Families of maps Singularities and its Gauss maps

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Abstract: This paper mainly studies the Singularities of smooth mapping. The singularities of the families of Gauss maps corresponding to the family of mappings are studied and the shape of these families and their singularities using mathematica program are illustrated and plotted.

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1. Families of Maps (Scaler Function)

Let

$$z = f(u^1,\dots,u^n,a^1,\dots,a^r)$$

be the family of r parameter of hypersurface in R^n.

Where $u^\alpha \in U \subset R^{n+1}, a = a^r \subset R^r$, the vector a is a vector of control parameter.

The discriminate set of these families can be calculate by the solution of these (n) equations $p^\alpha = 0$ such that $p^\alpha = \frac{\partial z}{\partial u^\alpha}$ and finding u^α's as function in a^r say $u^\alpha = u^\alpha(a^\alpha)$ and substitution in the relation f = 0 we obtain the discriminate set for the considered formal as in the following form

$$\Delta = f(a^1,\dots,a^r) = 0$$

By changing the control parameters we find some singular points for the family which can be classified according to the famous theorems in singularity theory. Using the Hessian matrix we can obtain the singular points and singular set. Geometrically these singularities can be plotted but the classification of them can't be a valuable for all points. Using the terminology of level set which tell us the type of singular points like folds (level sets is start line), cusp (level set is semicubical parabola). In general there is no existance of some famous types.

Remark 1 For the simple ideas for finding and calculated singular points and singular sets see [1, 2, 3].

Definition 1 [1] The level set attached to the hypersurface M is defined as the following: let $u_n = Z(u_1, u2, \dots, u_{n-1}) = c$, c is constant, if c = 0 that given the level set $V_0 = \{(u_1, u2, \dots, u_{n-1}): u_n = 0\}$ and the other level sets are

$$V_c=\{(u_1, u2, \dots, u_{n-1}): u_n = 0\}, \quad c \neq 0$$

Another version of the definition of level sets is contours as given in the following

Definition 3 We say the point p on a surface M with a parametric representation is a contour point if and only if N.pc=0

where N is the normal vector field on the surface M and c is the view point. The contour line or contour, for short, of a surface is the set of all its contour points.

The determination of the contour line of a surface in the general case involves a numerical method to find the zeros of a real-valued function of n real variables in a domain $(u^1, u^2, \dots, u^n) \in U$. An algorithm and its implementation can be found in [6].

2. Application

Let us consider a 3-parameter family of surfaces defined by mong's form as the following:

$$Z=f(u, v; a, b, c) = au^4+bu^2v+u^2+cv^2, \quad a, b, c \in R \quad (1)$$

This family is denoted by σ, where a, b, c are the parameters of the family σ. The bifurcation diagram of the zeroes of this family (of functions of u; v depending on the parameters (a, b, c) is given from:

$$\Delta = c = 0 \qquad (2)$$

where Δ is the discriminant set and it is a plan.

Remark 4 The family of contours is given from z = k (constant), and the family of zero level set corresponding to k = 0 are given through the figurers [1, 2, 3].

The families (1) are classified into serval subclasses as the following:

1) a = b = c = 0, in this case the family f has A_1 singularity (fold).
2) b = c = 0, a < 0 then f has a singularity (cusp).
3) b = c = 0, a > 0 in this case the family f has A_1 singularity (fold).
4) a = c = 0, b < 0 or b > 0 in this case the family f has a fold.
5) a = b = 0, c < 0 or c > 0 in this case the family f has $\pm$B2 singularity (the normal form of this set is $B_k = x^k + y^2; k \geq 2$).
6) a; b; c > 0, a; b; c < 0. 7) a = 0, b < 0; c < 0.

8) a = 0, b > 0; c > 0. 9) a = 0, b < 0; c > 0.
10) a = 0, b > 0; c < 0. 11) b = 0, a < 0; c < 0.
12) b = 0, a > 0; c > 0. 13) b = 0, a < 0; c > 0.
14) b = 0, a > 0; c < 0. 15) c = 0, a < 0; b < 0.
16) c = 0, a > 0; b > 0. 17) c = 0, a < 0; b > 0.
18) c = 0, a > 0; b < 0.

The previous families are plotted by σ_i, i = 1, 2,…,18 which corresponding to the conditions (1), 2),..., 18)) and their geometric interpretations are given through the figures [4]- [16].

we denote the previous families by σ_i i = 1, 2,…,18 which corresponding to the codomain (1), 2),..., 18)).

The normal vector field to the family of surfaces (1), is given by:

$N(u, v; a, b, c)=\{-2(u+2au^3+buv),-(bu^2+2cv),1\}$ (3)

For the subfamily σ_i, it is easy to see that the normal vector fields has a planer swallowtail when a, b, c > 0 or a, b, c < 0 as shown in figure (17).

Since the family (3) of function lies in the plane z =1, so we can make a modification to the families (N(u, v; a, b, c)) as in the following:

$N_{mod}(u, v; a, b, c)=\{-2(u+2au^3+buv),-(bu^2+2cv)\}$ (4)

The singularities of this family can be deferral using the rank of its Jacobian matrix which is given by:

$$\begin{pmatrix} -2(1+6au^2+bv) & -2bu \\ -2bu & -2c \end{pmatrix} \qquad (5)$$

so the discriminant set(singular set) is given as:

$S := 4c-4b^2u^2+24acu^2+4bcv = 0$ (6)

From which we have :

$$S:=u^2=\frac{bc}{b^2-6ac}\left(v+\frac{1}{b}\right), \quad b^2 \neq 6ac, b \neq 0 \qquad (7)$$

This relation represents a parabola in the plan u, v with $\left(0, \frac{-1}{b}\right)$, i. e, is non defined for all points lies on the hyperbolic paraboloid. Thus the image of singular set under the modified normal vector field N is

$$N_{mod}(S)=\left\{-\frac{2(b^2-4ac)}{c}u^3, \frac{2c}{b}-\frac{3(b^2-4ac)}{b}u^2\right\} \qquad (8)$$

It is easy to see that the normal vector field of the family has a cusp point for all points except for the points lies on the hyperbolic paraboloid $b^2-4ac=0$ with conditions b; c ≠ 0 see figures (18). The shapes of discriminate set, zero level sets, for the family subclasses σ_i of the given family and the normal vector field for the subclasses σ_i and singular point are shown in fig [1] to [18].

3. Gauss and Mean Curvature

The notion of curvature of a surface is a great deal complicated than the notion of curvature of a curve. Let α be a curve in R^3. and let p be a point in on the trace of α .The curvature of α at p measures the rate at which α leaves the tangent line to α at p. By analogy. the curvature of a surface $M \subset R^3$ at p $\in$ M should measure the rate at which M leaves the tangent plane to M at p. But for surface a difficulty arises that was not present for curves: although a curve can separate from one of its tangent lines in only two direction. a surface separates from one of its tangent planes in infinitely many directions. In general the rate of departure of a surface from one of its tangent planes depends on the direction.

It is usually possible to glance at almost any surface and recognize which points are elliptic, hyperbolic, parabolic or planer. at the planer points we find the Gauss curvature achieves to maximum value at thats points and mean curvature is planer. I. e Gauss curvature and mean curvature have a relation with elliptic, hyperbolic, parabolic and planer points.

Now we calculate the Gauss curvature and mean curvature of that family of functions. The shapes of Gauss curvature and mean curvature and their contour are plotted to show the singularities of these families.

The first fundamental coefficients of the family under consideration are given as:

$G_{11}=1+4(u+2au^3+buv)^2$, $G_{22}=1+(bu^2+2cv)^2$ and $G_{12}=2(bu^2+2cv)(u+2au^3+buv)$

and its discriminant (metric) is given by:

$$G= 1+16a^2u^6+4c^2v^2+u^4\left(b^2+16a(1+bv)\right) + 4u^2(1+bv(2+c+bv))$$

The second fundamental coefficients L_{ij} of this family are:

$L_{11}=2(1+6au^2+bv)Q$, $L_{22}=2cQ$, $L_{12}=2buQ$ (9)

Where $Q=\sqrt{1+(bu^2+2cv)^2+4(u+2au^3+buv)^2}$

and the discriminant $L = \det(L_{ij})$ is given by

$L=4(c-b^2u^2+6acu^2+bcv)Q$ (10)

From above equations and by simple calculation we obtain the Gauss and mean curvatures as the following:

$$K=\frac{4\left(c-b^2u^2+6acu^2+bcv\right)}{\left(1+16a^2u^6+4c^2v^2+u^4\left(b^2+16a(1+bv)\right)+4u^2(1+bv(2+c+bv))\right)^2} \qquad (11)$$

$$H= \left(\left(1+6au^2+4c^2v^2\left(1+6au^2+bv\right)+b\left(v-bu^4\left(3+2au^2+3bv\right)\right)\right)\right.$$

$$+c\left(1+4u^2\left(\left(1+2au^2\right)^2+b\left(1+6au^2\right)v\right)\right)$$
$$/\left(1+\left(bu^2+2cv\right)^2+4\left(u+2au^3+buv\right)^2\right)^{3/2}$$

respectively.

From (11) its easy to see that K = 0 when u; v lie on the parabola equation:

$$u^2=\frac{bc}{b^2-6ac}\left(v+\frac{1}{b}\right)$$

i. e., K = 0 when u; v lie on the singular set (6). Finally the shapes of Gauss Curvature and Mean Curvature and its contours corresponding some σ_i are plotted in figure [19] - [29].

Figures

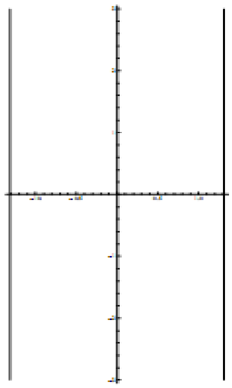

Figure 1: zero level set $b=c=0,\ a<0$

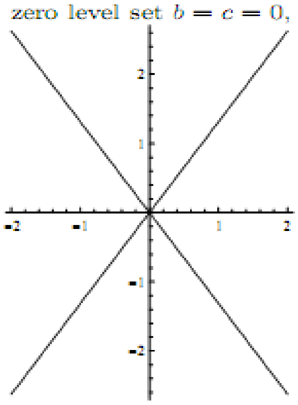

Figure 2: zero level set $a=b=0,\ c<0$

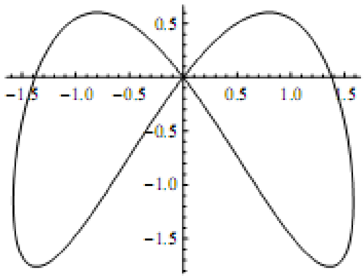

Figure 3: zero level set $a,b,c<0$

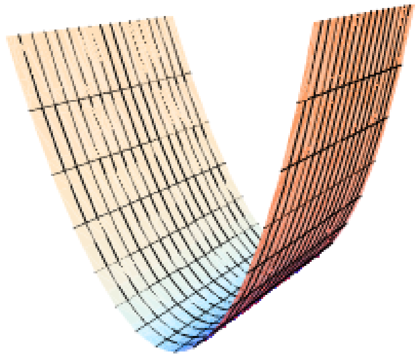

Figure 4: $\sigma_1 : a=b=c=0$ (fold)

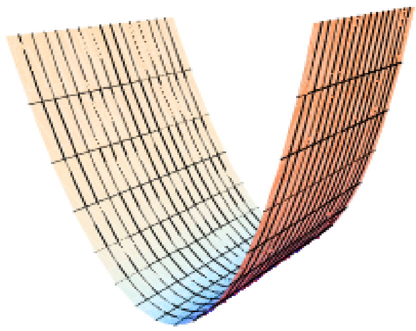

Figure 4: $\sigma_1 : a=b=c=0$ (fold)

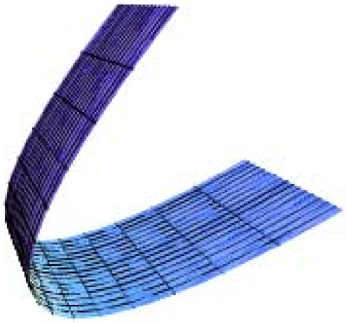

Figure 6: $\sigma_3 : b = c = 0,\ a > 0$

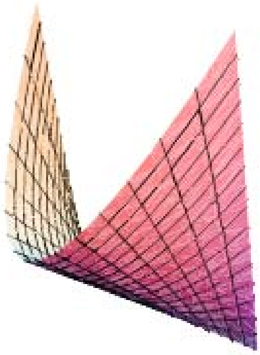

Figure 7: $\sigma_4 : a = c = 0,\ b < 0$

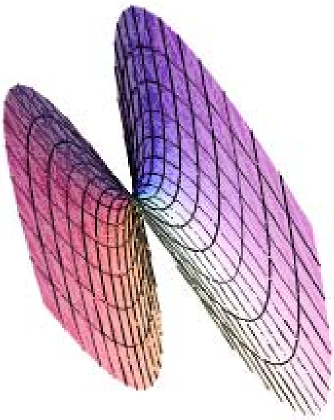

Figure 8: $\sigma_5 : a = b = 0,\ c < 0$

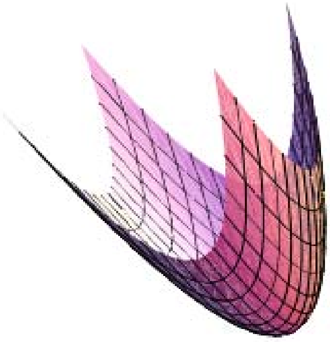

Figure 9: $\sigma_5 : a = b = 0,\ c > 0$

Figure 10: $\sigma_6 : a, b, c, > 0$ or $a, b, c, < 0$

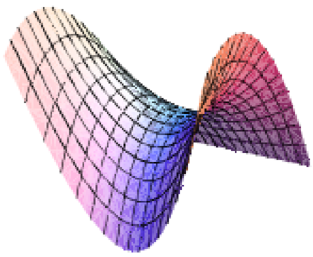

Figure 11: $\sigma_7 : a = 0,\ b < 0, \quad c < 0$

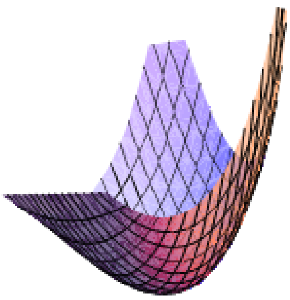

Figure 12: $\sigma_8 : a = 0,\ b > 0, \quad c > 0, \sigma_9 : a = 0, b < 0, \quad c > 0$

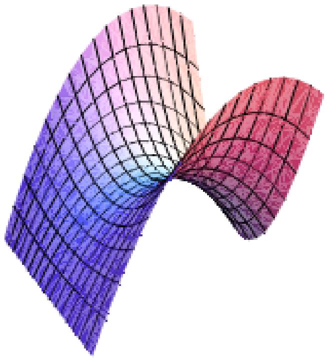

Figure 13: $\sigma_{10}: a = 0,\ b > 0,\ \ c < 0,\ \sigma_{14}: b = 0,\ a > 0,\ \ c < 0,\ \sigma_{18}: c = 0,\ a > 0,\ \ b < 0$

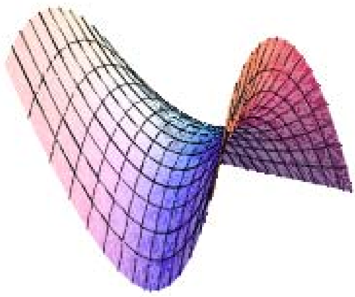

Figure 14: $\sigma_{11}: b = 0,\ a < 0,\ \ c < 0,\ \sigma_{15}: c = 0,\ a < 0,\ \ b < 0$

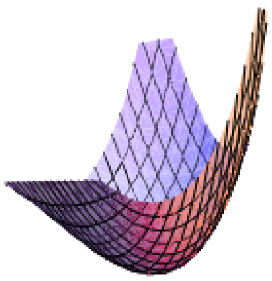

Figure 15: $\sigma_{12}: b = 0,\ a > 0,\ \ c > 0, \sigma_{16}: c = 0,\ a > 0,\ \ b > 0$

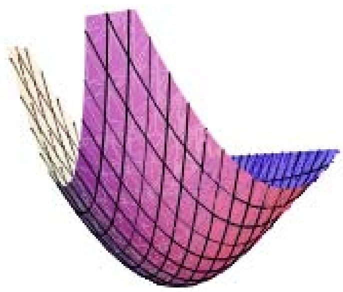

Figure 16: $\sigma_{13}: b = 0,\ a < 0,\ \ c > 0,\ \sigma_{17}: c = 0,\ a < 0,\ \ b > 0$

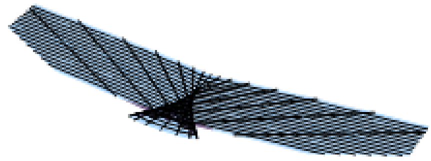

Figure 17: normal vector field when $a, b, c > 0$

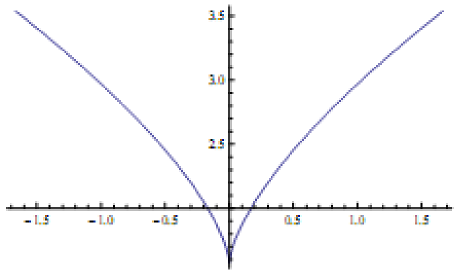

Figure 18: singular set under the N_{mod}

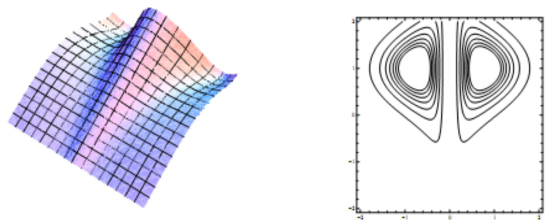

Figure 19: Gauss curvature $a = c = 0,\ b < 0,\ a = c = 0,\ b > 0$

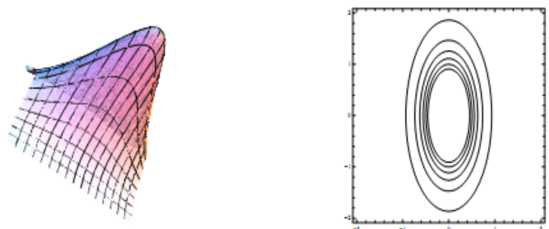

Figure 20: Gauss curvature $a = b = 0,\ c < 0$ or $c > 0$

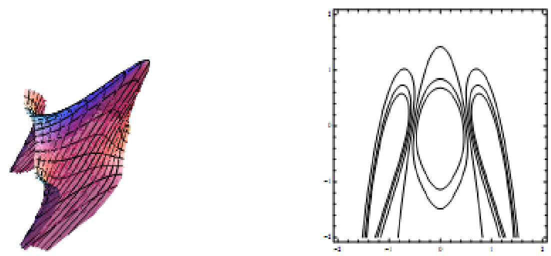

Figure 21: Gauss curvature $a, b, c < 0$

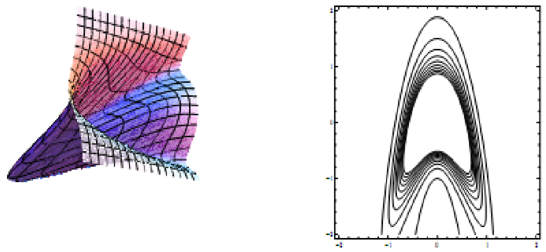

Figure 22: Gauss curvature $a, b, c > 0$

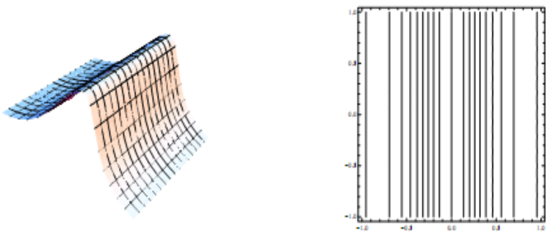

Figure 23: Mean curvature $a = b = c = 0$

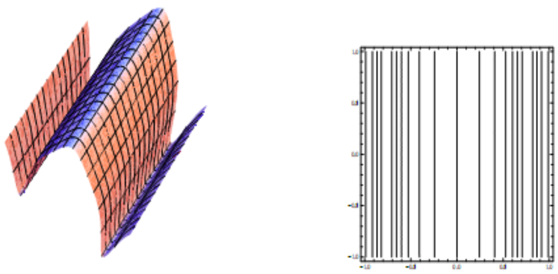

Figure 24: Mean curvature $b = c = 0,\ a < 0$

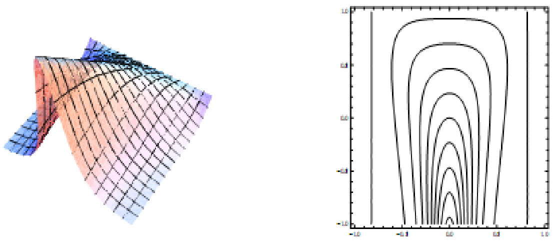

Figure 25: Mean curvature $a = c = 0,\ b < 0$ or $b > 0$

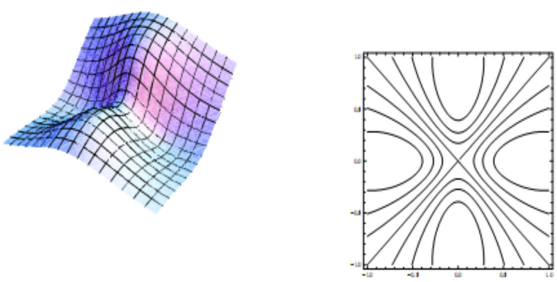

Figure 26: Mean curvature $a = b = 0,\ c < 0$

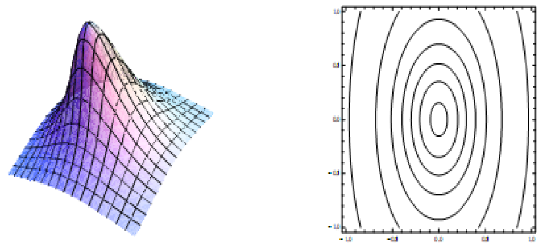

Figure 27: Mean curvature $a = b = 0,\ c > 0$

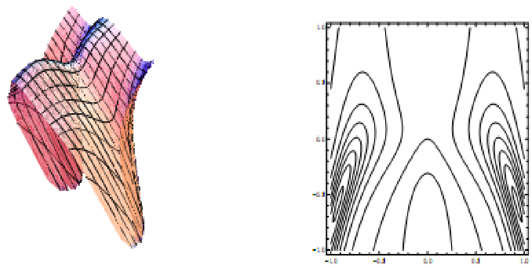

Figure 28: Mean curvature $a, b, c < 0$

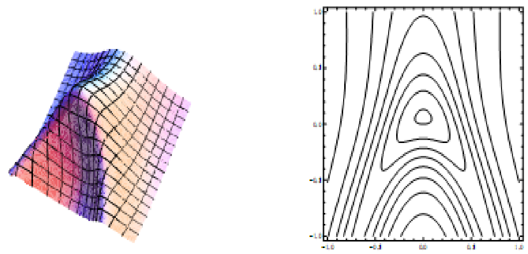

Figure 29: Mean curvature $a, b, c > 0$

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