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Error Analysis of a Rational Interpolation Spline

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Abstract. This paper deals with the approximation properties of a rational cubic interpolation with linear denominator. The rational cubic spline based on function values only and the two families of parameters, in the description of the rational interpolant, modify the shape of the curve freely. Error expression of the values and the derivatives of interpolating functions are derived, which shows the interpaltion is stable.

Keywords: Data visualization; Rational spline; Interpolation; Error analysis

I. INTRODUCTION

Smooth curve representation, to visualize the scientific data, is of great significance in the area of computer graphics. Smoothness is one of the very important requirements for a pleasing visual display. Since most of splines are polynomial splines, local modification of the interpolating curve is difficult under the conditions that the interpolating data are not changed. The cubic rational spline mentioned in this paper has two families of parameters which can change easily. The more important mathematical achievement of this method is that the uniqueness of the interpolating function for the given data would be replaced by the uniqueness of the interpolation for the given data and the selected parameters. For the given interpolation data, the change of the parameters causes the change of the interpolation curve. When given suited parameters, the needed interpolation curve can be obtained.

There are rational cubic splines with linear, quadratic or cubic denominator, which are effectively used in the design and modification of curves, such as region control, convexity control and monotonicity control[1-7]. So rational cubic spline is flexible in shape control, but it is hard to consider the approximation properties because of the appearance of the parameters. There are few paper in the literature on the approximation properties of the rational spline. In this paper, the approximation properties of a typical rational cubic spline restated in Section 2 are discussed.

II. THE RATIONAL CUBIC INTERPOLANT

Let $\{(x_i, f_i), i = 1, 2, \cdots, n\}$ be a given set of data points, where $x_1 < x_2 < \cdots < x_n$, and f_i are the function values at the knots x_i. For $x \in [x_i, x_{i+1}]$, Let $h_i = x_{i+1} - x_i$, $\theta = (x - x_i)/h_i$, $(0 \le \theta \le 1)$ .A piecewise rational cubic function $S(x)$ is defined for $x \in [x_i, x_{i+1}]$ as

$$S_i(x) = \frac{\begin{array}{c}(1-\theta)^3 u_i f_i + \theta(1-\theta)^2[(2u_i + v_i) f_i + u_i h_i d_i] \\ +\theta^2(1-\theta)[(u_i + 2v_i) f_{i+1} - v_i h_i d_{i+1}] + \theta^3 v_i f_{i+1}\end{array}}{(1-\theta)u_i + \theta v_i}. \qquad (1)$$

where d_i are the derivative values at the knots x_i , u_i, v_i are named shape parameters, and $u_i, v_i > 0$ .The rational cubic interpolant has the following interpolatory properties:

$$S(x_i) = f_i, S(x_{i+1}) = f_{i+1}, S'(x_i) = d_i, S'(x_{i+1}) = d_{i+1},$$

In most application, derivative parameters d_i are not given and hence must be determined from the data (x_i, f_i). In [4], Duan etc. had dicussed, this kind of spline where $d_i = (f_{i+1} - f_i)/h_i$, so $f_i' - d_i = O(h), h = \max(h_i)$. Now we give another choice. An obvious choice is mentioned here:

$$\begin{aligned} &d_i = (h_i\Delta_{i-1} + h_{i-1}\Delta_i)/(h_i + h_{i-1}), \qquad i = 2, \cdots, n-1, \\ &d_1 = \Delta_1 + (\Delta_1 - \Delta_2)h_1/(h_1 + h_2), \\ &d_n = \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2})h_{n-1}/(h_{n-2} + h_{n-1}). \end{aligned} \qquad (2)$$

This method is based on three-point difference approximation for the d_i. A Newton series expansion analysis shows that $f_i' - d_i = O(h^2), h = \max(h_i)$ [6]. For given bounded data, the derivative approximation using Eq.2 are bounded. Hence, for bounded values of the appropriate shape parameters u_i, v_i, the interpolant Eq.1 is bounded and unique.

III. ERROR ESTIMATION OF THE INTERPOLATION FUNCTION

For the error estimation of the piecewise rational cubic interpolant Eq.1, we deal with the case when the knots are equally spaced, let $h = x_{i+1} - x_i$. Without loss of generality, it is necessary to consider the subinterval $[x_i, x_{i+1}]$ .When $f(x) \in C^1[x_1, x_n]$, and $S(x)$ is the rational cubic interpolatory function of $f(x)$, using the Peano-Kernal Theorem [4] gives

$$R(f) = f(t) - S(t) = \int_{x_{i-1}}^{x_{i+2}} f'(\tau) R_x\left[(x-\tau)_+^0\right] d\tau$$

where

$$R_x\left[\left(x-\tau\right)_+^0\right]=\begin{cases}-\dfrac{\theta(1-\theta)^2u_i}{2[(1-\theta)u_i+\theta v_i]}, & x_{i-1}<\tau<x_i;\\ \dfrac{(1-\theta)[(1-\theta)(2+\theta)u_i+\theta(2-\theta)v_i]}{2[(1-\theta)u_i+\theta v_i]}, & x_i<\tau<x;\\ -\dfrac{\theta[(1-\theta)(1+\theta)u_i+\theta(3-\theta)v_i]}{2[(1-\theta)u_i+\theta v_i]}, & x<\tau<x_{i+1};\\ -\dfrac{\theta^2(1-\theta)v_i}{2[(1-\theta)u_i+\theta v_i]}, & x_{i+1}<\tau<x_{i+1}.\end{cases}$$

$$=\begin{cases}p(\tau), & x_{i-1}<\tau<x_i;\\ q(\tau), & x_i<\tau<x;\\ r(\tau), & x<\tau<x_{i+1};\\ k(\tau), & x_{i+1}<\tau<x_{i+2}.\end{cases}$$

Obviously,

$$p(\tau)\le 0, x_{i-1}<\tau<x_i; \qquad q(\tau)\ge 0, x_i<\tau<x;$$
$$r(\tau)\le 0, x<\tau<x_{i+1}. \qquad k(\tau)\ge 0, x_{i+1}<\tau<x_{i+2}.$$

After complicated calculation, we obtain

$$\|R(f)\|\le\|f'(x)\|[\int_{x_{i-1}}^{x_i}|p(\tau)|d\tau+\int_{x_{i+1}}^{x_{i+2}}|k(\tau)|d\tau+\int_{x_i}^{x}|q(\tau)|d\tau+\int_{x}^{x_{i+1}}|r(\tau)|d\tau]$$
$$\le\|f'(x)\|\{\int_{x_{i-1}}^{x_i}[-p(\tau)]d\tau+\int_{x_{i+1}}^{x_{i+2}}k(\tau)d\tau+\int_{x_i}^{x}q(\tau)d\tau+\int_{x}^{x_{i+1}}[-r(\tau)]d\tau\}$$
$$\le\frac{\|f'(x)\|\theta(1-\theta)h}{2\left(\left(1-\theta\right)u_i+\theta v_i\right)}\{\left(1-\theta\right)u_i+\left(1-\theta\right)\left(3+2\theta\right)u_i+\theta\left(5-2\theta\right)v_i+\theta v_i\}$$
$$\le\frac{\|f'(x)\|\theta\left(1-\theta\right)h}{\left(1-\theta\right)u_i+\theta v_i}\{\left[\left(1-\theta\right)\left(2+\theta\right)u_i+\theta\left(3-\theta\right)v_i\right]\}.$$

Theorem 1 Suppose $f(x)\in C^1[x_1,x_n]$, and $x_1<x_2<\cdots<x_n$ is an equal-knot spacing. For the given u_i,v_i, $S(x)$ is the corresponding rational cubic interpolation function given in Eq. 1, where d_i are replaced with Eq. 2. Then for $x\in[x_i,x_{i+1}],(i=2,\cdots,n-2)$ there holds

$$\|R(f)\|\le h\|f'(t)\|c_i$$

where $c_i=\max\limits_{0\le\theta\le1}\dfrac{\theta(1-\theta)}{(1-\theta)u_i+\theta v_i}\left[(1-\theta)(2+\theta)u_i+\theta(3-\theta)v_i\right]$, and c_i is called the optimal error constant.

As shown in Table1, although u_i and v_i vary in [0.01, 1000], the optimal error coefficients c_i change little. This means that this kind of interpolation is stable.

TABLE I. VALUES OF c_i FOR VARIOUS OF u_i AND v_i

i	u_i	v_i	θ	c_i
1	1000.0	1.0	0.5480	0.6311
2	100.0	1.0	0.5470	0.6308
3	10.0	1.0	0.5380	0.6289
4	5.0	1.0	0.5300	0.6275
5	1.5	1.0	0.5080	0.6252
6	1.0	1.0	0.5000	0.6250
7	1.0	1.5	0.4920	0.6252
8	1.0	5.0	0.4700	0.6275
9	1.0	10.0	0.4620	0.6289
10	1.0	100.0	0.4530	0.6308
11	1.0	1000.0	0.4520	0.6311

Theorem 2 For given u_i and v_i, the optimal error constant c_i is bounded , and

$$\frac{5}{8} \le c_i \le \frac{\left(\sqrt{7}-1\right)\left(4-\sqrt{7}\right)\left(5+\sqrt{7}\right)}{27} = 0.881917617996....$$

Proof. Let $\omega(u_i, v_i, \theta) = \frac{\theta(1-\theta)}{(1-\theta)u_i + \theta v_i}\left[(1-\theta)(2+\theta)u_i + \theta(3-\theta)v_i\right]$ and $v_i = \lambda_i u_i$,

then $\omega(u_i, v_i, \theta) = U(\lambda_i, \theta) = \frac{\theta(1-\theta)\left[(1-\theta)(2+\theta) + \theta(3-\theta)\lambda_i\right]}{(1-\theta) + \theta\lambda_i}$.

Since

$$\frac{\partial U(\lambda_i, \theta)}{\partial \lambda_i} = \frac{\theta^2(1-\theta)^2(1-2\theta)}{((1-\theta)+\theta\lambda_i)^2} \ge 0,$$

we can get when $\theta \in [0, \frac{1}{2}]$, $U(\lambda_i, \theta)$ is increasing about the parameter $\lambda_i \in (0, +\infty)$, and $U(\lambda_i, \theta)$ is decreasing about the parameter $\lambda_i \in (0, +\infty)$ when $\theta \in [\frac{1}{2}, 1]$. It is easy to known that

$$\theta \in [0, \frac{1}{2}), \lim_{\lambda_i \to 0} U\left(\lambda_i, \theta\right) = \theta(1-\theta)(2+\theta); \lim_{\lambda_i \to +\infty} U\left(\lambda_i, \theta\right) = \theta(1-\theta)(3-\theta).$$

$$\theta \in (\frac{1}{2}, 1], \lim_{\lambda_i \to 0} U\left(\lambda_i, \theta\right) = \theta(1-\theta)(3-\theta); \lim_{\lambda_i \to +\infty} U\left(\lambda_i, \theta\right) = \theta(1-\theta)(2+\theta).$$

Next we only to look for the maximum and the minimum of $\theta(1-\theta)(2+\theta)$ and $\theta(1-\theta)(3-\theta)$ on $\theta\in[0,1]$, after complicate calculation, we obtain

$$\min_{0\le\theta\le1}\left(U(\lambda_i,\theta)\right)=\frac{5}{8}\text{, where }\theta=0.5000.$$

$$\max_{0\le\theta\le1}\left(U(\lambda_i,\theta)\right)=\frac{\left(\sqrt{7}-1\right)\left(4-\sqrt{7}\right)\left(5+\sqrt{7}\right)}{27}\text{, where }\theta=\frac{4-\sqrt{7}}{3}.$$

IV. ERROR ESTIMATION OF THE DERIVATIVE

Since the interpolation is local, without generality it is necessary only to consider the error in the subinterval $[x_i,x_{i+1}]$.Using the Peano-Kernel Theorem gives the following

$$R(f)=f(t)-S(t)=\int_{x_{i-1}}^{x_{i+2}}f'(\tau)R_x\left[(x-\tau)_+^0\right]d\tau$$

then

$$\begin{aligned}R'(f)&=f'(x)-P'(x)\\&=\int_{x_{i-1}}^{t_i}f'(\tau)p_x'(\tau)d\tau+\int_{x_i}^{x_{i+1}}f'(\tau)q_x'(\tau)d\tau+\int_{x_{i+1}}^{x_{i+2}}f'(\tau)k_x'(\tau)d\tau+f'(x)\end{aligned}$$

where

$$p_x'(\tau)=-\frac{(1-\theta)u_i[(1-\theta)(1-2\theta)u_i-2\theta^2v_i]}{2[(1-\theta)u_i+\theta v_i]^2h}$$

$$q_x'(\tau)=-\frac{(1-\theta)^2(1+2\theta)u_i^2+\theta^2(3-2\theta)v_i^2+6\theta(1-\theta)u_iv_i}{2[(1-\theta)u_i+\theta v_i]^2h},$$

$$k_x'(\tau)=\frac{\theta v_i[2(1-\theta)^2u_i+\theta(1-2\theta)v_i]}{2[(1-\theta)u_i+\theta v_i]^2h}.$$

Consider $p_x'(\tau)$, $\tau\in[x_i,x_{i+1}]$, we found that we should first consider the equation

$$(1-\theta)(1-2\theta)u_i-2\theta^2v_i=0$$

It can be proved that for any arbitrary positive parameters u_i and v_i, the equation above has one and only one root in [0,1], that is $\theta_*=\dfrac{3u_i-\sqrt{u_i^2+8u_iv_i}}{4(u_i-v_i)}$.It is easy to shown that, when $\theta \le \theta_*$, $p'_x(\tau) \le 0$ and when $\theta \ge \theta_*$, $p'_x(\tau) \ge 0$. Thus, when $\theta \le \theta_*$,

$$\int_{x_{i-1}}^{x_i} |\, p'_x(\tau)\,|\, d\tau = \int_{x_{i-1}}^{x_i} -p'_x(\tau) d\tau = \frac{(1-\theta)u_i[(1-\theta)(1-2\theta)u_i - 2\theta^2 v_i]}{2[(1-\theta)u_i + \theta v_i]^2}.$$

And when $\theta \ge \theta_*$, we get

$$\int_{x_{i-1}}^{x_i} |\, p'_x(\tau)\,|\, d\tau = \int_{x_{i-1}}^{x_i} p'_x(\tau) d\tau = \frac{(1-\theta)u_i[(1-\theta)(1-2\theta)u_i - 2\theta^2 v_i]}{2[(1-\theta)u_i + \theta v_i]^2}.$$

Secondly, we consider the property of $q'_x(\tau)$, $\tau \in [x_i, x_{i+1}]$ as a function of τ. It is obvious that for all $\theta \in [0,1]$, $q'_x(\tau) \le 0$. So we get

$$\int_{x_i}^{x_{i+1}} \left|q'_x(\tau)\right| d\tau = -\int_{x_i}^{x_{i+1}} q'_x(\tau) d\tau = \frac{(1-\theta)^2(1+2\theta)u_i^2 + \theta^2(3-2\theta)v_i^2 + 6\theta(1-\theta)u_i v_i}{2[(1-\theta)u_i + \theta v_i]^2} \quad .$$

Thirdly, in the similar way, let $2(1-\theta)^2 u_i + \theta(1-2\theta)v_i = 0$, θ^* is the only root on [0,1], and

$$\theta^* = \frac{4u_i - v_i - \sqrt{v_i^2 + 8u_i v_i}}{4(u_i - v_i)}$$

Obviously, when $\theta \le \theta^*$, $k'_x(\tau) \ge 0$ and when $\theta \ge \theta^*$, $k'_x(\tau) \le 0$.

And when $\theta \le \theta^*$,

$$\int_{x_{i+1}}^{x_{i+2}} |\, k'_x(\tau)\,|\, d\tau = \int_{x_{i-1}}^{x_i} k'_x(\tau) d\tau = \frac{\theta v_i[2(1-\theta)^2 u_i + \theta(1-2\theta)v_i]}{2[(1-\theta)u_i + \theta v_i]^2}.$$

when $\theta \ge \theta^*$, we can get

$$\int_{x_{i+1}}^{x_{i+2}} |\, k'_x(\tau)\,|\, d\tau = \int_{x_{i-1}}^{x_i} -k'_x(\tau) d\tau = -\frac{\theta v_i[2(1-\theta)^2 u_i + \theta(1-2\theta)v_i]}{2[(1-\theta)u_i + \theta v_i]^2}.$$

By the analysis above, the following is easily to get.

$$0 < \theta_* = \frac{3u_i - \sqrt{u_i^2 + 8u_i v_i}}{4\left(u_i - v_i\right)} < \frac{4u_i - v_i - \sqrt{v_i^2 + 8u_i v_i}}{4(u_i - v_i)} = \theta^* < 1.$$

Theorem 2 Suppose $f(x) \in C^1[x_1, x_n]$, and $x_1 < x_2 < \cdots < x_n$ is an equal-knot spacing. For the given u_i, v_i, $S(x)$ is the corresponding rational cubic interpolation function given in Eq. 1, where d_i are replaced with Eq. 2. Then for $x \in [x_i, x_{i+1}], (i = 2, \cdots, n-2)$, the error of the derivative $S'(x)$ satisfies

$$\| f'(x) - S'(x) \| \le \| f'(x) \| (c_i + 1),$$

Where $c_i = \max_{0 \le \theta \le 1} W(u_i, v_i, \theta)$.

$$W(u_i, v_i, \theta) = \begin{cases} 1 + \dfrac{\theta v_i [2(1-\theta)^2 u_i + \theta(1-2\theta) v_i]}{[(1-\theta)u_i + \theta v_i]^2}, & 0 \le \theta \le \theta_*; \\ 2 - \dfrac{2[(1-\theta)^3 u_i^2 + \theta^3 v_i^2]}{[(1-\theta)u_i + \theta v_i]^2}, & \theta_* \le \theta \le \theta^*; \\ 1 + \dfrac{(1-\theta)u_i[-(1-\theta)(1-2\theta)u_i + 2\theta^2 v_i]}{[(1-\theta)u_i + \theta v_i]^2}, & \theta^* \le \theta \le 1. \end{cases}$$

Some values of c_i for different $u_i > 0, v_i > 0$ are given in Table 2 .As shown in Table 2, although u_i and v_i varies, the optimal error coefficients c_i change little. This means that this kind of interpolation is stable.

Table 2 : Values of c_i for various of u_i and v_i

i	u_i	v_i	θ	c_i
1	10.0	5.0	0.6132	1.5195
2	5.0	10.0	0.3868	1.5195
3	1.5	1.2	0.5971	1.5021
4	1.1	0.9	0.5335	1.5017
5	0.9	1.1	0.4665	1.5017
6	0.5	0.4	0.5371	1.5021
7	0.3	0.2	0.5670	1.5068
8	0.2	0.3	0.4530	1.5068

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