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Some Higher-Order Families of Methods for Finding Simple Roots of Nonlinear Equations

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Abstract

In this paper, a new fifth-order family of methods free from second derivative is obtained. This new iterative family contains the King's family, which is one of the most well-known family of methods for solving nonlinear equations, and some other known methods as its particular case. To illustrate the efficiency and performance of proposed family, several numerical examples are presented. Numerical results illustrate better efficiency and performance of the presented methods in comparison with the other compared fifth-order methods. Due to that, they can be effectively used for solving nonlinear equations.

Keywords: *Iterative methods, Simple-root of nonlinear equations, Newton's method.*

1 Introduction

In this paper, we propose an iterative method to find a simple root a of a nonlinear equation $f(x)=0$. i.e., $f(a)=0$ and $f'(a)\neq 0$.

Nonlinear equations arise in a wide variety of forms in all branches of science, engineering, and technology. In recent years, a large number of methods of different order have been proposed and analyzed for solving nonlinear equations. For example, we refer the readers to [1-8] and the references therein.
It's well-known that the Newton's method is the most widely used (second-order) method for solving such equations, giving by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

Here, our approach is based on fifth-order method defined in [5], as

$$x_{n+1} = y_n - \frac{f'(y_n)+3f'(x_n)}{5f'(y_n)-f'(x_n)}.\frac{f(y_n)}{f'(x_n)}, \tag{2}$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{3}$$

Throughout the rest of this section, y_n is defined by (3).
This paper is organized as follows: In Section 2, we consider a general iterative scheme, analyze it to present a family of fifth-order methods then several known special cases of this family are listed. Section 3 is devoted to numerical comparisons between the results obtained in this work and some known iterative methods. Finally, conclusions are drawn in the last section.

2 Development of Method and Convergence Analysis

In this section to derive a fifth order family of methods, we suggest and analyze the following iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - H(u_n)\frac{f(y_n)}{f'(x_n)} \tag{4}$$

where $H(u)$ is a function to be determined such that the iterative method defined by (4) has the order of convergence five, and

$$u_n = \frac{f'(y_n)}{f'(x_n)}. \tag{5}$$

It can be obviously followed that for $H(u)=\dfrac{3+u}{-1+5u}$, the iterative scheme (4), reduces to the method of Kou and Li (2).
For the proposed family of methods (4) we have following analysis of convergence.

Theorem 1. *Let $a\in I$ be a simple root of a sufficiently differentiable function $f:I\to\Re$ on an open interval which contains x_0 as a close initial approximation to a. If satisfy $H(u)$ the conditions*

$$H(1)=1,\ H'(1)=-1,\ H''(1)=5/2 \qquad (6)$$

then the method defined by (2) and (3) is of order at least five.

Proof. Let a be a simple zero of f. Consider the iteration function F defined by

$$F(x)=x-\frac{f(x)}{f'(x)}-H(u(x))\frac{f(y(x))}{f'(x)}$$

where

$$y(x)=x-\frac{f(x)}{f'(x)}.\text{ and } u(x)=\frac{f'(y(x))}{f'(x)}.$$

In view of an elementary, tedious evaluation of derivatives of F, we employ the symbolic computation of the Maple package to compute the Taylor expansion of $F(x_n)$ around $x=a$. We find after simplifying that

$$x_{n+1}=F(x_n)=a+K_2e_n^2+K_3e_n^3+K_4e_n^4+O(e_n^5).$$

Where

$$K_2=\left(1-H(1)\right)c_2,$$
$$K_3=\left(2-2H(1)\right)c_3+\left(4H(1)+2H'(1)-2\right)c_2^2,$$
$$K_4=\left(3-3H(1)\right)c_4+\left(4-13H(1)-14H'(1)-2H''(1)\right)c_2^3+\left(14H(1)+7H'(1)-7\right)c_2c_3,$$

It can be easily verified that K_2,K_3 and K_4 can be vanished, when

$$H(1)=1,\ H'(1)=-1,\ H''(1)=5/2$$

This completes the proof.
From Theorem 1, we can see that the order of Newton's method can be improved three units with additional evaluations of the one function and one derivative. So

the order of convergence and computational efficiency of the method are greatly improved.
Iterative scheme (4) with some special choices for the function of $H(u)$ leads to the some known fifth-order methods, as follows:
Case1: For the function H, defined by

$$H(u)=\frac{3+u}{-1+5u}, \tag{7}$$

we obtain the fifth-order scheme (2).
Case2: For the function H giving by

$$H(u)=\frac{5+3u^2}{1+7u^2}, \tag{8}$$

in (4), a fifth-order method is obtained which has been introduced Fang et al. [6] .
Case3: For the function H giving by

$$\begin{aligned}
&H(u)=\frac{5-2u+u^2}{4u},\\
&H(u)=\frac{3+u^2}{2+2u^2},\\
&H(u)=\frac{-4}{1-6u+u^2},\\
&H(u)=\frac{5+3u^2}{1+7u},\\
&H(u)=\frac{6+u+u^2}{-1+7u+2u^2},
\end{aligned} \tag{9}$$

respectively, which all have been introduced by Fang et al. in [7].
It is worthy mentioned that the functions defined by in [8] as

$$\begin{aligned}
&H(u)=\frac{1+u}{-1+3u},\\
&H(u)=\frac{2u^2}{1-4u+5u^2},\\
&H(u)=-\frac{2}{1-4u+u^2},
\end{aligned}$$

do not satisfy the conditions (6). So, against what claimed in [8] the methods results with them aren’t the methods of order five. This fact has been also mentioned in [7].

To obtain a general family, let's consider the function H , as

$$H(u) = \frac{A + Bu + Cu^2}{D + Eu + Fu^2}. \tag{10}$$

It can easily be shown that $H(u)$ satisfies the conditions of Theorem 1, when

$$\begin{aligned} A &= \frac{13D + 5E + F}{4}, \\ B &= -\frac{7D + E - F}{2}, \\ C &= \frac{5D + E + F}{4}, \end{aligned} \tag{11}$$

where D, E, F are the real parameters that can be freely chosen.
It can easily be verified that (10) and (11) covers all the functions defined in (7)-(9).
If we take $D = 1, E = 0$ and $F = 0$ in (10) and (11), we obtain the following fifth-order method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left[\frac{13}{4} - \frac{7}{2}\frac{f'(y_n)}{f'(x_n)} + \frac{5}{4}\frac{f'(y_n)^2}{f'(x_n)^2}\right]\frac{f(y_n)}{f'(x_n)}. \tag{12}$$

If we take $D = 0, E = 0$ and $F = 1$ in (10) and (11), we obtain the following fifth-order method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left[\frac{1}{4} + \frac{1}{2}\frac{f'(x_n)}{f'(y_n)} + \frac{1}{4}\frac{f'(x_n)^2}{f'(y_n)^2}\right]\frac{f(y_n)}{f'(x_n)}. \tag{13}$$

Per iteration in the methods defined by (4) requires two function and two first derivative evaluations. If we consider the definition of efficiency index [11] as $\sqrt[r]{p}$, where p is the order of the method and r is the number of functional evaluations per iteration required by the method, we have that the iteration formula defined by (10) and (11) has the efficiency index equal to $\sqrt[4]{5} \approx 1.5874$, which is better than the one of Newton's method $\sqrt{2} \approx 1.4142$.

3 Numerical examples

In this section, some numerical test of some various root-finding methods as well as our new methods and Newton's method are presented. Compared methods were Newton's method (1) (NM), Chun's method (2)(CM), the Grau et al. method (GM) [9] defined by

$$x_{n+1} = x_n - \left[1 + \frac{f''(x_n)(f(x_n) + f(z_n))}{2f'(y_n)^2}\right]\frac{f(x_n) + f(z_n)}{f'(x_n)},$$

Where $z_n = x_n - \left[1 + \frac{1}{2}\frac{f''(x_n)f(x_n)}{f'(x_n)^2}\right]\frac{f(x_n)}{f'(x_n)},$

the method of Kou et al. (KM)[10] defined by

$$x_{n+1} = z_n - \left[1 + \frac{M(x_n)}{1 + M(x_n)}\right]\frac{f(z_n)}{f'(x_n)},$$

Where $z_n = x_n - \left[1 + \frac{1}{2}\frac{t(x_n)}{1 - t(x_n)}\right]\frac{f(x_n)}{f'(x_n)},$

and $M(x_n) = \frac{f''(x_n)(f(x_n) - f(z_n))}{f'(x_n)^2},$

with the new presented methods by Eqs.(12)(BGM1) and (13)(BGM2), introduced in this contribution. All computations were done using MAPLE with 128 digit floating point arithmetics (Digits =128). Displayed in Table 1 are the number of iterations and functional evaluations required such that $|f(x_n)| < 10^{-15}$. The following functions (which are taken from [5-7]), are used for the comparison and we display the approximate zeros x_* found, up to the 28th decimal place.

$$f_1(x) = x^3 + 4x^2 - 10,\ x_* = 1.3652300134140968457608068290,$$
$$f_2(x) = x^2 - e^x - 3x + 2,\ x_* = 0{,}2575302854398607604553673049 4,$$
$$f_3(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5,\ x_* = -1.2076478271309189270094167584,$$
$$f_4(x) = \sin(x)e^x + \ln(x^2 + 1),\ x_* = 0,$$
$$f_5(x) = (x-1)^3 - 2,\ x_* = 2.2599210498948731647672106073,$$
$$f_6(x) = (x+2)e^x - 1,\ x_* = -0.4428544010023885831413280000 0,$$
$$f_7(x) = \sin^2(x) - x^2 + 1, x_* = 1.4044916482153412260350868178.$$

The results presented in Table 1 show that for the functions we tested, the new methods introduced in this contribution need reduce the number of iterations and needed functional evaluations show that this family can be competitive to the known fifth-order methods and Newton's method and converge more quickly than the other compared methods.

Table 1: Comparison of Number of iterations of various fifth-order convergent iterative methods

	NM	CM	GM	KM	BGM1	BGM2
$f_1,\ x_0 = -0.3$	55	11	27	24	8	7
$f_1,\ x_0 = 1$	6	3	4	4	3	3
$f_2,\ x_0 = 0$	5	3	3	3	2	2
$f_2, x_0 = 1$	5	3	4	4	3	3
$f_3,\ x_0 = -1$	6	4	4	4	3	4
$f_3,\ x_0 = -2$	9	5	5	Div	6	5
$f_4,\ x_0 = 2$	6	4	4	4	4	4
$f_4,\ x_0 = -5$	8	4	6	4	5	5
$f_5,\ x_0 = 3$	7	4	4	4	3	3
$f_5,\ x_0 = 4$	8	4	5	4	4	4
$f_6,\ x_0 = 2$	9	5	5	Div	5	5
$f_6,\ x_0 = 3.5$	11	5	6	Div	6	6
$f_7,\ x_0 = 1$	7	4	4	5	6	4
$f_7,\ x_0 = 2$	6	4	2	4	3	3

4 Conclusion

In this paper, we have constructed a new general fifth-order iterative family of methods for solving nonlinear equations. This proposed iterative family contains several well-known methods as special case. It is noteworthy that the presented methods show better performance and faster convergence than the King's method and some recent its variants. A further research to find the optimal values of function to achieve faster convergence is required.

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