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TWENTY-SIXTH ANNUAL MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

The twenty-sixth annual meeting of the Association for Symbolic Logic was held on Wednesday, December 27, 1961 at the Chalfonte-Haddon Hall Hotel in Atlantic City, New Jersey, in conjunction with the annual meetings of the American Philosophical Association.

At the afternoon session Professor Frederic B. Fitch of Yale University delivered an invited address on *A logical analysis of some value concepts*. Professor Raymond Smullyan presided.

Six twenty-minute contributed papers were delivered at the morning session, with Professor Frederic B. Fitch presiding. The remaining eight papers were presented by title.

The Council of the Association met at lunch. NUEL D. BELNAP, Jr.

FREDERIC B. FITCH. *A logical analysis of some value concepts.*

The concepts of *striving for*, *doing*, *believing*, *knowing*, and *proving* are treated as two-termed relations between an agent and a proposition, or more simply, if the agent is disregarded, as classes of propositions. A class of propositions will be said to be closed with respect to conjunction elimination if, necessarily, whenever $p \& q$ is in the class so are p and q; and it will be said to be closed with respect to conjunction introduction if, necessarily, whenever p and q are in the class, so is $p \& q$. A "truth-class" is a class of propositions which is such that, necessarily, if p is in the class, then p is true. The assumption is made that the following concepts are closed with respect to conjunction elimination and conjunction introduction: *striving for*, *doing*, *believing*, *knowing*, *proving*, *truth*, *causal necessity* (in the sense of Burks), *logical necessity*, *obligation*, and *desiring*. The following are assumed to be closed with respect to conjunction elimination: *causal possibility*, *logical possibility*, and *permission* (*deontic possibility*). The following are assumed to be truth-classes: *truth*, *causal necessity*, *logical necessity*, *doing*, *knowing*, and *proving*. It is shown that if C is a truth-class closed with respect to conjunction elimination, and if p is a proposition, then it is logically impossible for the proposition $p \& {\sim}Cp$ to be a member of C. In the case where p is true but not a member of C, the proposition $p \& {\sim}Cp$ is true but it is still logically impossible for $p \& {\sim}Cp$ to be a member of C. From these results it is shown that if some agent is all-powerful, he must in fact have done everything that is the case, and if some agent is not omniscient, then there is at least one true proposition that is logically impossible for him to know.

Using a relation of *partial causation* which is a modification of Burks' *causal implication*, a definition of *doing* is given in terms of *striving for*, and a definition of *knowing* in terms of *believing*. A definition of *ability to do* is given in terms of *striving for*, and a definition of *desiring* is given in terms of *striving for*, *believing*, and *ability to do*. Finally, a definition of *absolute value* is given in terms of *knowing* and *striving for*. The definitions are viewed as merely tentative, but nevertheless as having heuristic value. (*Received January 5, 1962.*)

J. H. BENNETT. *On the constructive arithmetic relations.*

The class of constructive arithmetic relations, CA, may be described (Smullyan, ***Theory of formal systems***) as the least class containing the ternary relations of addition and multiplication over the positive integers and closed under "and," "not," "there is an x less than y such that," and explicit transformation (replacing by constants, permuting, and identifying arguments). The (numerical) relation of m-adic concatenation ($m \geqq 2$) is the ternary relation holding on x, y, z iff the concatenation of the m-adic notations for x and y is the m-adic notation for z. Replacing addition and multiplication by the single relation of m-adic concatenation in the above description of CA, we have (Smullyan, *ibid.*) a description of the m-rudimentary relations. It is shown that these are merely alternative descriptions of the same class of relations; i.e., for each m, the class of m-rudimentary relations is precisely CA. The ternary relation of exponentiation is in CA. In fact, the quaternary relation $\mathrm{SP}(n, x, y) = z$ is in CA where SP is the doubly but not primitive recursive function defined by $\mathrm{SP}(0, x, y) = x + y$; $\mathrm{SP}(n + 1, x, 0) = x$; and $\mathrm{SP}(n + 1, x, y + 1) = \mathrm{SP}(n, x, \mathrm{SP}.(n + 1, x, y))$. However, CA is a small subclass of the primitive recursive relations in the sense that there is a fixed positive integer K such that the characteristic function of every relation in CA is definable from the successor function by at most K uses of primitive recursion. Progress in relating CA to other small classes of relations defined in the literature is also described. (*Received November 28, 1961.*)

R. B. ANGELL. *A logical notation with two primitive signs.*

This paper presents two systems of logical notation, each using just two typographic signs or shapes, namely, the left- and right-hand parentheses. The first system is easily shown adequate for Quine's ***Mathematical Logic*** (1940, 1958). It contains: I. *Primitive signs*:) (. II. *Symbols* (or well-formed signs): 1. '()' is a symbol. 2. If S and S′ are symbols, ⌜SS′⌝ is a symbol. 3. If S is a symbol, ⌜(S)⌝ is a symbol. A. *Atomic symbols*: 1. '(())' is an atomic symbol. 2. If ⌜(S)⌝ is an atomic symbol, ⌜(S())⌝ is an atomic symbol. (*Variables* are atomic symbols containing 2n $(n > 1)$ symbols '()'.) B. *Well-formed formulae*: 1. If α and β are variables, ⌜((())($\alpha\beta$))⌝ is a wff (in this case alone, an *atomic formula*). 2. If Φ is a wff, and α is a variable, ⌜((α)Φ)⌝ is a wff. 3. If Φ and Ψ are wffs, then ⌜((Φ)(Ψ))⌝ is a wff. It is shown that Quine's variables, atomic formulae, quantified expressions, and stroke applications can be correlated unambiguously with expressions resulting from formation rules 3A, 3B1, 3B2 and 3B3 respectively. The class of wffs defined above are syntactically equivalent to Quine's class of logical formulae (cf. Quine, p. 124) and are thus adequate for all his definitions and statements.

The second system is syntactically simpler, eliminating a special notation for quantifiers in favor of a single class constant. Formation rules are the same except that B, 1–3 are replaced by B′1. If α is a variable and β is either a variable or '(()())', then ⌜((())($\alpha\beta$))⌝ is a wff. B′2. If Φ is a wff, ⌜(Φ)⌝ is a wff. B′3. If Φ and Ψ are wffs, then ⌜($\Phi\Psi$)⌝ is a wff. Rules B′2 and B′3 are associated with denial and conjunction respectively. By the following definitions membership is introduced, as well as a class constant 'E' which eliminates the need of quantifier notation: Dl. ⌜$\alpha \varepsilon \beta$⌝ for ⌜((())($\alpha\beta$))⌝. D2. ⌜Eα⌝ for ⌜$\alpha \varepsilon$ E⌝ or ⌜((())(α(()())))⌝. An interpretation is provided such that every standard quantified statement is equivalent on the usual interpretation to a statement in this notation, and every statement in this notation is either equivalent to a standard quantified statement in its usual interpretation or else an unobjectionable addition. (The interpretation introduced requires a new interpretation of expressions usually called "propositional functions"). The paper does not attempt to provide a set of axioms for this system, as it establishes only syntactical equivalence of symbolisms. (*Received April 23, 1962.*)

G. KREISEL and W. W. TAIT. *Induction and recursion.*

Let R be primitive recursive arithmetic (PRA) with the rule

$$(\prec) \qquad \frac{A(0),\ t(a') \prec a',\ A(t(a)) \rightarrow A(a)}{A(a)}$$

of induction on the PR predicate $\prec$, and the axiom $0 \prec a'$ added. Let S(S′) be obtained by adding to PRA the axiom $0 \prec a'$ and the schema for introducing functions by ordinal recursion on $\prec$ (by ordinal recursion on $\prec$ when the auxiliary functions are PR). The rule ($\prec$) is derivable in S, so that $R \subseteq S$ (K. Gödel). In fact, if $A(a)$, $t(a)$ are PR expressions, the rule ($\prec$) is derivable in S′. Let $E(f, a, b)(E'(f, a, b))$ be the usual defining equations for the enumeration $f(0, b), f(1, b), \ldots$ of all PR functions (of all functions in S′) of a single variable.

Theorem 1. If for some function constant φ of S, $\vdash_S E(\varphi, a, b)$, then $\vdash_S$ Consis R.

For under the hypothesis, we can also derive $E'(\psi, a, b)$ in S for some ψ. But then $\vdash_S$ Consis S′ by Bernays' general consistency theorem. Finally, by arithmetizing in S the derivation of ($\prec$) in S′ for PR expressions $A(a)$ and $t(a)$, Consis S′ $\vdash_S$ Consis R.

Consequence: $R \subset S$. Note that the hypothesis of the theorem holds when $\prec \; = \; < \# <$, where $\prec_1 \# \prec_2$ denotes the (natural) exponentiation operation ($\prec_1$ to the power $\prec_2$).

Let $\prec_0 = \; < \# (< + <)$, and let R_0 be PRA with ($\prec_0$) added.

Theorem 2. $\vdash_{R_0}$ Consis S.

In the special case of $\prec \; = \; <$, we have the stronger result (also proved by J. Guard) that

Theorem 3. $\vdash_{R_1}$ Consis S

where R_1 is PRA with ($< \# <$) added.

Note: None of these results assume that $\prec$ is a well-ordering or even a simple ordering. (*Received November 28, 1961.*)

E. M. Fels. *Some algorithm theories since Markov.*

Markov's ***Teoriya Algorifmov*** [6] (for a summary in English cf. [10]) contains a way of marshalling the concept of effective calculability whose equivalence with recursiveness – and hence with other approaches like lambda – definability (cf. Davis [9, pp. 10–11]) – has been established by Detlovs [2]. While the latter's result in some respect diminishes the autarky of Markov's theory of normal algorithms (NA), a stock-taking of the derivatives of NA theory becomes topical beyond algorithm theory proper.

One class of such contributions is computer-oriented. For the practical algorithmization of mathematical problems (as opposed to its epitheory) NA theory has certain shortcomings: (i) it does not make allowances for storage devices — this practically removes it from contemporary hardware realizations; (ii) it is uneconomical: even for rather trivial problems an inordinately large number of replacement instances and very long intermediate words are needed; (iii) the composition of NA from (normal) subalgorithms, while abstractly provided for, is complicated.

Attacks on the first shortcoming are made in Korolyuk's work on address algorithms [5] and in Belyakin's contribution [1], which envisages potentially infinite exterior storage capacities and develops the concept of one metaprogram for computing values of any general-recursive function. Remedies for the second and third shortcoming are offered by Kaluzhnin's graph schemata [4] with "discerner" nodes and "operator" nodes and the replaceability of the latter by entire graph schemata; with certain provisions, such schemata are interpretable as NA. Yanov's "logical" schemata [8], in lieu of the (replacement, arrow) schemata for NA, provide improved means for deciding the equivalence of algorithm schemata, while Ershov's operator algorithms [3] move the theory of algorithm construction even closer to practical flow charting.

In another class of work stemming from NA theory, mainly generalizations of the NA concept by Nagorny [7] merit closer examination.

[1] Н. В. Белякин, "Универсальность вычислительной машины с потенциально бесконечной внешней памятью", **Проблемы кибернетики** 5, под. ред. А. А. Ляпунова, М.: Физматиздат, 1961, стр. 77–86. [2] В. К. Детловс, "Эквивалентность нормальных алгорифмов и рекурсивных функций", **Труды математического института имени В. А. Стеклова** 52, 1958, стр. 75–139. [3] А. П. Ершов, "Операторные алгорифмы. 1.", **Проблемы кибернетики** 3, 1960, стр.

5–48. [4] Л. А. Калужнин, "Об алгоритмизации математических задач", **Проблемы кибернетики** 2, 1959, стр. 51–67. [5] В. С. Королюк, "О понятии адресного алгоритма", **Проблемы кибернетики** 4, 1960, стр. 95–110. [6] А. А. Марков, **Теория алгорифмов, Труды математического института имени В. А. Стеклова** 42. 1954. [7] Н. М. Нагорный, "Некоторые обобщения понятия нормального алгорифма", **Труды математического института имени В. А. Стеклова** 52, 1958, стр. 7–65. [8] Ю. И. Янов, "О логических схемах алгоритмов", **Проблемы кибернетики** 1, 1958, стр. 75–127. [9] Martin Davis, ***Computability and Unsolvability***, New York: McGraw-Hill Book Co., 1958. [10] A. A. Markov, "The Theory of Algorithms", translated by Edwin Hewitt, ***American Mathematical Society Translations***, Series 2, Vol. 15, 1960, pp. 1–14. (*Received November 1, 1961.*)

GEORGE GOE. *On the simplification of quantificational formulae.*

Often one can find a simpler equivalent of a given quantificational formula. Thus, '$(x)(\exists y)((Fyz \supset Fyx) \supset Fux)$' is equivalent to '$Fuz$.' Yet the possibility of devising routine techniques for the simplification of quantificational formulae does not seem to have been ever investigated. The purpose of this paper is to initiate such an investigation.

We define 'simplicity' for certain *canonical* formulae, namely for prenex normal forms wherein what follows the prefix is a conjunctive or alternational normal form. A technique is then described for the simplification of canonical formulae analogous to the so-called cut-and-try method for the simplification of truth functions, and its power is illustrated in a variety of examples. The method is valuable, since it has the same kind of *de facto* effectiveness for comparatively simple formulae as does the corresponding method for truth functions.

As is the case for truth functions, a cut-and-try method is not adequate for the simplification of the more complex quantificational formulae. The next question of theoretical and practical interest then is: is it possible to have a general, theoretically effective simplification procedure for quantificational formulae, such as exists for truth functions? The answer is 'no.' In fact it can be easily shown that if we had such an effective simplification procedure for quantificational formulae, we should also possess a decision procedure for quantificational logic, which we know to be impossible. But effective simplification procedures may be devised for special classes of quantificational formulae. A mechanical procedure is described and shown to be effective for the simplification of formulae with at most one quantifier. The simplification even of such formulae is by no means always a trivial matter, and its theory appears to be susceptible of generalization for canonical formulae with only universal or only existential quantifiers, or with only monadic predicates. (*Received October 21, 1961.*)

LEON HENKIN. *An extension of the Craig-Lyndon interpolation theorem.*

For formulas of suitable predicate calculi Craig's theorem asserts that if $A_1 \vdash A_2$ then there is a B, every predicate symbol of which appears in both A_1 and A_2, such that $A_1 \vdash B$ and $B \vdash A_2$. If, in the hypothesis (but not the conclusion) of this proposition, the symbol "$\vdash$" for formal derivability is replaced by one denoting semantical consequence, the resulting sentence yields the completeness theorem as an immediate corollary. We give a proof for this version of the interpolation theorem, strengthened (as originally done by Lyndon) to distinguish between positive and negative occurrences of predicate symbols. But we go further, by relating the pattern in which different predicate symbols appear jointly in B to the corresponding patterns of A_1 and A_2. For example, if two predicate symbols occur with given signs in B, within the scope of a positive occurrence of $\wedge$ in B, then the same symbols, each with the same sign in A_1 as in B, will occur witin the scope of a positive occurrence of $\wedge$ or a negative occurrence of $\vee$ in A_1. We establish such results by showing that the given formulas A_1 and A_2 can be transformed into equivalent formulas A_1^* and A_2^*, by certain simple transformations, in such a way that B, except for its existential quantifiers, can be obtained from A_1^* by erasing certain parts of the latter and changing the remaining individual symbols in prescribed ways, and except for its universal quantifiers B can be similarly obtained from A_2^*. (*Received January 3, 1962.*)

ALAN ROSS ANDERSON. *Entailment shorn of modality.*

Ackermann's axioms (XXII 324) for the (presumably) pure theory of *strenge Implikation* are $A \to A$, $A \to B \to. B \to C \to. A \to C$, $A \to B \to. C \to A \to. C \to B$, and $(A \to. A \to B) \to. A \to B$. In the presence of *modus ponens* and Ackermann's rule (δ) (from B and $A \to. B \to C$ to infer $A \to C$), these lead to the *pure calculus of entailment* (as in a forthcoming paper in this JOURNAL by Belnap and the present writer). If we drop (δ), then these axioms can again be shown to have a completeness property, analogous to that of *entailment* (see the author's paper in ***Zeitschrift für mathematische Logik,*** vol. 6 (1960), pp. 201–216). Using the notation of that paper, if we restrict applications of *modus ponens* in subproofs in such a way that we may deduce $B_{a \cup b}$ from A_a and $(A \to B)_b$ only if $\max a \geqq \max b$ (where subscripts are placed on hypotheses in the "natural" way), then the subproof formulation is equivalent to the four axioms above (together with *modus ponens* unrestricted). Axioms for extensional "and," "or," and "not" may then be added as in the cited paper. In the resulting system we do not have $A \to A \to A \to A$ (i.e., $NA \to A$, where "N" is necessity), hence no theory of necessity is (seems to be) available in the theory.

The intuitive content of the restriction is that an entailment imported by reiteration into a subproof *must* be used as an "inference ticket," and may not be used as a minor premiss for further deductions. (*Received October 20, 1961.*)

NUEL D. BELNAP, JR. *First degree formulas.*

In ***First degree entailments*** (Technical Report No. 10, Office of Naval Research Contract SAR/Nonr-609(16); also forthcoming elsewhere), Anderson and the present writer developed a semantics for first degree entailments (*fde*), i.e., entailments between formulas involving only truth-functions (defined in terms of "or" and "not") and quantifiers. The key ideas were (i) the notion of a *frame* $\langle P, F_{IP}, I\rangle$, where P is a set of (intensional) propositions closed under negation and multiple disjunction, I is a domain of individuals, and F_{IP} is the set of functions from I to P; and (ii) the semantic relation of *cons* (consequence), as obtaining between a set of propositions taken conjunctively, and a set taken disjunctively; and (iii) the notion of an atomic frame, i.e., a frame generated by a set of propositions X closed under negation, such that for any disjoint subclasses Y and Z of X, Y does not bear *cons* to Z. Consistency and completeness proofs were forthcoming for the *fde* fragment of the system EQ of entailment with quantifiers.

These semantics also extend to yield a definition of "valid" for *first degree formulas* (*fdf*), i.e., the set of formulas which contains all *fde* and purely truth-functional formulas, and is closed under disjunction, negation, and quantification. It was conjectured that the *fdf* fragment of EQ is consistent and complete, and problems concerning the effectiveness of the notion of validity for the quantifier-free fragment of *fdf* were raised.

The conjecture is correct. It also turns out that there is a decision procedure for validity (and hence for provability in EQ) of quantifier-free *fdf*. There is also the following curious version of the Löwenheim-Skolem theorem: if a set of *fdf* are simultaneously satisfiable, then they are so in a frame $\langle P, F_{IP}, I\rangle$, where P and I are at most denumerable. We remark that the denumerability of P is essential: there are *fdf* not satisfiable in any frame with P finite, but satisfiable with P denumerable. (*Received October 30, 1961.*)

E. W. Beth. *Observations on an independence proof for Peirce's Law.*

The independence of Peirce's Law $[(A \rightarrow B) \rightarrow A] \rightarrow A$ with respect to the positive implication calculus can be proved by means of a pseudo-valuation w which associates with each formula U a truth value $w(U) = 0$ (false) or $= 2$ (true), as follows. We have $w(A) = w(B) = 0$, $w(M) = 2$ for each atom M different from A and B, $w(U \rightarrow V)$ as usual, unless U and V are formulas $U_1 \rightarrow (U_2 \rightarrow (\ldots \rightarrow (U_m \rightarrow A)\ldots))$ and $V_1 \rightarrow (V_2 \rightarrow (\ldots \rightarrow (V_n \rightarrow B)\ldots))$, $w(U_j) = w(V_k) = 2$ $[0 \leqq j \leqq m, 0 \leqq k \leqq n]$, in which case $w(U \rightarrow V) = 0$.

Further analysis of the idea suggests the following semantic construction of a logical system which is, at least from a classical point of view, identical with intuitionistic logic. An I-valuation is an ordered triple $[W, \leqq, w^0]$ composed of a set W of functions w, a partial ordering $\leqq$ of W, and the largest element w^0 in W. The functions w associate truth values $w(U) = 0$ or $= 2$ with formulas U in accordance with the following semantic rules: (S1) If A is an atom, $w' \leqq w$ and $w(A) = 2$, then $w'(A) = 2$; (S2) If, for every $w' \leqq w$, $w'(U) = 0$ or $w'(V) = 2$, then $w(U \rightarrow V) = 2$; otherwise $w(U \rightarrow V) = 0$. Theorem: the following conditions are equivalent: (i) $w^0(U) = 2$ for every I-valuation $[W, \leqq, w^0]$; (ij) the deductive tableau for the sequent $\varnothing / U$ is closed; (iij) U is a theorem of the inferential (= positive) implication calculus. — Presumably the construction is similar to one previously announced by S. A. Kripke. The above results were obtained through. investigations carried out under Contract No. 010-60-12 DOH between Euratom and the University of Amsterdam. (*Received October 24, 1961.*)

G. Kreisel. *Explicit definability in intuitionistic logic.*

Let $\vdash_0$ ($\vdash_1$) denote provability in intuitionistic propositional (predicate) logic. $\mathfrak{Q}$ is a list of the non-logical constants of the formula A besides P, A′ is obtained from A by replacing P in A by P′ ($\mathrm{P}' \notin \mathfrak{Q}$), and x does not occur in A. *Theorem 1. If* (*) $\vdash_0 (\mathrm{A} \wedge \mathrm{A}') \rightarrow (\mathrm{P} \leftrightarrow \mathrm{P}')$ *then* $\vdash_0 \mathrm{A} \rightarrow (\mathrm{P} \leftrightarrow \mathrm{A_t})$, *where* [the 'explicit definition' (e.d)] $\mathrm{A_t}$ [of P] *is obtained by replacing* P *in* A *by the constant* t (= True). *Proof.* From (*), $\vdash_0 (\mathrm{A} \wedge \mathrm{A_t}) \rightarrow (\mathrm{P} \leftrightarrow \mathrm{t})$, whence $\vdash_0 \mathrm{A} \rightarrow (\mathrm{A_t} \rightarrow \mathrm{P})$, and, generally, $\vdash_0 (\mathrm{A} \wedge \mathrm{P}) \rightarrow \mathrm{A_t}$, whence $\vdash_0 \mathrm{A} \rightarrow (\mathrm{P} \rightarrow \mathrm{A_t})$. *Remark.* The analogue to Theorem 1 applies to all fragments of classical propositional logic in the sense of Henkin, XIV 197, since these fragments contain $\rightarrow$, hence the constant t, and so $\mathrm{A_t}$ is in the fragment if A is. For the full classical propositional calculus a (different) e.d. was given by A. Robinson XXV 174; here, the mere existence of an e.d. is immediate because (*) means that P is a truth function of $\mathfrak{Q}$, and therefore definable by composition of the usual truth functions. For predicate logic we have only a partial result. Let A^- be obtained by replacing P in A by $\neg\neg$P, and $\mathrm{A_G}$ by eliminating $\vee$, V and putting $\neg\neg$ in front of every prime part.

Theorem 2. If (i) $\vdash_1 \mathrm{A} \rightarrow \mathrm{A}^-$, (ii) $\vdash_1 \mathrm{A} \rightarrow \mathrm{A_G}$ *and* (iii) $\vdash_1 (\mathrm{A} \wedge \mathrm{A}') \rightarrow [\mathrm{P}(x) \leftrightarrow \mathrm{P}'(x)]$ *then there is a* $\mathrm{B}(x)$, *whose non-logical constants are in* $\mathfrak{Q}$, *such that* $\vdash_1 \mathrm{A} \rightarrow [\mathrm{P}(x) \leftrightarrow \mathrm{B_G}(x)]$.

Lemma. If (i) *and* (iii), $\vdash_1 \mathrm{A} \rightarrow [\mathrm{P}(x) \leftrightarrow \neg\neg\mathrm{P}(x)]$ (P is *stable*). For, by (iii), $\vdash_1 \mathrm{A} \wedge \mathrm{A}^- \rightarrow [\mathrm{P}(x) \leftrightarrow \neg\neg\mathrm{P}(x)]$, by (i), $\vdash_1 \mathrm{A} \rightarrow (\mathrm{A} \wedge \mathrm{A}^-)$. To prove Theorem 2: By (iii), Beth's definability theorem for classical logic, XXI 194, and Gödel's translation of the latter (mapping A into $\mathrm{A_G}$): $\vdash_1 \mathrm{A_G} \rightarrow [\neg\neg\mathrm{P}(x) \leftrightarrow \mathrm{B_G}(x)]$; hence, by (ii) and the lemma $\vdash_1 \mathrm{A} \rightarrow [\mathrm{P}(x) \leftrightarrow \mathrm{B_G}(x)]$. Conditions (i) and (ii) are satisfied e.g. if $\vdash_1 \mathrm{A} \leftrightarrow \mathrm{A_G}$ or, if A is prenex and disjunction-free. It is not known whether Craig's interpolation lemma XXIV 243, holds for intuitionistic logic except in the trivial case of $\vdash_0 \mathrm{A} \rightarrow \mathrm{B}$ where A and B have no symbols in common. In this case either $\vdash_0 \neg\mathrm{A}$ or $\vdash_0 \mathrm{B}$; for if non $\vdash_0 \neg A$ there exist substitutions t or $\neg$ t for the letters of A yielding A_s such that $\vdash_0 A_s$. Since these substitutions leave B unchanged, $\vdash_0 A_s \rightarrow B$, and hence $\vdash_0 B$. Note that the results above for intuitionistic logic include the corresponding results for classical logic as special cases, via Gödel's translation above. (*Received November 3, 1961.*)

G. Kreisel. *Status of the first ε-number in first order arithmetic.*

Gentzen, IX 70, considered a particular (natural) ordering $<$ (of the natural numbers) of ordinal ε_0 and the (conservative) extension Z_P of classical arithmetic Z obtained by adding a free predicate variable P to the notation, without altering axioms or schemata. Formulating transfinite induction (TI) up to b (well-ordering of $\{x : x < b\}$) by

$$(\Lambda x)[x < b \rightarrow \{(\Lambda y)[y < x \rightarrow P(y)] \rightarrow P(x)\}] \rightarrow (\Lambda x)[x < b \rightarrow P(x)],$$

he showed that (TI) could be proved in Z_P for each value of b, but not with free variable b. Schütte (***Beweistheorie***) showed that if (TI) is provable for any decidable $<$ in Z_P, even when arbitrary true recursive statements are added as additional axioms, the ordinal of $<$ is less than ε_0 (*Satz* 23.12; Schütte also states the converse, but this is false). Let Z^* denote the following (conservative) extension of Z: add free and bound function and predicate variables to the notation of Z, 'first order' comprehension axioms: $(VP)(\Lambda x)[P(x) \leftrightarrow A(x)]$, where A does not contain P nor bound variables of higher type, and induction is applied to first order formulae only (of course, containing free variables of higher type). Substitution of μ-terms $\mu_x A(x)$ for function variables is allowed provided $A(x)$ is of first order. A formula of Z^* is called Σ_1^1 if it consists of an existential function quantifier followed by an expression of first order. Consider any Σ_1^1 ordering $<\cdot$, with $a <\cdot b$ defined by $(Vf)(\Lambda x)R(f, x, a, b)$, R recursive.

Theorem. If $<\cdot$ can be proved to be a well-ordering in Z^*, with all true Σ_1^1 statements as additional axioms, the ordinal of $<\cdot$ is less than ε_0. For, let $<_n$ be an enumeration of primitive recursive relations, and let $\mathbf{W}(n)$ express: $<_n$ is a well-ordering (in Gentzen's form, or $(\Lambda f)(Vx)\neg[f(x+1) <_n f(x)]$, which is equivalent in Z^*). Then $\mathbf{W}(n)$ is a (provably) complete Π_1^1-form. Let $\mathbf{W}_1(n)$ express: $<_n$ is a section of $<\cdot$. Then, by contradiction of quantifiers and axiom of choice, $\mathbf{W}_1(n)$ is equivalent to a Σ_1^1 form, $\mathbf{W}_1^*(n)$, and, without axiom of choice, $\vdash^* (\Lambda x)[\mathbf{W}_1^*(x) \rightarrow \mathbf{W}_1(x)]$, and, since $<\cdot$ is assumed to be a provable well-ordering, $\vdash^* (\Lambda x)[\mathbf{W}_1^*(x) \rightarrow \mathbf{W}(x)]$. Also, by completeness, for some numeral $\bar{n}$, $\vdash^* [\mathbf{W}_1^*(\bar{n}) \leftrightarrow \neg\mathbf{W}(\bar{n})]$: so, $\vdash^* \mathbf{W}(\bar{n})$ and $\vdash^* \neg\mathbf{W}_1^*(\bar{n})$. Since, by $\vdash^* \mathbf{W}(\bar{n})$ and (the correct half of) *Satz* 23.12, $<_n$ has ordinal $< \varepsilon_0$, $\neg\mathbf{W}_1^*(\bar{n})$ means that $<\cdot$ has ordinal $< \varepsilon_0$ too.

Corollary. All Z^*-provable arithmetic well-orderings have ordinal $< \varepsilon_0$.

Remarks. The theorem is best possible in the sense that there are Π_1^1 orderings of ordinal ω_1 which can be proved to be well-orderings in Z^*; e.g., trivially, the one given by Gandy [***Bull. Acad. Pol. Sc.,*** vol. 9 (1960), 571–575].

Naturally, the addition of axioms to Z^* makes (TI) provable for some $<\cdot$, for which (TI) is not provable in Z^* itself: it is only the least upper bound on the size of the ordinals of provable well-orderings which is unchanged. (*Received November 3, 1961.*)

G. Kreisel. *Ordinals of ramified analysis.*

$\prec$ is the ordering of Schütte's book ***Beweistheorie*** (BT). As in BT, Γ_n is ramified analysis of n levels; $Z(a)$ first order classical arithmetic Z with (TI) (cf. preceding abstract) applied to formulae $P(x)$ of Z, for each b, $b \prec a$; $Z^+(a)$, the same, for each b, $b \prec a$; PRD(a) is primitive recursive arithmetic (pra) with definition by quantifier-free transfinite recursion over each proper initial segment of $\{x : x \prec a\}$, as in abstract XXIV 322–323; PRP(a) is *pra* with proof by quantifier-free (TI) over $\{x : x \prec b\}$, for each $b \prec a$, but no additional (function) constants; $\mathrm{PRD}^+(a)$, $\mathrm{PRP}^+(a)$ are defined analogously. If $|a|$ denotes the ordinal of $\{x : x \prec a\}$, $|3| = \omega$, $|5| = \varepsilon_0$; let e_n be defined by: $|e_0| = \varepsilon_0$, $|e_{n+1}| = \varepsilon_{|e_n|}$. For $3 \prec a$, $Z(a)$, $\mathrm{PRD}(a)$, $\mathrm{PRP}(a)$ are evidently of non-increasing strength, and, in general, decreasing: (for comparison, introduce in $Z(a)$ the constants of $\mathrm{PRD}(a)$ by explicit definition) e.g. for $3 \prec a \prec 5$, $Z^+(3) = Z(a)$, $Z(a)$ is a non-conservative extension of $\mathrm{PRD}(a)$ by Gentzen, $\mathrm{PRD}(a)$ is a non-conservative extension of $\mathrm{PRP}(a)$, e.g. from results of Tait [***Math. Ann.***, vol. 143 (1961) 235–250].

Theorem 1. (i) Γ_n *is a conservative extension of* $Z(e_n)$, (ii) $Z(e_n)$ *of* $\mathrm{PRD}(e_n)$, (iii) $\mathrm{PRD}(e_n)$ *of* $\mathrm{PRP}(e_n)$.

Theorem 2. *The consistency of* Γ_n *can be proved in* $\mathrm{PRD}^+(e_n)$.

To see (i), transform a given proof in Γ_n of the formula A of Z into a *Normalbeweis* N_A (as in BT) in Γ^* of order $< \omega.2$; there is a formal proof in *pra* (i.e., explicit description of the infinite proof tree N_A and a formal proof in *pra*) that N_A is built up according to the rules of Γ^*. *Satz* 28.10 (BT, p. 263) gets from N_A a proof tree N_A of A in Γ^* which is either cut-free or of level $< n$, and a formal proof of this fact in $\mathrm{PRD}(e_n)$. Finally, *Satz* 28.8 yields [again in $\mathrm{PRD}(e_n)$] a cut-free *Grundherleitung* G_A of A in order $< |e_n|$. G_A being cut free, no formula of G_A contains more quantifiers than A, and hence, as in Kreisel-Wang, XXI 404, one proves in $Z(e_n)$: for all formulae F, if $F \in G_A$ then F is true. Since $A \in G_A$, we have a proof of A in $Z(e_n)$. To see (ii), use *Satz* 23.9 (BT, p. 219) and the fact that, for $a \prec e_n$, the first ε-number $> |a|$ is $< |e_n|$. To see (iii), use Tait's results and the fact that, for $a \prec e_n$, $\omega^{|a|+\omega} < |e_n|$. It is of interest to observe that a formalization of the Gentzen-Schütte style consistency proofs for arithmetic uses $\mathrm{PRP}^+(5)$, while a straight formalization of the ε-substitution method uses $\mathrm{PRD}^+(5)$: this can be reduced for consistency as in (iii) above; but the proper frame work of the ε-substitution method is $\mathrm{PRD}^+(5)$ since it yields 1-consistency of Z, and this cannot even be stated in $\mathrm{PRP}^+(a)$, for any a. It should be mentioned that the results of abstract XXIV 322–323 are due to Gödel.

Note that the proofs of Theorem 1(i)–(iii) are finitist. The explicit analysis of Γ_n in terms of $\mathrm{PRD}(a)$, $\mathrm{PRP}(a)$ is not to be regarded as a formalistic exercise. When the precise study of informal distinctions within constructive mathematics (finitist, predicative, etc.), which is just beginning, is developed into a hierarchy, results of the kind above will have the following use: since, presumably, it will be easy to find the level, if any, in the hierarchy of (the intuitive proofs codified in) $\mathrm{PRD}(a)$, $\mathrm{PRP}(a)$, the constructive status of Γ_n will be read off from Theorems 1 and 2. (*Received November 3, 1961.*)

E. J. LEMMON. *Quantified S4 and the Barcan formula.*

Let QS4 (QS5) be the system obtained by adding quantifiers in the usual way to the modal system S4 (S5), and let β be $(x)\Box Fx \rightarrow \Box(x)Fx$ (then β is deductively equivalent in QS4 to $\Diamond(\exists x)Fx \rightarrow (\exists x)\Diamond Fx$, called by Prior, ***Time and Modality,*** the 'Barcan formula'). Prior XXII 91 in effect shows that $\vdash_{QS5} \beta$. Hence Myhill, ***Logique et analyse,*** 1958, pp. 74–83, is inconsistent in advocating QS5 whilst rejecting β. On the other hand, Myhill's argument against β (p. 80) is strong. Hence it is of interest that β is not a theorem of QS4, which can be shown from Theorem 5.1 of Rasiowa XVIII 72 by putting $I_0 =$ the set of positive integers and $\kappa_0 =$ the usual closure algebra on the Euclidean line. Then, if $Fm = \left\{x : -\dfrac{1}{m} \leqq x \leqq \dfrac{1}{m}\right\}$, so that F is a function whose arguments run through I_0 and values belong to κ_0, it is easily seen that $\prod_{m \in I_0} \mathbf{I}Fm \subseteq \mathbf{I} \prod_{m \in I_0} Fm$ is false. Hence the (I_0, κ_0) functional Φ_β is not identically equal to the unit element of κ_0, and β is not provable in QS4. In fact, if '$\Box$' is interpreted as 'It is provable that', β holds only for ω-complete systems. We may consider the system QS4(β) which results from adding β as an axiom to a normal formulation of QS4. Let us say that a closure algebra is *saturated* if for any subset S of closed elements the sum of S is closed. Then we show by induction on the length of α that, if $\vdash_{QS4(\beta)} \alpha$, then for all sets I and closure algebras κ with κ complete and saturated the (I, κ) functional Φ_α is identically equal to the unit element of κ. (The converse, which is conjectured, would, if true, yield a completeness result for QS4(β).) Since S4 has the finite model property and since all finite closure algebras are saturated, it follows that QS4 and QS4(β) do not differ in their propositional fragment; hence in particular QS4(β) has the same 14 irreducible modalities as QS4. (*Received November 3, 1961.*)

A. A. Mullin. *Logico-philosophical comments on the philosophies of Charles Sanders Peirce and Ludwig Wittgenstein.*

This brief monograph (68 pp.), copies of which are available from the author, deals with, primarily and among other things, contrasts and similarities in some of the logico-mathematical aspects of the philosophies of Charles Sanders Peirce and Ludwig Wittgenstein. In particular it treats of a critical philosophical comparison of their classifications of mathematics and logic together with a study of some of their ideas concerning the nature of *negation, relation* and *computation.*

The reference material is mainly from among the posthumously published *Collected Papers of Charles Sanders Peirce* (especially volumes 2, 3 and 4), Wittgenstein's *Logisch-philosophische Abhandlung* and three of his posthumously published works: *Philosophische Untersuchungen, Bemerkungen über die Grundlagen der Mathematik* and *The Blue and Brown Books.*

Among other things the report presents arguments to defend the theses that (i) logic and mathematics occupy cardinal positions in the philosophies of Peirce and Wittgenstein, (ii) their philosophical approaches and styles are more complementary than they are similar and (iii) one of the functions of science (including mathematics and logic) and technology is their continual attempt, vindicated by means of their own methods, to bring new, and occasionally unwarranted, interpretations to conceptual words, which provides one of the dynamic features to language. (*Received September 18, 1961.*)

PRELIMINARY NOTICE

It is planned to hold an international symposium on the theory of models in Berkeley, California, from June 25, 1963 through July 11, 1963. The symposium is being organized by the Association for Symbolic Logic, which is asking the International Union for the History and Philosophy of Science to serve as co-sponsor. Supporting funds are being requested to enable The Organizing Committee to provide travel expenses for invited speakers.

If possible, one or more sessions for contributed papers will also be scheduled. It is hoped that younger logicians, including students, will be encouraged to attend, and The Organizing Committee hopes to find some funds to furnish partial support for a few of these.

A more detailed announcement will be sent in several months when further information is available.

ANNOUNCEMENT
TO SUBSCRIBERS TO THE JOURNAL OF SYMBOLIC LOGIC

The Association for Symbolic Logic is proud to announce that it is preparing a CUMULATIVE INDEX for the 25-year period 1936–1960. In addition to the customary indices, this volume will contain a subject index constituting a *complete* coverage of the field of logic for that period. The CUMULATIVE INDEX will be an indispensable tool to research workers in the field of logic.

The CUMULATIVE INDEX will be published as volume 26 (1961) of the JOURNAL OF SYMBOLIC LOGIC, and is scheduled to appear late in 1963.

The schedule of publication of the JOURNAL OF SYMBOLIC LOGIC will be approximately as follows:

Volume 27,	No. 1	November 1962
	No. 2	January 1963
	No. 3	March 1963
	No. 4	May 1963
Volume 26,	INDEX VOLUME	Fall 1963

Members of the Association for Symbolic Logic will receive the INDEX VOLUME under their dues payment for 1961. Subscribers to the JOURNAL OF SYMBOLIC LOGIC who have paid their subscription before September 1962 will receive the INDEX VOLUME for the pre-publication price of $6.50. After September 1962 the list price will be at least $8.00.

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