's open mapping theorem see e.g. [16, 24.30]). □

Remark: If we set $K_n = \ker \rho_n$ then we obtain a reduced projective spectrum with $\mathrm{limproj}_n K_n = J_X(\mathbb{R}^d)$. However we have not necessarily $K_n = J_V(D_n) = \{f \in H(D_n) : f|_X = 0\}$. As an example we may take a shifted "Cartan's umbrella", that is we set $P(x_1, x_2, x_3) = (x_1^2 + x_2^2)(x_3 - 2) = x_2^3$ and $X = \{x \in \mathbb{R}^3 : P(x) = 0\}$. Then $V = \{z \in \mathbb{C}^3 : P(z) = 0\}$, $K_1 = P \cdot H(D_1)$, but $J_X(D_1) = x_1 \cdot H(D_1) + x_2 \cdot H(D_1)$.

We have to recall some definitions. Let V_a be the germ of a complex variety in the real point a and identify, by abuse of language, V_a with a representation, which is always assumed to be bounded. We define

$$\omega_{a,V}(z) = \limsup_{\zeta \to z} \sup\{u(\zeta) : \text{ u plurisubharmonic on } V_a,\ u \le 1, u \le 0 \text{ on } X_a\}.$$

Let $V_a \sim \tilde{V}_a$. For two germs of functions ω_a on V_a and $\tilde{\omega}_a$ on $\tilde{V}_a$ we set $\omega_a \prec \tilde{\omega}_a$ if there is a constant $C > 0$ so that $\omega_a \leq C\tilde{\omega}_a$ in a neighborhood of a in $V_a \cap \tilde{V}_a$. If $\omega_a \prec \tilde{\omega}_a$ and $\tilde{\omega}_a \prec \omega_a$ we write $\omega_a \sim \tilde{\omega}_a$ and call such germs equivalent. Then we obtain: Up to equivalence the germ of $\omega_{a,V}$ depends only on the germ of V_a.

The following condition was introduced by Hörmander in [14] in connection with his study of surjectivity of linear partial differential operators with constant coefficients on the space of real analytic functions. Later on it was studied extensively in papers of Braun, Meise, Taylor and the second named author in connection with a characterization of those linear partial differential operators with constant coefficients which admit linear continuous right inverse on spaces of smooth functions.

Definition 2.2 *V_a satisfies PL_{loc} if $\omega_{a,V} \prec |\mathrm{Im}\, z|$.*

A complex variety V satisfies PL_{loc} in the real point a if its germ in a satisfies PL_{loc}. The germ X_a of a real analytic variety in the real point a is of type PL if its complexification satisfies PL_{loc}. X is of type PL if is of type PL in every point.

Examples are summarized in [21, Section 6] and [22, Section 8]. In particular, the following holds (see [21, Lemma 1.3, Theorem 1.9]):

Proposition 2.3 *If V at some $a \in \mathbb{R}^d$ satisfies PL_{loc}, then V_a is the complexification of X_a and X_a is coherent. In particular, if V satisfies PL_{loc} at every real point, then X is coherent and V is its global complexification.*

From now on we can proceed as in [22] and we arrive at the main result of the paper:

Theorem 2.4 *If $J_V(\mathbb{R}^d)$ is complemented in $A(\mathbb{R}^d)$, then X has finitely many connected components and V satisfies PL_{loc} in every $a \in X$.*

Proof: If (1) splits then $H_V(X)$ is a complemented subspace of $A(\mathbb{R}^d)$. If X has infinitely many connected components this is impossible since $H_V(X)$ has no continuous norm while $A(\mathbb{R}^d)$ always has a continuous norm.

The proof of PL_{loc} is analogous to that of [22, Proposition 3.4] but for the sake of convenience we give its short overview. We start with some notation. By ω_r we denote the pluricomplex Green function of D_r (see [15, Section 5, p. 207]). Then

$$D_{r,\alpha} := \{z \in \mathbb{C}^d : \omega_r(z) < \alpha\} \qquad V_{r,\alpha} := D_{r,\alpha} \cap V$$

and we denote by $|\cdot|_{r,\alpha}$, $\|\cdot\|_{r,\alpha}$ norms in $H^\infty(D_{r,\alpha})$ and $H^\infty(V_{r,\alpha})$ respectively.

For $0 < \alpha_1 < \alpha_2 < \alpha_3$ we have

$$|\cdot|_{r,\alpha_2}^{\alpha_3-\alpha_1} \leq |\cdot|_{r,\alpha_1}^{\alpha_3-\alpha_2} |\cdot|_{r,\alpha_3}^{\alpha_2-\alpha_1}$$

on $H^\infty(D_{r,\alpha_3})$ — this is due to Zahariuta [26], see the proof in [21, Lemma 3.3] where we use ω_r instead of ω. Similarly, for $0 < \alpha_1 < \alpha_2' < \alpha_2 < \alpha_3$ and for some constant $C > 0$ we have

$$\left(\|\cdot\|_{r,\alpha_2}^*\right)^{\alpha_3-\alpha_1} \leq C\left(\|\cdot\|_{r,\alpha_1}^*\right)^{\alpha_3-\alpha_2'}\left(\|\cdot\|_{r,\alpha_3}^*\right)^{\alpha_2'-\alpha_1}$$

on $H^\infty(V_{r,\alpha_1})'$ — for the latter see [22, Lemma 3.3] and the argument before it.

Clearly, if φ is a continuous linear right inverse for ρ then for every $r > 0$ there is $R > r$ such that φ induces a continuous map $\tilde{\varphi} : H_V(X \cap D_R) \longrightarrow H(D_r)$. Then, using the proof of [21, Lemma 5.3], we can prove that there is $\varepsilon > 0$ such that for all $\alpha \leq r$ there is $C_\alpha > 0$ satisfying

$$|\tilde{\varphi}(f)|_{r,\varepsilon\alpha} \leq C_\alpha \|f\|_{R,\alpha}$$

for all $f \in H^\infty(V_{R,\alpha})$.

Now, we apply the proof on page 822–823 of [22] in order to get for any $0 < \rho < r$ and β a constant A such that for any plurisubharmonic function u of the form $c \log|f(\cdot)|$ where $c > 0$ and $f \in H(V_{R,\beta})$ we have: if

$$u(z) < 0 \quad \text{for } z \in V_{R,\beta} \cap \mathbb{R}^d \qquad \text{and} \qquad u(z) < 1 \quad \text{for } z \in V_{R,\beta}$$

then

$$u(z) \leq A|\mathrm{Im}\, z|, \quad \text{for } z \in V_{r,\varepsilon\beta} \cap \{z : |\mathrm{Re} z| \leq \rho\}.$$

The previous condition holds for arbitrary plurisubharmonic functions u on $V_{R,\beta}$ — to prove that we extend u from $V_{R,\beta'}$ for $\beta' < \beta$ onto its Stein open neighbourhood Ω as a plurisubharmonic function λ using the proof of [8, Th. 5.3.1], then we find, by [15, Th. 2.5.5 and its proof], a continuous plurisubharmonic function $\tilde{u}$ on a slight smaller Stein open set Ω_1 such that $\lambda \leq \tilde{u} \leq \lambda + \delta$ for a given arbitrary $\delta > 0$ and finally, by the proof of Bremermann's theorem [2, Th. 2], we find a function w, $\tilde{u} - \delta \leq w \leq \tilde{u}$ on $V_{R,\gamma}$ for $\gamma < \beta'$, w of the form $\max_{j=1,\ldots,p} c_j \log|f_j(\cdot)|$ for f_j holomorphic. We apply the previously proved condition for w to get similar condition for u.

Finally, the variety V satisfies PL_{loc} at its arbitrary real point — to prove that apply [21, Lemma 1.2] or, more precisely, the proof of (b)⇒(a) of [1, Lemma 3.3] □

Our main result above strengthens the necessity part of [21, Theorem 2.2] (use Proposition 2.3):

Corollary 2.5 *Let X be an arbitrary subset of $\mathbb{R}^d$. If the ideal $J_X(\mathbb{R}^d)$ is complemented in $A(\mathbb{R}^d)$ then the global complexification V of the $\mathbb{C}$-analytic set $\hat{X}$ satisfies PL_{loc} at all real points, i.e, at all points in $\hat{X}$. In particular, if X is a $\mathbb{C}$-analytic set and $J_X(\mathbb{R}^d)$ is complemented in $A(\mathbb{R}^d)$ then X is of type PL and coherent.*

The main result also implies [22, Proposition 3.4]

Corollary 2.6 *Let P be a polynomial. If the principal ideal (P) generated by P is complemented in $A(\mathbb{R}^d)$ then the zero variety V of P satisfies PL_{loc} at every real point of V.*

Proof: As is shown in [22, Section 2], it suffices to prove the result for irreducible P. Then $(P) = J_V(\mathbb{R}^d)$, where V is the complex zero variety of P. Apply Theorem 2.4. □

Notice that there is a striking difference between the both corollaries above. In the latter corollary V means the complex variety of the polynomial P and in the former one V means the (global) complexification of a $\mathbb{C}$-analytic set X. To exhibit the difference we might consider the polynomial $P(x) = |x|^2$. Then V as the zero variety is $\{z \in \mathbb{C}^d : \sum_j z_j^2 = 0\}$, which does not have PL_{loc} in 0, and (P) is not complemented in $A(\mathbb{R}^d)$, however for $X = \{x \in \mathbb{R}^d : P(x) = 0\} = \{0\}$ we have $J_X(\mathbb{R}^d) = \{f \in A(\mathbb{R}^d) : f(0) = 0\}$ which is clearly complemented, and its complexification $V = \{0\}$ trivially has PL_{loc}. A more substantial example can be found in Section 4 of [22].

Let J be an ideal in $A(\mathbb{R}^d)$, then the $\mathbb{C}$-analytic set

$$\mathrm{Loc}_{\mathbb{R}}(J) := \{x \in \mathbb{R}^d : f(x) = 0 \quad \forall f \in J\}$$

is called the *real locus of the ideal J*. Finally, our main result gives:

Corollary 2.7 *Let J be a closed ideal in $A(\mathbb{R}^d)$ which is locally radical, i.e.,* $\mathrm{Rad}(J_a) = J_a$ *for every $a \in \mathrm{Loc}_{\mathbb{R}}(J)$. If J is complemented then* $\mathrm{Loc}_{\mathbb{R}}(J)$ *is coherent and of PL type, $J = \{f \in A(\mathbb{R}^d) : f|_{\mathrm{Loc}_{\mathbb{R}}(J)} = 0\}$.*

Proof: By Theorem 1.1, $J = J_V(\mathbb{R}^d)$, where V is the germ of the common zero set of J in $\mathbb{C}^d$ around $\mathbb{R}^d$. Apply Theorem 2.4. □

In general there are many different locally radical closed ideals with the same locus. For instance, let

$$J_1 := \{f \in \mathscr{A}(\mathbb{R}^2) : f(0) = 0\}, \qquad J_2 := \{f \in \mathscr{A}(\mathbb{R}^2) : f|_V \equiv 0\},$$

where $V := \{(z_1, z_2) : z_1^2 + z_2^2 = 0\}$. Then $\mathrm{Loc}_{\mathbb{R}}(J_1) = \mathrm{Loc}_{\mathbb{R}}(J_2) = \{0\}$. The first one is complemented, the second not. More generally, if $V_1 \supsetneq V_2$ are two complex analytic varieties such that $V_1 \cap \mathbb{R}^d = V_2 \cap \mathbb{R}^d$ then $J_{V_1}(\mathbb{R}^d)$ is never complemented.

3 Sufficient conditions

In case $d = 1$ the closed ideal $J(M)$ (see Remark (3) after Theorem 1.1) is complemented if and only if the set I is finite, which means that the necessary conditions from Theorem 2.4 are also sufficient.

For higher dimensions d the problem of complementedness of $J_X(\mathbb{R}^d)$ is solved for algebraic curves (or more generally for analytic covers regular at infinity) in [23] and it is proved that also in that case conditions from Theorem 2.4 are sufficient.

It was observed in [22, p. 827] that the condition $\mathrm{Proj}^1_{n\in\mathbb{N}} L(A(X), J(K_n)) = 0$ or $\mathrm{Proj}^1_{n\in\mathbb{N}} L(A(X), H(K_n)) = 0$ for a fundamental sequence (K_n) of compact subsets of Ω would be helpful in proving sufficient conditions. Unfortunately as the following example shows this is not always true.

Example. Let $X \subseteq \mathbb{R}^d$ be a set of points where the first coordinate is integer. Then $V = X$ is the complexification of X and X is of PL-type. It is easy to see that $M_f : A(\mathbb{R}^d) \to A(\mathbb{R}^d)$, $M_f(g) := f \cdot g$, where $f(z) = \sin(\pi z_1)$ for $z = (z_1, \ldots, z_d)$, is a topological embedding onto the principal ideal $J_V(\mathbb{R}^d) = (f)$ and also a topological embedding of $H(K_n)$ onto $J_V(K_n)$, where $K_n := [-n, n]^d$. Thus for every n we have the following commutative diagram with topologically exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H(K_n) & \xrightarrow{M_f} & H(K_n) & \xrightarrow{\rho|_{H(K_n)}} & A(X \cap K_n) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A(\mathbb{R}^d) & \xrightarrow{M_f} & A(\mathbb{R}^d) & \xrightarrow{\rho} & A(X) & \longrightarrow & 0
\end{array}$$

where ρ is the restriction map as in Lemma 2.1. Clearly, by the interpolation formulas we can construct a continuous linear right inverse for $\rho|_{H(K_n)}$ and the upper row splits. Thus if $\mathrm{Proj}^1_{n\in\mathbb{N}} L(A(X), H(K_n)) = 0$ then the lower sequence would split as well (this is a standard fact, for elementary explanation see, for instance, [5, p. 320], for the functor Proj^1 see [24]). On the other hand, if the lower sequence split then $A(X)$ would be a subspace of $A(\mathbb{R}^d)$. This is not the case since $A(\mathbb{R}^d)$ has a continuous norm while $A(X)$ has no continuous norm. We have proved that $\mathrm{Proj}^1_{n\in\mathbb{N}} L(A(X), J_V(K_n)) = \mathrm{Proj}^1_{n\in\mathbb{N}} L(A(X), H(K_n)) \neq 0$.

In fact, a similar example can be found in arbitrary open set $\Omega \subseteq \mathbb{R}^d$. By a slight modification of [4, Lemma 5.5], without loss of generality we may assume that that $(-1, 0)^d \subset \Omega \subset (-R, 0)^d$. We set $f(x) = \sin\left(\pi / \sum_{j=1}^d x_j\right)$ and $x = \{x \in \Omega : f(x) = 0\}$. Then $X = \bigcup_{k=1}^\infty X_k$ where $X_k = \{x \in \Omega : \sum_{j=1}^d x_j = -\frac{1}{k}\}$. To get for $K \subset \Omega$ compact a map in $\varphi \in L(A(X), H(K)$ such that $\rho \circ \varphi(f) = f|_{X\cap K}$, we extend for

$f \in H(X)$ the function $f|_{X_k}$ constant on rays through 0 to get a function f_k in $A(\Omega)$ and then set $\varphi f(x) = \sum_{k=1}^m f_k(x) P_{k,m}(1/\sum_1^d x_j)$ where $P_{k,m}$ is a polynomial with $P_{k,m}(j) = \delta_{j,k}$ for $j = 1, \dots, m$, $m \in \mathbb{N}$ large enough.

Then by the same method as above one can prove that $\mathrm{Proj}^1_{n\in\mathbb{N}} L(A(X), J_V(K_n)) = \mathrm{Proj}^1_{n\in\mathbb{N}} L(A(X), H(K_n)) \neq 0$.

The problem, whether for every *connected* real analytic variety of type PL the ideal $J_X(\mathbb{R}^d) = J_V(\mathbb{R}^d)$, V the complexification of X, is complemented or, which is the same, there exists a continuous linear extension operator $A(X) \longrightarrow A(\mathbb{R}^d)$ is still unsolved, however there are two special cases where we can prove this, one is the case of compact X, which is shown in [21] the other that of a homogeneous variety, where variation of the arguments in [22] leads to success.

For the latter we need some preparation. For $0 \leq r < \rho \leq \infty$ we put $D_r^\rho = \{x : r \leq \|x\| \leq \rho\}$. Here $\| \ \|$ denotes the euclidean norm. We use and quote here for the purpose to fix notation [22, Lemma 6.2]:

Lemma 3.1 *For any $0 < r < \rho$ there are σ_1, σ_2 with $0 < \sigma_1 < r < \rho < \sigma_2$ and continuous linear maps $\psi_0 : H(D_{\sigma_1}^{\sigma_2}) \longrightarrow H(D_0^\rho)$ and $\psi_\infty : H(D_{\sigma_1}^{\sigma_2}) \longrightarrow H(D_r^\infty)$ so that $\psi_0 f + \psi_\infty f = f$ on D_r^ρ.*

Like in [21, §4] using the tame splitting theorem of [19, Th. 6.1] we obtain:

Lemma 3.2 *For every $0 \leq r < \rho < \infty$ and for any closed finitely generated ideal $J = (f_1, \dots, f_m)$ in $H(D_r^\rho)$, $f_1, \dots, f_m \in A(\mathbb{R}^d)$, there exist a continuous linear map $\chi_r^\rho : J(D_r^\rho) \to H(D_r^\rho)^m$ such that with $\chi_r^\rho(f) = (g_1, \dots, g_m)$ we have $\sum_{j=1}^m g_j f_j = f$, for every $f \in J(D_r^\rho)$.*

Proof: We give the proof for the sake of convenience. Let us define the map:

$$S : H(D_r^\rho)^m \longrightarrow J(D_r^\rho), \qquad S(g_1, \dots, g_m) := g_1 \cdot f_1 + \dots + g_m \cdot f_m.$$

Clearly, S is continuous and, by the remarks above, surjective. Since the domain and the range are LS-spaces, S is open.

The kernel of S on $A(\mathbb{R}^d)$ generates a coherent sheaf of modules by the theorem of Oka [11, Vol. III Th. B10]. By the Cartan-Oka theory, there are finitely many sections $h_1, \dots, h_n$ of this sheaf such that the map:

$$T : H(D_r^\rho)^n \longrightarrow \ker S(D_r^\rho), \qquad T(g_1, \dots, g_n) := g_1 \cdot h_1 + \dots + g_n \cdot h_n$$

is surjective. By the same arguments as for S the map T is continuous and open. If $\hat{T} : H(D_r^\rho)^n / \ker T \longrightarrow H(D_r^\rho)^m$ is the map induced by T, then we get the following topologically exact sequence of LS-spaces:

$$0 \longrightarrow H(D_r^\rho)^n / \ker T \xrightarrow{\hat{T}} H(D_r^\rho)^m \xrightarrow{S} J(D_r^\rho) \longrightarrow 0. \tag{2}$$

We will show that it splits using the tame splitting theorem [19, Th. 6.1] for the dual sequence

$$0 \longrightarrow (J(D_r^\rho))' \xrightarrow{S'} (H(D_r^\rho)^m)' \xrightarrow{\hat{T}'} (H(D_r^\rho)^n / \ker T)' \longrightarrow 0. \tag{3}$$

We define χ_r^ρ as a linear continuous right inverse for S.

Let ω be the pluricomplex Green function of D_r^ρ (see [15]). We represent $H(D_r^\rho)$ as inductive limit of $H^\infty(D_{r,\alpha}^\rho)$ for $\alpha > 0$ where

$$D_{r,\alpha}^\rho := \{z : \omega(z) < \alpha\}$$

and analogous spectra will be used for other spaces. The sup-norms in the spaces $J(D_{r,\alpha}^\rho)$, $H^\infty(D_{r,\alpha}^\rho)^m$ and $H^\infty(D_{r,\alpha}^\rho)^n$ will be denoted respectively by $|\cdot|_\alpha$, $\|\cdot\|_\alpha$ and $|||\cdot|||_\alpha$. By the Cartan-Oka theory, $S(H^\infty(D_{r,\beta}^\rho)^m) \supseteq J(D_{r,\alpha}^\rho)$ for any $0 < \beta < \alpha$. Thus, by the closed graph theorem,

$$C\|\cdot\|_\beta \le |S(\cdot)|_\alpha \le D\|\cdot\|_\alpha \qquad \text{for some constants } C,\ D \text{ and any } 0 < \beta < \alpha. \tag{4}$$

Analogously, we get

$$c|||\cdot|||_\beta \le \|T(\cdot)\|_\alpha \le d|||\cdot|||_\alpha \qquad \text{for some constants } c,\ d \text{ and any } 0 < \beta < \alpha. \tag{5}$$

This implies that the sequence (3) is tame if we equip the spaces in the sequence with sequences of seminorms

$$\left(|\cdot|^*_{\frac{1}{k}}\right)_{k\in\mathbb{N}} \qquad \left(\|\cdot\|^*_{\frac{1}{k}}\right)_{k\in\mathbb{N}} \qquad \left(|||\cdot|||^*_{\frac{1}{k}}\right)_{k\in\mathbb{N}}.$$

By [21, Lemma 4.1], for any $0 < \alpha_1 < \alpha_2 < \alpha_2' < \alpha_3$ we have for dual norms

$$\left(|||\cdot|||^*_{\alpha_2}\right)^{\alpha_3-\alpha_1} \le C\left(|||\cdot|||^*_{\alpha_1}\right)^{\alpha_3-\alpha_2'}\left(|||\cdot|||^*_{\alpha_3}\right)^{\alpha_2'-\alpha_1}.$$

Clearly the same inequality holds for norms dual to quotient norms on $H(D_r^\rho)^n / \ker T$. As in [21, Lemma 3.3] we get also for any $0 < \alpha_1 < \alpha_2 < \alpha_3$ for norms on $H(D_r^\rho) \supseteq J(D_r^\rho)$ and thus also for norms on $J(D_r^\rho)$:

$$\left(|\cdot|_{\alpha_2}\right)^{\alpha_3-\alpha_1} \le C\left(|\cdot|_{\alpha_1}\right)^{\alpha_3-\alpha_2}\left(|\cdot|_{\alpha_3}\right)^{\alpha_2-\alpha_1}.$$

Thus for $\theta_k = \frac{k+1}{2k}$ and $\frac{k}{2(k-1)} > \tau_k > \frac{k+1}{2k}$ we get:

$$|\cdot|_{\frac{1}{k}} \leq c_k \left(|\cdot|_{\frac{1}{k-1}}\right)^{1-\theta_k} \left(|\cdot|_{\frac{1}{k+1}}\right)^{\theta_k}$$

and

$$|||\cdot|||^*_{\frac{1}{k}} \leq c_k \left(|||\cdot|||^*_{\frac{1}{k-1}}\right)^{1-\tau_k} \left(|||\cdot|||^*_{\frac{1}{k+1}}\right)^{\tau_k}.$$

Since $\theta_k > \tau_{k+1}$ the assumptions of [19, Th. 6.1] are satisfied and (3) splits. In fact, we need to replace sup-norms by hilbertian norms by slightly changing sets $D^{\rho}_{r,\alpha}$ and taking Bergman-Hilbert norms on these new sets. By reflexivity, also (2) splits. □

Theorem 3.3 *Let $V \subset \mathbb{C}^d$ be a complex analytic set and let $X := V \cap \mathbb{R}^d$ be $\mathbb{C}$-analytic. If either X is compact or V homogeneous then the following are equivalent:*

1. *$J_V(\mathbb{R}^d)$ is complemented.*
2. *$J_X(\mathbb{R}^d)$ is complemented and V is a global complexification of X or, equivalently $J_X(\mathbb{R}^d) = J_V(\mathbb{R}^d)$.*
3. *There exists a continuous linear extension operator $A(X) \longrightarrow A(\mathbb{R}^d)$ and V is the global complexification of X.*
4. *X is of type PL and V is the global complexification of X.*

Each of these (equivalent) conditions implies that X is coherent.

Proof: 2. ⇒ 1. ⇒ 4. holds by Theorem 2.4 and X is coherent by Proposition 2.3.

3. ⇒ 2. If we have 3. then, in particular, the restriction map $\rho : A(\mathbb{R}^d) \longrightarrow A(X)$ is surjective, hence we have the exact sequence

$$0 \longrightarrow J_X(\mathbb{R}^d) \hookrightarrow A(\mathbb{R}^d) \xrightarrow{\rho} A(X) \longrightarrow 0.$$

By 3. it splits, that is $J_X(\mathbb{R}^d) = J_V(\mathbb{R}^d)$ is complemented in $A(\mathbb{R}^d)$.

4. ⇒ 3. If X is compact then this is [21, Theorem 2.2] based on the tame splitting theorem [19, Th. 6.1]. So we assume that X is homogeneous, by 4. and Proposition 2.3 it is coherent and $H_V(X) = A(X)$. We adapt the proof of [22, Proposition 6.3] and set $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$ and $X_* = \mathbb{R}^d_* \cap X$. By [22, Theorem 1.5] there exists a continuous linear extension operator $\varphi_\infty : A(X) \longrightarrow A(\mathbb{R}^d_*)$. We fix some $0 < r < \rho$ and, according to Lemma 3.1, we find $0 < \sigma_1 < r < \rho < \sigma_2$ and operators ψ_0, ψ_∞. Since X is of type

PL and by [22, Theorem 8.1], there is a continuous linear map $\varphi_0 : A(X) \longrightarrow H(D_0^{\sigma_2})$ such that $\varphi_0 f = f$ on $X \cap D_0^{\sigma_2}$.

The ideal $J_V(\mathbb{R}^d) = J_X(\mathbb{R}^d)$ is finitely generated by $(f_1, \ldots, f_m)$ (see the remark after Corollary 1.2) and we can apply Lemma 3.2. For $f \in A(X)$ we put $(g_1, \ldots, g_m) := \chi_{\sigma_1}^{\sigma_2}(\varphi_0 f - \varphi_\infty f)$ and define the map $\varphi : A(X) \to A(\mathbb{R}^d)$ by the following formulas:

$$\varphi f := \varphi_0 f - \sum_{j=1}^m f_j\, \psi_0(g_j) \text{ on } D_0^\rho \text{ and } \varphi f := \varphi_\infty f + \sum_{j=1}^m f_j\, \psi_\infty(g_j) \text{ on } D_r^\infty.$$

On D_r^ρ we have

$$\sum_{j=1}^m f_j\, \psi_0(g_j) + \sum_{j=1}^m f_j\, \psi_\infty(g_j) = \sum_{j=1} f_j g_j = \varphi_0 f - \varphi_\infty f.$$

Therefore $\varphi : A(X) \longrightarrow A(\mathbb{R}^d)$ is a well defined linear continuous right inverse for ρ in (1) and the assertion is proved. □

Comments. (1) The ideals $J_V(\mathbb{R}^d)$ are finitely generated under the assumptions of Theorem 3.3. Let $f_1, \ldots, f_m$ be generators of an arbitrary ideal J in $A(\mathbb{R}^d)$ then the map

$$S : A(\mathbb{R}^d)^m \longrightarrow J, \qquad S(g_1, \ldots, g_m) := g_1 \cdot f_1 + \cdots + g_m \cdot f_m,$$

is surjective. If there is a linear continuous map $T : A(\mathbb{R}^d) \longrightarrow A(\mathbb{R}^d)^m$ such that for $T(f) = (g_1, \ldots, g_m)$ for every $f \in J$ we have

$$f = f_1 g_1 + \cdots + f_m g_m.$$

then $S \circ T$ is a projection from $A(\mathbb{R}^d)$ onto J. In [22, Lemma 2.1] it is observed that for principal ideals (i.e. $m = 1$) the converse holds as well.

Applying Lemma 3.2 we can construct for an arbitrary finitely generated complemented ideals J and $n \in \mathbb{N}$ a map $T_n : A(\mathbb{R}^d) \longrightarrow H(D_n)^m$ such that for $(g_1, \ldots, g_m) = T(f)$ holds $f = f_1 g_1 + \cdots + f_m g_m$ but only on D_n. To glue T_n to one map T we need vanishing of $\operatorname{Proj}^1 L(A(\mathbb{R}^d), \ker S) = 0$ which unfortunately is not known.

(2) By the above and [21], [22], [23], we observe that the necessary conditions of Theorem 2.4 are also sufficient whenever the "compact part" of X is essential while an extension "around infinity" we get by some other reasons, somehow for free. This suggests that there is an additional necessary condition describing the behavior of X or V at infinity which must be added to Theorem 2.4 in order to obtain a characterization.

References

[1] R. Braun, R. Meise, B. A. Taylor, The geometry of analytic varieties satisfying the local Phragmén-Lindelöf condition and a geometric characterization of the partial differential operators that are surjective on $\mathscr{A}(\mathbb{R}^4)$, *Trans. Amer. Math. Soc.* **356** (2004), 1315–1383.

[2] H. J. Bremermann, On the conjecture of the equivalence of the plurisubharmonic functions and the Hartogs functions, *Math. Ann.***131** (1956), 76–86.

[3] H. Cartan, Variétés analytiques réelles et variétés analytiques complexes, *Bull. Soc. Math. France* **85** (1957), 77–99.

[4] P. Domański, L. Frerick, D. Vogt, Fréchet quotients of spaces of real analytic functions, *Studia Math.* **159** (2003), 229–245.

[5] P. Domański, M. Mastyło, Characterization of splitting for Fréchet-Hilbert spaces via interpolation, *Math. Ann.* **339** (2007), 317–340.

[6] P. Domański, D. Vogt , Linear topological properties of the space of real analytic functions on the real line, in: *Recent Progress in Functional Analysis* (K. D. Bierstedt, J. Bonet, M. Maestre, J. Schmets, eds.), Proc. Valencia 2000, North-Holland Math. Studies, Amsterdam 2001, pp. 113–132.

[7] K. Floret, Lokalkonvexe Sequenzen mit kompakten Abbildungen, *J. reine angew. Math.* **247** (1971), 155–195.

[8] J. E. Fornaess, R. Narasimhan, The Levi problem and complex spaces with singularities, *Math. Ann.* **248** (1980), 47–72.

[9] J. Frisch, Points de platitude d'un morphisme d'espaces analytiques complex, *Invent. Math.* **4** (1967), 118–138.

[10] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1953).

[11] R. Gunning, *Introduction to Holomorphic Functions of Several Variables*, Wadsworth & Brooks/Cole, Pacific Grove 1990.

[12] R. Gunning, H. Rossi, *Analytic functions of several complex variables,* Prentice Hall, Englewood Cliffs 1965.

[13] L. Hörmander, *An introduction to complex analysis in several complex variables*, North-holland, Amsterdam 1990.

[14] L. Hörmander, On the existence of real analytic solutions of partial differential equations with constant coefficients, *Invent. Math.* **21** (1973), 151–182.

[15] M. Klimek, *Pluripotential Theory*, Clarendon Press, Oxford 1991.

[16] R. Meise, D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford 1997.

[17] R. Narasimhan, *Introduction to the theory of analytic spaces,* Lecture notes in Mathematics **25,** Springer-Verlag, Berlin-Heidelberg-New York 1966.

[18] G. Nardelli, A. Tancredi, A note on the extension of analytic functions off real analytic subset, *Revista Matématica de la Universidad Complutense de Madrid* **9** (1996), 85–97.

[19] M. Poppenberg, D. Vogt, A tame splitting theorem for exact sequences of Fréchet spaces, *Math. Z.* **219** (1995), 141–161.

[20] Y.-T. Siu, Noetherianness of rings of holomorphic functions on Stein compact subsets, *Proc. Amer. Math. Soc.* **21** (1969), 483–489.

[21] D. Vogt, Extension operators for real analytic functions on compact subvarieties of $\mathbb{R}^d$, *J. reine angew. Math.* **606** (2007), 217–233.

[22] D. Vogt, A division theorem for real analytic functions, *Bull. London Math. Soc.* **39,** no. 5, 818–828.

[23] D. Vogt, Complemented ideals in $A(\mathbb{R}^d)$ of algebraic curves, *Arch. Math.* to appear.

[24] J. Wengenroth, Derived functors in functional analysis, *Lect. Notes Math.* **1810***,* Springer, Berlin 2003.

[25] H. Whitney, F. Bruhat, Quelques propriétés fondamentales des ensembles analytiques-réels, *Comm. Math. Helv.* **33** (1959), 132–160.

[26] V. P. Zahariuta, Extremal plurisubharmonic functions, Hilbert scales and isomorphisms of spaces of analytic functions of several variables, I, II, *Theory of functions, functional analysis and their applications, Kharkov* **19** (1974), 133-157; **21** (1974), 65–83 (Russian).

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